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# EQUIVALENCE RELATIONS AMONG SOME INEQUALITIES ON OPERATOR MEANS

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*Dedicated to Professor Kichisuke Saito.*

Abstract. We will consider some inequalities on operator means for more than three operators, for instance, ALM and BMP geometric means will be considered. Moreover, log-Euclidean and logarithmic means for several operators will be treated.

## 1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $A$ is said to be positive semi-definite (resp. positive definite) if and only if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$ (resp. $\langle Ax, x\rangle > 0$ for all non-zero $x \in \mathcal{H}$). We denote positive semi-definite operator $A \in B(\mathcal{H})$ by $A \geq 0$. Let $B(\mathcal{H})_+$ and $B(\mathcal{H})_{sa}$ be the sets of all positive definite and self-adjoint operators, respectively. We can consider the order among $B(\mathcal{H})_{sa}$, i.e., for $A, B \in B(\mathcal{H})_{sa}$,

$$A \leq B \quad \text{if and only if} \quad 0 \leq B - A.$$

A real valued function $f$ on an interval $J \subset \mathbb{R}$ is called an *operator monotone function* if and only if

$$A \leq B \quad \text{implies} \quad f(A) \leq f(B)$$

for all $A, B \in B(\mathcal{H})_{sa}$ whose spectra are contained in $J$.

For two positive definite operators, the operator mean is important in the operator theory.

**Definition 1** (Operator mean [6])**.** *A binary operation $\sigma : B(\mathcal{H})_+^2 \to B(\mathcal{H})_+$ is called an operator mean if and only if the following conditions are satisfied.*

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(1) *If* $A \leq C$ *and* $B \leq D$, *then* $A\sigma B \leq C\sigma D$,
(2) $X^*(A\sigma B)X \leq (X^*AX)\sigma(X^*BX)$ *for* $X \in B(\mathcal{H})$,
(3) $A_n\sigma B_n \downarrow A\sigma B$ *when* $A_n \downarrow A$ *and* $B_n \downarrow B$ *in the strong operator topology,*
(4) $I\sigma I = I$, *where* $I$ *means the identity operator on* $\mathcal{H}$.

We notice that operator means can be defined for positive semi-definite operators by (3) in Definition 1. Kubo-Ando [6] have shown the following important result:

**Theorem A** ([6])**.** *For each operator mean* $\sigma$, *there exists the unique operator monotone function* $f : (0,\infty) \longrightarrow (0,\infty)$ *such that* $f(1) = 1$ *and*

$$f(t)I = I\sigma(tI) \quad \textit{for all } t \in (0,\infty).$$

*Moreover for* $A \in B(\mathcal{H})_+$ *and* $B \geq 0$, *the formula*

$$A\sigma B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$$

*holds, where the right hand side is defined via the analytic functional calculus. An operator monotone function* $f$ *is called the representing function of* $\sigma$.

Typical examples of operator means are weighted harmonic, geometric and arithmetic means denoted by $!_w$, $\sharp_w$ and $\nabla_w$ for $w \in [0,1]$, respectively. Their representing functions are $[(1-w) + wt^{-1}]^{-1}$, $t^w$ and $1 - w + wt$, respectively. In fact, we can define $A!_wB = [(1-w)A^{-1} + wB^{-1}]^{-1}$, $A\sharp_wB = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^wA^{\frac{1}{2}}$ and $A\nabla_wB = (1-w)A + wB$.

Extending Kubo-Ando theory to the theory for three or more operators was a long standing problem, in particular, we did not have any nice definition of geometric mean for three operators. Recently, Ando-Li-Mathias have given a nice definition of geometric mean for $n$-tuples of positive definite matrices in [1]. Then many authors study about operator means for $n$-tuples of positive definite operators, and now we have three definitions of geometric means which are called ALM, BMP and the Karcher means. Moreover, we have an extension of the Karcher mean which is called the power mean.

M. Uchiyama and one of the authors have obtained equivalence relations between inequalities for the power and arithmetic means as extensions of a converse of Loewner-Heinz inequality [13].

In this paper, we shall investigate the previous research [13] to other operator means for $n$-tuples of operators. In fact, we shall treat ALM and BMP means, moreover we shall discuss some types of logarithmic means of several operators. This paper is organized as follows. In Section 2, we will introduce some definitions and notations which will be used in this paper. Then we shall consider weighted operator means in the view point of their representing functions in Section 3. In Section 4, we shall consider generalizations of the results by M. Uchiyama and one

of the authors [13]. Especially, we shall consider the log-Euclidean mean which is a kind of geometric mean for $n$-tuples of positive definite operators. In the last section, we shall introduce some properties of the $M$-logarithmic mean which is generated from an arbitrary operator mean via an integration.

## 2. Preliminaries

Let $OM$ be the set of all operator monotone functions on $(0,\infty)$, and let $OM_1 = \{f \in OM : \ f(1) = 1\}$. For $f \in OM_1$, there exists an operator mean $\sigma_f$ such that

$$A\sigma_f B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$$

for $A, B \in B(\mathcal{H})_+$. It is well known that for $w \in [0,1]$, if

$$A!_w B \leq A\sigma_f B \leq A\nabla_w B$$

holds for all $A, B \in B(\mathcal{H})_+$, then

$$\left[(1-w) + wt^{-1}\right]^{-1} \leq f(t) \leq (1-w) + wt$$

holds for all $t > 0$.

