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A NOTE ON SMOOTH MULTIPLE FIBERS IN PENCILS OF ALGEBRAIC CURVES

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Abstract. Multiple fibers of the simplest kind in a pencil of algebraic curves are studied, in order to clarify the influence on the gonality and the base locus of the canonical linear system.

1. Introduction

We shall work over the complex number field $\mathbb{C}$. Let S be a non-singular projective surface and $f : S \to B$ a surjective morphism with connected fibers to a non-singular projective curve B. We assume that f is relatively minimal (i.e., there are no smooth rational curves with self-intersection number -1 contained in fibers) and a general fiber is a non-singular projective curve of genus $g \geq 2$. Let F be a fiber of f. Then there exist a positive integer m and a numerically 1-connected effective divisor D such that $F = mD$, since the intersection form is negative semi-definite on the support of F by Zariski's lemma. When $m > 1$, F is called a *multiple fiber* and m its multiplicity. In this case, $[D]|_D$ is a torsion line bundle of order m, where $[D]$ denotes the line bundle on S associated to D.

We often localize the situation by shrinking B to a small open disk. That is, we let $\Delta = \{|t| < \epsilon\} \subset \mathbb{C}$ be a sufficiently small open disk and consider a relatively minimal fibration $f : S \to \Delta$ of curves of genus g, usually assuming that f is smooth over $\Delta \setminus \{0\}$ and, hence, the central fiber $F = f^{-1}0$ is the only possible singular fiber. Suppose that F is a multiple fiber of multiplicity m and write $F = mD$. There is a canonical way for reducing f to a fibration without multiple fibers. Let $\widetilde{\Delta}$ be another open disk with parameter s and $\phi : \widetilde{\Delta} \to \Delta$ the m-sheeted cyclic covering defined by $s \mapsto t = s^m$. We denote by $\widetilde{S}$ the normalization of the fiber product $S \times_\Delta \widetilde{\Delta}$. Then it is a non-singular surface and the induced fibration $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ does not have multiple fibers. The central fiber $\widetilde{F}$ of $\widetilde{f}$ is a 1-connected curve obtained as

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the unramified m-sheeted covering of D associated to the m-torsion bundle $[D]|_D$. In this paper, we call $\widetilde{F}$ the *reduction* of F.

The presence of a multiple fiber imposes some restrictions on numerical invariants. Here we recall an apparent one: $g - 1 = m(p_a(D) - 1)$, where $p_a(D)$ denotes the arithmetic genus of D. In particular, we have $g \equiv 1 \pmod{m}$, e.g., g should be odd when a double fiber occurs. For small values of g, we can list the possible pairs of $m(\geq 2)$ and $p_a(D)$ as follows:

g	3	4	5		6	7			8	9			10	
m	2	3	2	4	5	2	3	6	7	2	4	8	3	9
$p_a(D)$	2	2	3	2	2	4	3	2	2	5	3	2	4	2

We also remark that the multiplicity is influenced by a general fiber. If f is a hyperelliptic fibration, for example, then $m \leq 2$ (cf. Proposition 3.1).

In this paper, we are concerned with multiple fibers of the simplest kind that the numerical cycle D is a non-singular curve. We call such singular fibers *smooth multiple fibers*. From a remarkable theorem of Serrano [10], we know that the set of multiplicities of multiple fibers in f are invariant under small deformations of the fibration. Furthermore, a smooth multiple fiber is stable under deformations, in other words, an atomic fiber (cf. [4]). This is one of the main reasons why we are interested in such special fibers.

In this paper, among other things, we shall show the following:

Theorem 1.1. *Let $F = mD$ be a smooth multiple fiber in a pencil of curves of genus $g \geq 3$, and $\widetilde{F}$ the reduction of F. Let p be a prime number satisfying $g > (p-1)^2$. Then, $\widetilde{F}$ is a p-gonal curve if and only if $m = p$ and $|K_F|$ has a base point.*

In the final section, we compute the local invariants of smooth multiple fibers in a general pencil of curves of genus 3, 5.

2. Unramified covering and the gonality

Let C be a non-singular projective curve of genus $g \geq 3$ which is obtained as an unramified m-sheeted covering of another non-singular curve D of genus $h \geq 2$. We denote by $\pi : C \to D$ the covering map. Note that we have $g - 1 = m(h-1)$ by the Hurwitz theorem. When π is a cyclic covering, we have the eigen-space decomposition $\pi_* \mathcal{O}_C \simeq \bigoplus_{i=0}^{m-1} \mathcal{O}_D(-iL)$ under the action of the Galois group $\mathbb{Z}_m$, where L is a line bundle on D that is a torsion element of order m in the Picard group of D. We call L the *torsion line bundle associated to π*.

