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MAPS ON THE SPHERE OF THE ALGEBRAS OF MATRICES

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ABSTRACT. Let S_n be the unit sphere with respect to the operator norm of the algebra of $n \times n$ complex matrices. We give a complete description of the form of surjections on S_n which preserve the metric induced by a unitarily invariant strictly convex norm.

1. Introduction and statement of the main result

The celebrated Mazur-Ulam theorem asserts that every surjective isometry between normed spaces is automatically affine. It is natural to consider isometries between certain subsets of normed spaces. A local Mazur-Ulam theorem due to Mankiewicz [1] states that every isometry between connected open subsets of normed spaces has a unique isometric extension. Tingley [4] proposed the problem if a surjective isometry between spheres in normed spaces is affine. Several attempts are done for the problem, but the problem seems to be still open. This paper concerns with a problem in the same vein. Although the main theorem of the paper is not even a partial answer to the Tingley problem, the author believes that the proof of it gives an ingredient for the future development of the problem.

Let n be a positive integer. Throughout the paper $M_n(\mathbb{C})$ denotes the algebra of all complex matrices of the degree n. A norm $\|\cdot\|$ on $M_n(\mathbb{C})$ is called unitarily invariant if $\|VAV'\| = \|A\|$ for every $A \in M_n(\mathbb{C})$ and any unitary matrices V and V'. For $A \in M_n(\mathbb{C})$, A^* is the adjoint of A; A^{tr} is the transpose of A, $\bar{A}$ is the matrix whose entries are the complex conjugate of the corresponding one's. It is well known that for a unitarily invariant norm $\|\cdot\|$ on $M_n(\mathbb{C})$, $\|A\| = \|A^*\| = \|A^{\mathrm{tr}}\| = \|\bar{A}\|$ for every $A \in M_n(\mathbb{C})$. The Schatten p-norm $(1 \le p < \infty)$ $\|\cdot\|_p$ on $M_n(\mathbb{C})$ is defined by $\|A\|_p = (\sum_{j=1}^n |s_j|^p)^{\frac{1}{p}}$ for $A \in M_n(\mathbb{C})$ with the singular-values $s_1 \ge s_2 \ge \cdots \ge s_n$ for A. Note that $\|\cdot\|_p$ for $1 \le p < \infty$ is unitarily invariant. A matrix is considered as the bounded operator on the Euclidean n-space $\mathbb{C}^n$ with the usual Euclidean

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norm $\|\cdot\|_E$ defined by $\|x\|_E = (\sum_{j=1}^n |x_j|^2)^{\frac{1}{2}}$ for $x = (x_1, \dots, x_n) \in \mathbb{C}^n$. The norm $\|A\|_\infty$ for $A \in M_n(\mathbb{C})$ denotes the operator norm of A as the bounded operator on $(\mathbb{C}^n, \|\cdot\|_E)$. It is well known that $\|A\|_\infty$ coincides with the largest singular value of A. The norm $\|\cdot\|$ on a normed space is called strictly convex if $x = y$ whenever $\|x\| = \|y\| = \frac{1}{2}\|x+y\|$. A norm $\|\cdot\|$ on a normed space is called uniformly convex if for every $0 < \varepsilon \le 2$ there exists a $\delta > 0$ so that for any x and y with $\|x\| = \|y\| = 1$, the condition $\|x+y\| \ge 2(1-\delta)$ implies $\|x-y\| \le \varepsilon$. Note that a uniformly convex norm is strictly convex. Note also that $\|\cdot\|_p$ $(1 < p < \infty)$ is uniformly convex for every $1 < p < \infty$ [2], hence it is strictly convex. On the other hand $\|\cdot\|_\infty$ is not strictly convex.

Let

$$S_n = \{A \in M_n(\mathbb{C}) : \|A\|_\infty = 1\},$$

the unit sphere of $M_n(\mathbb{C})$ with respect to $\|\cdot\|_\infty$. Suppose that V and V' are unitary matrices and a map $T : S_n \to S_n$ is defined by one of the following;

(1) $T(A) = VAV', \quad A \in S_n$,
(2) $T(A) = VA^*V', \quad A \in S_n$,
(3) $T(A) = VA^{\mathrm{tr}}V', \quad A \in S_n$,
(4) $T(A) = V\bar{A}V', \quad A \in S_n$.