Let $A, B \in B(\mathcal{H})_+$. The Thompson metric $d(A,B)$ is defined by

$$d(A,B) = \max\{\log M(A/B), \log M(B/A)\},$$

where $M(A/B) = \inf\{\alpha > 0 \mid B \leq \alpha A\}$. It is known that a cone of positive definite operators is a complete metric space for the Thompson metric. In what follows, we will consider "limit" of operator sequences or "continuous" of operator valued functions in the Thompson metric without any explanation.

For $n$-tuples of positive definite operators, the ALM and BMP (geometric) means are defined as follows.

**Theorem B** (ALM mean [1])**.** *For $\mathbb{A} = (A_1, A_2) \in B(\mathcal{H})_+^2$, the ALM (geometric) mean $\mathfrak{G}_{ALM}(\mathbb{A})$ of $\mathbb{A}$ is defined by $\mathfrak{G}_{ALM}(\mathbb{A}) = A_1 \sharp_{1/2} A_2$. Assume that the ALM (geometric) mean $\mathfrak{G}_{ALM}(\cdot)$ on $B(\mathcal{H})_+^{n-1}$ is defined. Let $\mathbb{A} = (A_1, \ldots, A_n) \in B(\mathcal{H})_+^n$ and $\{A_i^{(r)}\}_{r=0}^{\infty}$ $(i = 1, ..., n)$ be the sequences of positive definite operators defined by*

$$A_i^{(0)} = A_i \quad \text{and} \quad A_i^{(r+1)} = \mathfrak{G}_{ALM}\left((A_j^{(r)})_{j\neq i}\right),$$

*where $(A_j^{(r)})_{j\neq i} = (A_1^{(r)}, ..., A_{i-1}^{(r)}, A_{i+1}^{(r)}, ..., A_n^{(r)})$. Then there exists $\lim_{r\to\infty} A_i^{(r)}$ $(i = 1, ..., n)$ and it does not depend on $i$. The ALM (geometric) mean $\mathfrak{G}_{ALM}(\mathbb{A})$ for $n$-tuples of positive definite operators $\mathbb{A} \in B(\mathcal{H})_+^n$ is defined by $\lim_{r\to\infty} A_i^{(r)}$.*

A vector $\omega = (w_1, ..., w_n) \in (0,1)^n$ is said to be a *probability vector* if and only if $\sum_k w_k = 1$. Let $\Delta_n$ be the set of all probability vectors in $(0,1)^n$.

**Theorem C** (BMP mean [3, 5, 9]). *For $\mathbb{A} = (A_1, A_2) \in B(\mathcal{H})_+^2$ and $\omega = (1-w, w) \in \Delta_2$, the BMP (geometric) mean $\mathfrak{G}_{BMP}(\omega; \mathbb{A})$ of $\mathbb{A}$ is defined by $\mathfrak{G}_{BMP}(\omega; \mathbb{A}) = A_1 \sharp_w A_2$. Assume that the BMP (geometric) mean $\mathfrak{G}_{BMP}(\cdot; \cdot)$ on $\Delta_{n-1} \times B(\mathcal{H})_+^{n-1}$ is defined. Let $\mathbb{A} = (A_1, \ldots, A_n) \in B(\mathcal{H})_+^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$. Define the sequences of positive definite operators $\{A_i^{(r)}\}_{r=0}^{\infty}$ $(i = 1, ..., n)$ by*

$$A_i^{(0)} = A_i \quad \text{and} \quad A_i^{(r+1)} = \mathfrak{G}_{BMP}\left(\hat{\omega}_{\neq i}; (A_j^{(r)})_{j \neq i}\right) \sharp_{w_i} A_i^{(r)},$$

*where $\hat{\omega}_{\neq i} = \frac{1}{\sum_{j \neq i} w_j}(w_j)_{j \neq i}$. Then there exists $\lim_{r \to \infty} A_i^{(r)}$ $(i = 1, ..., n)$ and it does not depend on $i$. The BMP (geometric) mean $\mathfrak{G}_{BMP}(\omega; \mathbb{A})$ for $n$-tuples of positive definite operators $\mathbb{A} \in B(\mathcal{H})_+^n$ is defined by $\lim_{r \to \infty} A_i^{(r)}$.*

We remark that it is not known any weighted ALM mean. Let $\mathbb{A} = (A_1, ..., A_n), \mathbb{B} = (B_1, ..., B_n) \in B(\mathcal{H})_+^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$. Here we denote the above geometric means of $\mathbb{A}$ for the weight $\omega$ by $\mathfrak{G}(\omega; \mathbb{A})$, and they have at least 10 basic properties [1, 3, 5, 9] as follows (in the ALM mean case, we consider just only $\omega = (\frac{1}{n}, ..., \frac{1}{n})$ case).

(P1) If $A_1, ..., A_n$ commute with each other, then

$$\mathfrak{G}(\omega; \mathbb{A}) = \prod_{k=1}^{n} A_k^{w_k}.$$

(P2) For positive numbers $a_1, ..., a_n$,

$$\mathfrak{G}(\omega; a_1 A_1, ..., a_n A_n) = \mathfrak{G}(\omega; a_1, ..., a_n)\mathfrak{G}(\omega; \mathbb{A}) = \left(\prod_{k=1}^{n} a_k^{w_k}\right) \mathfrak{G}(\omega; \mathbb{A}).$$

(P3) For any permutation $\sigma$ on $\{1, 2, ..., n\}$,

$$\mathfrak{G}(w_{\sigma(1)}, ..., w_{\sigma(n)}; A_{\sigma(1)}, ..., A_{\sigma(n)}) = \mathfrak{G}(\omega; \mathbb{A}).$$

(P4) If $A_i \leq B_i$ for $i = 1, ..., n$, then $\mathfrak{G}(\omega; \mathbb{A}) \leq \mathfrak{G}(\omega; \mathbb{B})$.