Then T is a surjective isometry with respect to *any* unitarily invariant norm. For the converse statement we have

Theorem 1. *Let $\|\cdot\|$ be a unitarily invariant norm on $M_n(\mathbb{C})$. Suppose that $\|\cdot\|$ is strictly convex. Suppose that $T : S_n \to S_n$ is a surjective isometry with respect to $\|\cdot\|$. Then there exist unitary matrices V and V' such that one of the following holds.*

(1) $T(A) = VAV', \quad A \in S_n$,
(2) $T(A) = VA^*V', \quad A \in S_n$,
(3) $T(A) = VA^{\mathrm{tr}}V', \quad A \in S_n$,
(4) $T(A) = V\bar{A}V', \quad A \in S_n$.

As the Schatten p-norm is strictly convex for $1 < p < \infty$ we have

Corollary 2. *Let $1 < p < \infty$. Suppose that $T : S_n \to S_n$ is a surjective isometry with respect to the Schatten p-norm. Then there exist unitary matrices V and V' such that one of the following holds.*

(1) $T(A) = VAV', \quad A \in S_n$,
(2) $T(A) = VA^*V', \quad A \in S_n$,
(3) $T(A) = VA^{\mathrm{tr}}V', \quad A \in S_n$,
(4) $T(A) = V\bar{A}V', \quad A \in S_n$.

2. A condition of commutativity

For a normed real-linear space $(X, \|\cdot\|)$, the sphere with the centre $a \in X$ with the radius $r > 0$ is denoted by $\mathbb{S}_r(a) = \{x \in X | \|x - a\| = r\}$.

Lemma 3. *Let X be a real normed space with a strictly convex norm $\|\cdot\|$. Let a and b be different points in X. Put $\|a - b\| = 2r$. Then $\mathbb{S}_r(a) \cap \mathbb{S}_r(b) = \{\frac{a+b}{2}\}$.*

Proof. It is straightforward that $\frac{a+b}{2} \in \mathbb{S}_r(a) \cap \mathbb{S}_r(b)$.

Conversely let $c \in \mathbb{S}_r(a) \cap \mathbb{S}_r(b)$ be arbitrary. Suppose that $c \neq \frac{a+b}{2}$. Since $\left\|\frac{a+b}{2} - a\right\| = \|c - a\| = r$ and $\frac{a+b}{2} - a \neq c - a$ we have

$$\left\|\left(\frac{a+b}{2} - a\right) + (c - a)\right\| < 2r$$

as $\|\cdot\|$ is strictly covex. In the same way we have

$$\left\|\left(\frac{a+b}{2} - b\right) + (c - b)\right\| < 2r.$$

Adding the both of the inequalities and applying the triangle inequality we have

$$4r = 2\|a - b\| \leq \left\|\left(\frac{a+b}{2} - a\right) + (c - a)\right\| + \left\|\left(\frac{a+b}{2} - b\right) + (c - b)\right\| < 4r,$$

which is a contradiction. Thus we have $c = \frac{a+b}{2}$. $\square$

Proof of Theorem 1. Let $\mathbb{U}_n$ be the group of all unitary matrices in $M_n(\mathbb{C})$, which we call the unitary group. The operator norm of a unitary matrix is 1, so that $\mathbb{U}_n \subset S_n$.

Let $A \in S_n \setminus \mathbb{U}_n$. Let $A = U_A|A|$ be the polar decomposition of A, where $|A|$ is the positive part and U_A is the unitary part of A as usual. As A is not unitary, $|A| \neq E$, where E is the identity matrix. The norm $\|A\|_\infty$ coincides with the largest singular value of A, hence it equals to 1. Thus the largest singular value of $|A|^2$ is 1, so $E - |A|^2$ is positive semidefinite, and $(E - |A|^2)^{\frac{1}{2}}$ is a well defined. As $|A| \neq E$, we have $|A|^2 \neq E$ and $(E - |A|^2)^{\frac{1}{2}}$ is a non-zero matrix. Then

$$U_0 = |A| + i(E - |A|^2)^{\frac{1}{2}}$$

is well defined and U_0 is a unitary matrix. We have $U_0^* = |A| - i(E - |A|^2)^{\frac{1}{2}}$ and $U_0 \neq U_0^*$ since $(E - |A|^2)^{\frac{1}{2}} \neq 0$. Letting $U_1 = U_A U_0$ and $U_2 = U_A U_0^*$ we have different unitary matrices U_1 and U_2 with

$$A = U_A|A| = \frac{U_1 + U_2}{2}.$$

Next we prove that $T(A) \in S_n \setminus \mathbb{U}_n$. Since U_1 and U_2 are different we have $\|U_1 - U_2\| = 2r \neq 0$. Then by Lemma 3 $\mathbb{S}_r(U_1) \cap \mathbb{S}_r(U_2) = \{A\}$, where $\mathbb{S}_r(B) =$