(P5) $\mathfrak{G}(\omega; \cdot)$ is continuous on each operators. Especially,

$$d(\mathfrak{G}(\omega; \mathbb{A}), \mathfrak{G}(\omega; \mathbb{B})) \leq \sum_{i=1}^{n} w_i d(A_i, B_i).$$

(P6) For each $t \in [0, 1]$, $(1-t)\mathfrak{G}(\omega; \mathbb{A}) + t\mathfrak{G}(\omega; \mathbb{B}) \leq \mathfrak{G}(\omega; (1-t)\mathbb{A} + t\mathbb{B})$.

(P7) For any invertible $X \in B(\mathcal{H})$, $\mathfrak{G}(\omega; X^* A_1 X, ..., X^* A_n X) = X^* \mathfrak{G}(\omega; \mathbb{A}) X$.

(P8) $\mathfrak{G}(\omega; \mathbb{A}^{-1})^{-1} = \mathfrak{G}(\omega; \mathbb{A})$, where $\mathbb{A}^{-1} = (A_1^{-1}, ..., A_n^{-1})$.

(P9) If every $A_i$ is a positive definite matrix, then $\det \mathfrak{G}(\omega; \mathbb{A}) = \prod_{i=1}^{n} \det A_i^{w_i}$.

(P10)

$$\left[\sum_{i=1}^{n} w_i A_i^{-1}\right]^{-1} \leq \mathfrak{G}(\omega; \mathbb{A}) \leq \sum_{i=1}^{n} w_i A_i.$$

## 3. Operator means of two variables

In this section, we shall consider the weighted operator means in the view point of their weight.

**Theorem 1.** *Let $\Phi, f \in OM_1$ be non-constant, and let $\sigma$ be an operator mean whose representing function is $\Phi$. If $\Phi'(1) = w \in (0,1)$, then for $A, B \in B(\mathcal{H})_{sa}$, they are mutually equivalent:*

(1) $(1-w)A \leq wB$,
(2) $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$ *holds for all sufficiently small* $\lambda \geq 0$.

Theorem 1 is an extension of the following Theorem D in [13] by Lemma 2 introduced in the below. It was shown as a converse of Loewner-Heinz inequality.

**Theorem D** ([13])**.** *Let $f(t) \in OM_1$ be non-constant, and let $A, B \in B(\mathcal{H})_{sa}$. Let $\sigma$ be an operator mean satisfying $!_{1/2} \leq \sigma \leq \nabla_{1/2}$. Then $A \leq B$ if and only if $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$ for all sufficiently small $\lambda \geq 0$.*

To prove Theorem 1, we need the following lemma.

**Lemma 2.** *Let $\Phi \in OM_1$. Then for each $w \in (0,1)$, they are mutually equivalent:*

(1) $\Phi'(1) = w$,
(2) $[(1-w) + wt^{-1}]^{-1} \leq \Phi(t) \leq (1-w) + wt$ *for all* $t \in (0, \infty)$.

*Proof.* Proof of (1) $\Longrightarrow$ (2) has been given in [12, Lemma 2.2]. But we shall introduce its proof for the reader's convenience. Since every operator monotone function is operator concave, $\Phi$ is a concave function. We have

$$\Phi(t) \leq \Phi(1) + \Phi'(1)(t-1) = (1-w) + wt.$$

On the other hand, $\frac{t}{\Phi(t)}$ is also an operator monotone function, and

$$\frac{d}{dt}\frac{t}{\Phi(t)}\bigg|_{t=1} = 1 - w.$$

Then by the same argument as above, we have

$$\frac{t}{\Phi(t)} \leq w + (1-w)t,$$

that is,

$$[(1-w) + wt^{-1}]^{-1} \leq \Phi(t).$$

Conversely, we shall prove (2) $\Longrightarrow$ (1). Since the tangent line of $f(t) = [(1-w) + wt^{-1}]^{-1}$ at $t = 1$ is $y = (1-w) + wt$, $\Phi(t)$ has the same tangent line as that of $[(1-w) + wt^{-1}]^{-1}$ at $t = 1$. Therefore $\Phi'(1) = w$. □

Before proving Theorem 1, we introduce the following formulas. For any differential function $f$ on 1 satisfying $f(1) = 1$ and $w \in (0, 1)$, the following hold in the norm topology.

$$\lim_{\lambda \to 0} f(\lambda A + I)^{\frac{1}{\lambda}} = e^{f'(1)A} \text{ for } A \in B(\mathcal{H})_{sa}, \tag{3.1}$$

$$\lim_{p \to 0} \left[(1-w)A^p + wB^p\right]^{\frac{1}{p}} = \exp\left[(1-w)\log A + w \log B\right] \text{ for } A, B \in B(\mathcal{H})_+, \tag{3.2}$$

where (3.1) can be obtained by $\lim_{\lambda \to 0} f(\lambda a + 1)^{\frac{1}{\lambda}} = e^{f'(1)a}$ for $a \in \mathbb{R}$, and (3.2) is introduced in [11, (2.4)], for example.

*Proof of Theorem 1.* By Lemma 2, $\Phi'(1) = w$ is equivalent to

$$[(1-w) + wt^{-1}]^{-1} \leq \Phi(t) \leq (1-w) + wt \quad \text{for all } t > 0. \tag{3.3}$$

We shall prove (1) $\Longrightarrow$ (2). If $(1-w)A \leq wB$, then it is equivalent to $(1-w)(\lambda A + I) + w(-\lambda B + I) \leq I$ for all $\lambda \geq 0$. Since $f$ is an operator concave function with $f(1) = 1$, we have

$$\begin{aligned} I = f(I) &\geq f\left((1-w)(\lambda A + I) + w(-\lambda B + I)\right) \\ &\geq (1-w) f(\lambda A + I) + w f(-\lambda B + I) \\ &\geq f(\lambda A + I) \sigma f(-\lambda B + I), \end{aligned}$$

where the last inequality holds by (3.3).

Conversely, assume that $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$ for all sufficiently small $\lambda \geq 0$. By (3.3), we have

$$\begin{aligned} I &\geq f(\lambda A + I)\sigma f(-\lambda B + I) \\ &\geq \left[(1-w) f(\lambda A + I)^{-1} + w f(-\lambda B + I)^{-1}\right]^{-1} \\ &\geq \left[(1-w) f(\lambda A + I)^{\frac{-p}{\lambda}} + w f(-\lambda B + I)^{\frac{-p}{\lambda}}\right]^{\frac{-\lambda}{p}} \end{aligned}$$

for all $0 < \lambda \leq p$, where the last inequality follows from the operator concavity of $t^\alpha$ for $\alpha \in [0, 1]$. Then we have

$$\left[(1-w) f(\lambda A + I)^{\frac{-p}{\lambda}} + w f(-\lambda B + I)^{\frac{-p}{\lambda}}\right]^{\frac{-1}{p}} \leq I.$$

By letting $\lambda \to 0$ and (3.1), we have

$$\left[(1-w) e^{-pf'(1)A} + w e^{pf'(1)B}\right]^{\frac{-1}{p}} \leq I,$$

and $p \to 0$, we have

$$\exp\left(-(1-w) f'(1) A + w f'(1) B\right) \geq I$$

by (3.2). It is equivalent to $(1-w)A \leq wB$. □

A kind of a converse of Theorem 1 can be considered as follows.

**Proposition 3.** *Let $\Phi, f \in OM_1$ be non-constant, and let $\sigma$ be an operator mean whose representing function is $\Phi$. For $A, B \in B(\mathcal{H})_{sa}$ and $w \in (0,1)$, if $f(\lambda A + I)\sigma f(-\lambda B + I) \leq I$ holds for all sufficiently small $\lambda \geq 0$ whenever $(1-w)A \leq wB$. Then $\Phi'(1) = w$.*

*Proof.* We may assume $f'(1) > 0$. Let $A = wtI$ and $B = (1-w)tI$ for a real number $t$. Then we have $(1-w)A \leq wB$. By the assumption, we have $f(\lambda tw + 1)\sigma f(-\lambda t(1-w)+1) \leq 1$ holds for all sufficiently small $\lambda \geq 0$. It is equivalent to

$$\Phi\left(\frac{f(-\lambda t(1-w)+1)}{f(\lambda tw+1)}\right) \leq \frac{1}{f(\lambda tw+1)}.$$

For each $\lambda > 0$, we have

$$\frac{\Phi\left(\frac{f(-\lambda(1-w)t+1)}{f(\lambda tw+1)}\right) - 1}{\lambda} \leq \frac{\frac{1}{f(\lambda wt+1)} - 1}{\lambda}.$$

Letting $\lambda \to 0$, the right-hand side of the above inequality converges to

$$\frac{\partial}{\partial\lambda}\frac{1}{f(\lambda wt+1)}\bigg|_{\lambda=0} = \frac{-wtf'(\lambda wt+1)}{f(\lambda wt+1)^2}\bigg|_{\lambda=0} = -wtf'(1)$$

by the assumption $f(1) = 1$. On the other hand, the left-hand side is

$$\frac{\partial}{\partial\lambda}\Phi\left(\frac{f(-\lambda(1-w)t+1)}{f(\lambda tw+1)}\right)\bigg|_{\lambda=0} = -t\Phi'(1)f'(1)$$

by the assumption $f(1) = 1$. Hence we have $t\Phi'(1) \geq wt$ for all real number $t$. Hence we have $\Phi'(1) = w$. □

## 4. More than three operators case

Let $\mathbb{A} = (A_1, ..., A_n) \in B(\mathcal{H})_+^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$. Define

$$\mathfrak{A}(\omega; \mathbb{A}) = \sum_{i=1}^n w_i A_i \quad \text{and} \quad \mathfrak{H}(\omega; \mathbb{A}) = \left(\sum_{i=1}^n w_i A_i^{-1}\right)^{-1}.$$

As an extension of the Karcher mean, the power mean is given by Lim-Páifia [10] as follows. Let $\mathbb{A} = (A_1, ..., A_n) \in B(\mathcal{H})_+^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$. For $t \in (0, 1]$, the power mean $P_t(\omega; \mathbb{A})$ is defined by the unique positive definite solution of

$$X = \sum_{k=1}^n w_k X \sharp_t A_k,$$

and for $t \in [-1, 0)$, the power mean $P_t(\omega; \mathbb{A})$ is defined by $P_t(\omega; \mathbb{A}) = P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$ (see also [8]). We remark that $P_t(\omega; \mathbb{A})$ converges to the Karcher mean $\Lambda(\omega; \mathbb{A})$ as $t \to 0$, strongly. So we can consider $P_0(\omega; \mathbb{A})$ as $\Lambda(\omega; \mathbb{A})$. It is known that the

Karcher mean also satisfies all properties (P1) – (P10) in Section 2 (cf. [2, 7, 8]). It is easy to see that $P_1(\omega;\mathbb{A}) = \mathfrak{A}(\omega;\mathbb{A})$ and $P_{-1}(\omega;\mathbb{A}) = \mathfrak{H}(\omega;\mathbb{A})$. Moreover $P_t(\omega;\mathbb{A})$ is increasing on $t \in [-1,1]$. Hence the power mean interpolates arithmetic-geometric-harmonic means. In [13], we had a generalization of Theorem D as follows.