$\{X \in M_n(\mathbb{C}) : \|X - B\| = r\}$. As T is isometric with respect to $\|\cdot\|$ we have $T(A) \in \mathbb{S}_r(T(U_1)) \cap \mathbb{S}_r(T(U_2))$. By Lemma 3

$$T(A) = \frac{T(U_1) + T(U_2)}{2}.$$

Note that $T(U_1) \neq T(U_2)$ as T is injective. Suppose that $T(A) \in \mathbb{U}_n$. Choose any vector $x \in \mathbb{C}^n$ with $\|x\|_E = 1$. As $T(A)$ is assumed to be unitary we have

$$\begin{aligned} 2 = \|2T(A)x\|_E = \|T(U_1)x + T(U_2)x\|_E \leq \|T(U_1)x\|_E + \|T(U_2)x\|_E \\ \leq \|T(U_1)\|_\infty + \|T(U_2)\|_\infty = 2. \end{aligned}$$

Hence $\|T(U_1)x + T(U_2)x\|_E = 2$ and $\|T(U_1)x\|_E = \|T(U_2)x\|_E = 1$. By the parallelogram law

$$\|T(U_1)x + T(U_2)x\|^2 + \|T(U_1)x - T(U_2)x\|^2 = 2\|T(U_1)x\|^2 + 2\|T(U_2)x\|^2$$

we infer that $T(U_1)x = T(U_2)x$. As this equation holds for every $x \in \mathbb{C}^n$ with $\|x\|_E = 1$, we have that $T(U_1) = T(U_2)$, which is a contradiction proving that $T(A)$ is not unitary. Hence we have that $T(S_n \setminus \mathbb{U}_n) \subset S_n \setminus \mathbb{U}_n$.

Applying the similar argument for T^{-1} instead of T we see that $T^{-1}(S_n \setminus \mathbb{U}_n) \subset S_n \setminus \mathbb{U}_n$. It follows that $T(S_n \setminus \mathbb{U}_n) = S_n \setminus \mathbb{U}_n$, hence $T(\mathbb{U}_n) = \mathbb{U}_n$.

We obtain the surjection $T|_{\mathbb{U}_n} : \mathbb{U}_n \to \mathbb{U}_n$ and $T|_{\mathbb{U}_n}$ is an isometry with respect to $\|\cdot\|$. Then by a theorem of Molnár [3, Theorem 3] there exist $V, V' \in \mathbb{U}_n$ such that T is of one of the following forms:

(i) $T(U) = VUV', \quad U \in \mathbb{U}_n$,
(ii) $T(U) = VU^*V', \quad U \in \mathbb{U}_n$,
(iii) $T(U) = VU^{\mathrm{tr}}V', \quad U \in \mathbb{U}_n$,
(iv) $T(U) = V\bar{U}V', \quad U \in \mathbb{U}_n$.

Let $A \in S_n \setminus \mathbb{U}_n$. Then by the first part of the proof there exist unitaries U_1 and U_2 with $A = \frac{U_1+U_2}{2}$ and $T(A) = \frac{T(U_1)+T(U_2)}{2}$. Suppose that T has the form (i) above. Then

$$T(A) = \frac{T(U_1) + T(U_2)}{2} = \frac{VU_1V' + VU_2V'}{2} = V\frac{U_1 + U_2}{2}V' = VAV'.$$

By the same way for the rest of the case (ii), (iii) and (iv) we have that $T(A) = VA^*V'$ for (ii), $T(A) = VA^{\mathrm{tr}}V'$ for (iii) and $T(A) = V\bar{A}V'$ for (iv). It follows that T has of the one of the form of (1), (2), (3) or (4) of the statement of Theorem 1. □

References

[1] P. Mankiewicz, *On extension of isometries in normed linear spaces*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **20** (1972), 367–371.

[2] C. A. McCarthy, C_p, Israel J. Math. **5** (1967), 249–271.

[3] L. Molnár, *Jordan triple endomorphisms and isometries of unitary groups*, Linear Algebra Appl. **439** (2013), 3518–3531.
[4] D. Tingley, *Isometries of the unit sphere*, Geom. Dedicata **22** (1987), 371–378.

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