**Theorem E** ([13]). *Let $T_1,...,T_n$ be Hermitian matrices, and $\omega = (w_1,...,w_n) \in \Delta_n$. Let $f \in OM_1$ be non-constant. Then the following assertions are equivalent:*

(1) $\displaystyle\sum_{i=1}^n w_i T_i \le 0$,

(2) $P_1(\omega; f(\lambda T_1 + I),..., f(\lambda T_n + I)) = \displaystyle\sum_{i=1}^n w_i f(\lambda T_i + I) \le I$ *for all sufficiently small $\lambda \ge 0$,*

(3) *for each $t \in [-1,1]$, $P_t(\omega; f(\lambda T_1 + I),..., f(\lambda T_n + I)) \le I$ for all sufficiently small $\lambda \ge 0$.*

Here we shall generalize the above result into the following Theorem 4.

**Theorem 4.** *Let $f \in OM_1$ be non-constant, and let $\Phi : \Delta_n \times B(\mathcal{H})_+^n \times \mathcal{H} \to \mathbb{R}^+$ satisfying*

$$\|\mathfrak{H}(\omega;\mathbb{A})\| \le \sup_{\|x\|=1} \Phi(\omega;\mathbb{A};x) \le \|\mathfrak{A}(\omega;\mathbb{A})\| \tag{4.1}$$

*for all $\mathbb{A} \in B(\mathcal{H})_+^n$ and $\omega \in \Delta_n$. Then for $\mathbb{T} = (T_1,...,T_n) \in B(\mathcal{H})_{sa}^n$ and $\omega = (w_1,...,w_n) \in \Delta_n$, they are mutually equivalent:*

(1) $\displaystyle\sum_{i=1}^n w_i T_i \le 0$,

(2) $\Phi(\omega; f(\lambda T_1 + I),..., f(\lambda T_n + I); x) \le 1$ *for all sufficiently small $\lambda \ge 0$ and all unit vector $x \in \mathcal{H}$.*

In fact, we obtain Theorem E by putting $\Phi(\omega;\mathbb{A};x) = \langle P_t(\omega;\mathbb{A})x, x\rangle$ in Theorem 4.

*Proof of Theorem 4.* First of all, we may assume $f'(1) > 0$. Firstly, we shall prove (1) $\Longrightarrow$ (2). For each $\lambda > 0$, (1) is equivalent to

$$\sum_{i=1}^n w_i(\lambda T_i + I) \le I.$$

Since operator concavity of $f$ and $f(1) = 1$, we have

$$\begin{aligned} I = f(I) &\ge f\left(\sum_{i=1}^n w_i(\lambda T_i + I)\right) \\ &\ge \sum_{i=1}^n w_i f(\lambda T_i + I) = \mathfrak{A}(\omega; f(\lambda T_1 + I),..., f(\lambda T_n + I)). \end{aligned}$$

Here by (4.1),

$$\begin{aligned} 1 &\geq \|\mathfrak{A}(\omega; f(\lambda T_1 + I), ..., f(\lambda T_n + I))\| \\ &\geq \sup_{\|x\|=1} \Phi(\omega; f(\lambda T_1 + I), ..., f(\lambda T_n + I); x), \end{aligned}$$

we have

$$1 \geq \Phi(\omega; f(\lambda T_1 + I), ..., f(\lambda T_n + I); x)$$

for all unit vector $x \in \mathcal{H}$, i.e., (2).

Conversely, we shall prove (2) $\Longrightarrow$ (1). By (4.1), we have

$$\begin{aligned} 1 &\geq \sup_{\|x\|=1} \Phi(\omega; f(\lambda T_1 + I), ..., f(\lambda T_n + I); x) \\ &\geq \|\mathfrak{H}(\omega; f(\lambda T_1 + I), ..., f(\lambda T_n + I))\|. \end{aligned}$$

Then

$$\begin{aligned} I &\geq \mathfrak{H}(\omega; f(\lambda T_1 + I), ..., f(\lambda T_n + I)) \\ &= \left[\sum_{i=1}^n w_i f(\lambda T_i + I)^{-1}\right]^{-1} \geq \left[\sum_{i=1}^n w_i f(\lambda T_i + I)^{\frac{-p}{\lambda}}\right]^{\frac{-\lambda}{p}} \end{aligned}$$

for all $0 < \lambda \leq p$ since $t^\alpha$ is operator concave for $\alpha \in [0, 1]$. Hence we have

$$\left[\sum_{i=1}^n w_i f(\lambda T_i + I)^{\frac{-p}{\lambda}}\right]^{\frac{-1}{p}} \leq I.$$

By letting $\lambda \to 0$ and (3.1), we obtain

$$\left[\sum_{i=1}^n w_i e^{-pf'(1)T_i}\right]^{\frac{-1}{p}} \leq I,$$

and $p \to 0$, we have $f'(1) \sum_{i=1}^n w_i T_i \leq 0$, that is, (1). □

**Corollary 5.** *Let $f \in OM_1$ be non-constant. Then for $\mathbb{T} = (T_1, ..., T_n) \in B(\mathcal{H})_{sa}^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$, they are mutually equivalent:*

(1) $\displaystyle\sum_{i=1}^n w_i T_i \leq 0$,

(2) $\displaystyle\prod_{i=1}^n \|f(\lambda T_i + I)^{\frac{1}{2}} x\|^{w_i} \leq 1$ *for all sufficiently small $\lambda > 0$ and all unit vector $x \in \mathcal{H}$.*

*Proof.* For $\mathbb{A} = (A_1, ..., A_n) \in B(\mathcal{H})_+^n$, let $\Phi(\omega; \mathbb{A}; x) = \prod_{i=1}^n \|A_i^{\frac{1}{2}} x\|^{2w_i}$. We shall only check

$$\|\mathfrak{H}(\omega; \mathbb{A})\| \leq \sup_{\|x\|=1} \prod_{i=1}^n \|A_i^{\frac{1}{2}} x\|^{2w_i} \leq \|\mathfrak{A}(\omega; \mathbb{A})\|$$

for all $\mathbb{A} = (A_1, ..., A_n) \in B(\mathcal{H})_+^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$. Firstly, we shall show $\sup_{\|x\|=1} \prod_{i=1}^n \|A_i^{\frac{1}{2}} x\|^{2w_i} \leq \|\mathfrak{A}(\omega; \mathbb{A})\|$.

$$\prod_{i=1}^n \|A_i^{\frac{1}{2}} x\|^{2w_i} = \prod_{i=1}^n \langle A_i x, x\rangle^{w_i} \leq \sum_{i=1}^n w_i \langle A_i x, x\rangle = \langle \mathfrak{A}(\omega; \mathbb{A}) x, x\rangle.$$

Hence, we have $\sup_{\|x\|=1} \prod_{i=1}^n \|A_i^{\frac{1}{2}} x\|^{2w_i} \leq \|\mathfrak{A}(\omega; \mathbb{A})\|$.

Next, we shall prove $\|\mathfrak{H}(\omega; \mathbb{A})\| \leq \sup_{\|x\|=1} \prod_{i=1}^n \|A_i^{\frac{1}{2}} x\|^{2w_i}$.

$$\begin{aligned}
\prod_{i=1}^n \|A_i^{\frac{1}{2}} x\|^{2w_i} &= \prod_{i=1}^n \langle A_i x, x\rangle^{w_i} \\
&\geq \langle \Lambda(\omega; \mathbb{A}) x, x\rangle \quad \text{(by [14])} \\
&\geq \langle \mathfrak{H}(\omega; \mathbb{A}) x, x\rangle.
\end{aligned}$$

Therefore the proof is completed by Theorem 4. □

Corollary 5 is an extension of a part of the following result:

**Theorem F** ([13]). *Let $T_1, ..., T_n$ be Hermitian matrices, and let $f \in OM_1$ be non-constant. Then the following are equivalent:*

(1) $\sum_{i=1}^n T_i \leq 0$,

(2) $\|x\|^n \leq \prod_{i=1}^n \|f(\lambda T_i + I)^{\frac{-1}{2}} x\|$ *for all sufficiently small $\lambda \geq 0$ and all $x \in \mathcal{H}$.*

In fact, we can obtain the assertion (1) $\Longrightarrow$ (2) of Theorem F as follows: If $\sum_{i=1}^n T_i \leq 0$, then we have

$$\prod_{i=1}^n \|f(\lambda T_i + I)^{\frac{1}{2}} x\|^{\frac{1}{n}} \leq 1$$

for all sufficiently small $\lambda > 0$ and all unit vector $x \in \mathcal{H}$ by Corollary 5 and putting $\omega = (\frac{1}{n}, ..., \frac{1}{n})$. Then we have

$$\begin{aligned}
1 &\leq \prod_{i=1}^n \|f(\lambda T_i + I)^{\frac{1}{2}} x\|^{\frac{-1}{n}} \\
&= \prod_{i=1}^n \langle f(\lambda T_i + I) x, x\rangle^{\frac{-1}{2n}} \\
&\leq \prod_{i=1}^n \langle f(\lambda T_i + I)^{-1} x, x\rangle^{\frac{1}{2n}} \quad \text{(by the Hölder-McCarthy inequality)} \\
&= \prod_{i=1}^n \|f(\lambda T_i + I)^{\frac{-1}{2}} x\|^{\frac{1}{n}}.
\end{aligned}$$

By replacing $x$ into $\frac{x}{\|x\|}$, we have (2) of Theorem F.

From here we shall consider another geometric mean for $n$-tuples of positive definite operators which is called the log-Euclidean mean $\mathfrak{G}_E(\omega;\mathbb{A})$ for $\mathbb{A}=(A_1,...,A_n)\in B(\mathcal{H})_+^n$ and $\omega=(w_1,...,w_n)\in\Delta_n$. It is defined by

$$\mathfrak{G}_E(\omega;\mathbb{A})=\exp\left(\sum_{i=1}^n w_i\log A_i\right).$$

Log-Euclidean mean satisfies some of properties (P1)–(P10) in Section 2. However, log-Euclidean mean does not satisfy important properties (P4) and (P10).

**Corollary 6.** *Let $f\in OM_1$ be non-constant. For $\mathbb{A}\in B(\mathcal{H})_+^n$ and $\omega\in\Delta_n$, let $M(\omega;\mathbb{A})$ be ALM or weighted BMP or log-Euclidean mean (in the ALM mean case, $\omega$ should be $\omega=(\frac{1}{n},...,\frac{1}{n})$). Then for $\mathbb{T}=(T_1,...,T_n)\in B(\mathcal{H})_{sa}^n$ and $\omega=(w_1,...,w_n)\in\Delta_n$, the following assertions are equivalent:*

(1) $\displaystyle\sum_{i=1}^n w_iT_i\le 0$,

(2) $M(\omega;f(\lambda T_1+I),...,f(\lambda T_n+I))\le I$ *for all sufficiently small* $\lambda\ge 0$.

*Proof.* The cases of ALM and BMP means. Put $\Phi(\omega;\mathbb{A};x)=\langle M(\omega;\mathbb{A})x,x\rangle$. Then by (P10), $\Phi(\omega;\mathbb{A};x)$ satisfies the condition (4.1). So that we can prove the cases of ALM and BMP means by Theorem 4.

By the way, log-Euclidean mean satisfies

$$\log\mathfrak{H}(\omega;\mathbb{A})\le\log\mathfrak{G}_E(\omega;\mathbb{A})\le\log\mathfrak{A}(\omega;\mathbb{A}) \tag{4.2}$$

for $\mathbb{A}\in B(\mathcal{H})_+^n$ and $\omega\in\Delta_n$. In fact, by the operator concavity of $\log t$, we have

$$\begin{aligned}\log\mathfrak{G}_E(\omega;\mathbb{A})&=\log\left[\exp\left(\sum_{i=1}^n w_i\log A_i\right)\right]\\&=\sum_{i=1}^n w_i\log A_i\\&\le\log\left(\sum_{i=1}^n w_iA_i\right)=\log\mathfrak{A}(\omega;\mathbb{A}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\log\mathfrak{H}(\omega;\mathbb{A})&=\log\mathfrak{A}(\omega;\mathbb{A}^{-1})^{-1}\\&=-\log\mathfrak{A}(\omega;\mathbb{A}^{-1})\\&\le-\log\mathfrak{G}_E(\omega;\mathbb{A}^{-1})\\&=\log\mathfrak{G}_E(\omega;\mathbb{A}).\end{aligned}$$

Hence we have (4.2). We remark that if $\log A \leq \log B$ for $A, B \in B(\mathcal{H})_+$, then for each $p > 0$, there is a unitary operator $U_p$ such that $A^p \leq U_p^* B^p U_p$ in [4]. Hence we have $\|A\| \leq \|B\|$. By using this fact to (4.2), we have

$$\|\mathfrak{H}(\omega; \mathbb{A})\| \leq \|\mathfrak{G}_E(\omega; \mathbb{A})\| \leq \|\mathfrak{A}(\omega; \mathbb{A})\|.$$

Hence we can prove Corollary 6 by putting $\Phi(\omega; \mathbb{A}; x) = \langle \mathfrak{G}_E(\omega; \mathbb{A})x, x\rangle$ in Theorem 4. □

## 5. Logarithmic means

We shall consider some logarithmic means for $n$-tuples of positive definite operators. Since the representing function of logarithmic mean is $\frac{t-1}{\log t}$, logarithmic mean $A\lambda B$ of $A, B \in B(\mathcal{H})_+$ can be considered as

$$A\lambda B = \int_0^1 A \sharp_t B dt.$$

So it is quite natural to consider the similar type of integrated means as follows.

**Definition 2** ($M$-logarithmic mean)**.** *Let $M : \Delta_n \times B(\mathcal{H})_+^n \to B(\mathcal{H})_+$. Then for $\mathbb{A} \in B(\mathcal{H})_+^n$, the $M$-logarithmic mean $\mathfrak{L}(M)(\mathbb{A})$ of $\mathbb{A} \in B(\mathcal{H})_+^n$ is defined by*

$$\mathfrak{L}(M)(\mathbb{A}) := \int_{\Delta_n} M(\omega; \mathbb{A}) dp(\omega)$$

*if there exists, where $dp(\omega)$ means an arbitrary probability measure on $\Delta_n$.*

In what follows, we consider the case of $dp(\omega) = (n-1)!d\omega$.

**Proposition 7.** *Let $M : \Delta_n \times B(\mathcal{H})_+^n \to B(\mathcal{H})_+$ satisfying (P3), (P7), (P8) and (P10). Then $M$-logarithmic mean*

$$\mathfrak{L}(M)(\mathbb{A}) = (n-1)! \int_{\Delta_n} M(\omega; \mathbb{A}) d\omega$$

*satisfies (P3) and (P7) for $\omega = (\frac{1}{n}, ..., \frac{1}{n})$ if it exists. Especially, $\mathfrak{L}(M)$ satisfies (P10), i.e.,*

$$\mathfrak{H}(\mathbb{A}) \leq \mathfrak{L}(M)(\mathbb{A}) \leq \mathfrak{A}(\mathbb{A}),$$

*where $\mathfrak{H}(\mathbb{A}) = \mathfrak{H}(\omega; \mathbb{A})$ and $\mathfrak{A}(\mathbb{A}) = \mathfrak{A}(\omega; \mathbb{A})$ for $\omega = (\frac{1}{n}, ..., \frac{1}{n})$.*

We remark that $\mathfrak{L}(\mathfrak{A})(\mathbb{A}) = \mathfrak{A}(\mathbb{A})$. As for the preparation, we define some notations. Let $S$ be the cyclic shift operator on $\mathbb{C}^n$ and let $\mathbb{S}$ be also the cyclic shift operator on $B(\mathcal{H})^n$; namely,

$$S(w_1, w_2, ..., w_n) = (w_2, w_3, ..., w_n, w_1).$$
$$\mathbb{S}(A_1, A_2, ..., A_n) = (A_2, A_3, ..., A_n, A_1).$$

We claim that if $M$ satisfies (P3), then $M(S\omega; \mathbb{A}) = M(\omega; \mathbb{S}^*\mathbb{A})$.

*Proof of Proposition 7.* It is clear that $\mathfrak{L}(M)$ satisfies (P3) and (P7). The remain is to show (P10). Let $\mathbb{M}$ be the set of all maps $M : \Delta_n \times B(\mathcal{H})_+^n \to B(\mathcal{H})_+$. It is easy to show that $\mathfrak{L}$ is a linear map on $\mathbb{M}$, and $\mathfrak{L}(M)(\mathbb{A}) \geq 0$ for all $\mathbb{A} \in B(\mathcal{H})_+^n$ if $M \in \mathbb{M}$. Hence for $N_1, M, N_2 \in \mathbb{M}$, if $N_1(\omega; \mathbb{A}) \leq M(\omega; \mathbb{A}) \leq N_2(\omega; \mathbb{A})$ holds for all $\omega \in \Delta_n$ and $\mathbb{A} \in B(\mathcal{H})_+^n$, then

$$\mathfrak{L}(N_1)(\mathbb{A}) \leq \mathfrak{L}(M)(\mathbb{A}) \leq \mathfrak{L}(N_2)(\mathbb{A})$$

holds for all $\mathbb{A} \in B(\mathcal{H})_+^n$. Since $M(\omega; \mathbb{A})$ satisfies (P10), we have

$$\begin{aligned}
\mathfrak{L}(M)(\mathbb{A}) &= (n-1)! \int_{\Delta_n} M(\omega; \mathbb{A}) d\omega \\
&\leq (n-1)! \int_{\Delta_n} \mathfrak{A}(\omega; \mathbb{A}) d\omega \\
&= \mathfrak{L}(\mathfrak{A})(\mathbb{A}) = \mathfrak{A}(\mathbb{A}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathfrak{H}(\mathbb{A}) &= \mathfrak{A}(\mathbb{A}^{-1})^{-1} \\
&\leq \{\mathfrak{L}(M)(\mathbb{A}^{-1})\}^{-1} \\
&= \left( (n-1)! \int_{\Delta_n} M(\omega; \mathbb{A}^{-1}) d\omega \right)^{-1} \\
&= \left( (n-1)! \int_{\Delta_n} M(\omega; \mathbb{A})^{-1} d\omega \right)^{-1} \qquad \text{by (P8)} \\
&\leq (n-1)! \int_{\Delta_n} M(\omega; \mathbb{A}) d\omega = \mathfrak{L}(M)(\mathbb{A}).
\end{aligned}$$

□

**Remark 8.** The above theorem is valid for an arbitrary permutation (shift)- invariant probability measure $p$.

**Remark 9.** Let $M : \Delta_n \times B(\mathcal{H})_+^n \to B(\mathcal{H})_+$ be a map satisfying (P3), (P7), (P8) and (P10). We put

$$M_0(\omega; \mathbb{A}) := M\left( (\frac{1}{n}, ..., \frac{1}{n}); M(\omega; \mathbb{A}), M(S\omega; \mathbb{A}), ..., M(S^{n-1}\omega; \mathbb{A}) \right).$$

Then $M_0$ satisfies the assumption of Proposition 7. Precisely, $M_0$ satisfies (P3), (P7), (P8) and $\mathfrak{H}(\mathbb{A}) \leq M_0(\omega; \mathbb{A}) \leq \mathfrak{A}(\mathbb{A})$ for all $\omega \in \Delta_n$ and $\mathbb{A} \in B(\mathcal{H})_+^n$. So $\mathfrak{L}(M_0)$ also satisfies (P10). Moreover, the following inequalities hold

$$\mathfrak{H}(\mathbb{A}) \leq \mathfrak{L}(M_0)(\mathbb{A}) \leq \mathfrak{L}(M)(\mathbb{A}) \leq \mathfrak{A}(\mathbb{A}).$$

The second inequality can be shown as follows. Since $M(\omega;\mathbb{A})$ satisfies (P10), we have

$$M_0(\omega;\mathbb{A}) \leq \sum_{k=0}^{n-1} \frac{1}{n} M(S^k\omega;\mathbb{A}).$$

Then we obtain

$$\begin{aligned}
\mathfrak{L}(M_0)(\mathbb{A}) &= (n-1)! \int_{\Delta_n} M_0(\omega;\mathbb{A}) d\omega \\
&\leq (n-1)! \int_{\Delta_n} \left\{ \sum_{k=0}^{n-1} \frac{1}{n} M(S^k\omega;\mathbb{A}) \right\} d\omega \\
&= \frac{(n-1)!}{n} \sum_{k=0}^{n-1} \int_{\Delta_n} M(S^k\omega;\mathbb{A}) d\omega \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \mathfrak{L}(M)(\mathbb{A}) = \mathfrak{L}(M)(\mathbb{A}).
\end{aligned}$$

Since the weighted Karcher mean $\Lambda(\omega;\mathbb{A})$ is continuous on the probability vector in the Thompson metric [8], so $\mathfrak{L}(\Lambda)(\mathbb{A})$ exists.

**Proposition 10.**

$$\mathfrak{H}(\mathbb{A}) \leq \mathfrak{L}(\Lambda)(\mathbb{A}) \leq \mathfrak{A}(\mathbb{A}).$$

*Proof.* Since the weighted Karcher mean satisfies (P1)–(P10) in Section 2 [2, 7, 8], it is easy by Proposition 7. □

**Corollary 11.** *Logarithmic mean $\mathfrak{L}(\Lambda)(\mathbb{A})$ satisfies the same assertion to Corollary 6 for $\omega = (\frac{1}{n}, ..., \frac{1}{n})$, too.*

*Proof.* We can prove Corollary 11 by the same way to the proof of Corollary 6. □

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