Lemma 2.1. *In the above situation, assume that C is a hyperelliptic curve. Then D is also hyperelliptic, and $m = 2$ or 4. If $m = 2$, then the 2-torsion bundle associated to π is of the form $[P - Q]$ with two Weierstrass points P, Q on D.*

Proof. Since $\pi : C \to D$ is a finite unramified covering, we have $K_C = \pi^* K_D$ and see that the canonical linear system $|K_D|$ of D induces a linear subsystem of $|K_C|$. In other words, the canonical image of C dominates that of D. Hence, D is also hyperelliptic. Let $\widetilde{\phi} : C \to \mathbb{P}^1$ and $\phi : D \to \mathbb{P}^1$ be the canonical double coverings. Since they are induced by the canonical mappings of C and D, respectively, we have a morphism $\varpi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi \circ \pi = \varpi \circ \widetilde{\phi}$. Let $\{P_1, \ldots, P_{2h+2}\} \subset \mathbb{P}^1$ be the set of all branch points of ϕ. We put $Q_i = \phi^{-1}(P_i)$ for $i = 1, \ldots, 2h+2$. Then $\pi^{-1}(Q_i)$ consists of m distinct points for each i, and we have in total

$$2m(h+1) = 2(g+1) + 4(m-1)$$

such points on C. It is clear that $\bigcup_{i=1}^{2h+2} \pi^{-1}(Q_i)$ is the set of all ramification points of $\phi \circ \pi$ and the ramification index at each point is 2. Since $\phi \circ \pi = \varpi \circ \widetilde{\phi}$, the same is true for $\varpi \circ \widetilde{\phi}$. From this, one sees immediately that ϖ ramifies exactly at $2(m-1)$ points with ramification index 2, since $\widetilde{\phi}$ is a double covering branched at $2g+2$ points. We may assume that ϖ branches at $P_1, \ldots, P_\ell$ for some $\ell \le 2(m-1)$. Let k_i be the number of ramification points in $\varpi^{-1}(P_i)$, $1 \le i \le \ell$. Then $\varpi^{-1}(P_i)$ consists of $(m - 2k_i) + k_i = m - k_i$ points. Since $\varpi^{-1}(P_i)$ cannot contain a branch point of $\widetilde{\phi}$, comparing the cardinality of $(\phi \circ \pi)^{-1}(P_i)$ and $(\varpi \circ \widetilde{\phi})^{-1}(P_i)$, we get $m = 2(m - k_i)$, i.e., $k_i = m/2$. Recall that there are $2(m-1)$ ramification points of ϖ in total. Hence $\sum_{i=1}^{\ell} k_i = \ell(m/2) = 2(m-1)$, that is, $(4-\ell)m = 4$. It follows that $(\ell, m) = (3,4)$, $(2,2)$.

If $(\ell, m) = (2,2)$, then the 2-torsion bundle associated to $\pi : C \to D$ is $[Q_1 - Q_2]$. Indeed, we have $\pi^* Q_1 \sim \pi^* Q_2$ by the construction and, $2Q_1 \sim 2Q_2$ but $Q_1 \not\sim Q_2$, since Q_1 and Q_2 are distinct Weierstrass points, where the symbol $\sim$ means the linear equivalence of divisors. □

Remark 2.1. The rational function ϖ with $(\ell, m) = (3,4)$ exists. Without loosing generality, we can assume that ϖ ramifies at 0, 1, ∞ and $\varpi(0) = 0$, $\varpi(1) = 1$, $\varpi(\infty) = \infty$. Then

$$\varpi(z) = \frac{z^2(7z-8)^2}{-14z^2 + 16z - 1}$$

is such an example. When $(\ell, m) = (2,2)$, ϖ is assumed to be $w = z^2$.

As usual, we denote by g_d^r a linear system of dimension r and degree d, and $W_d^r(C)$ the set of all the g_d^r's on C. The *gonality* of C, which we denote by $\mathrm{gon}(C)$, is by definition the minimum of degrees of surjective morphisms from C to $\mathbb{P}^1$, that is, $\mathrm{gon}(C) = \min\{k \mid W_k^1(C) \ne \emptyset\}$. Recall that, when D and C are the numerical cycle of a smooth multiple fiber and its reduction, respectively, the natural map $\pi : C \to D$ is not merely a finite covering but also a cyclic covering.

Lemma 2.2. *Assume that $\pi : C \to D$ is an unramified m-sheeted cyclic covering, and let k be a positive integer such that $W_k^1(C) \neq \emptyset$. Then, either $W_k^1(C)$ is an infinite set or $\#W_k^1(C) \times k$ is a multiple of m, where $\#T$ denotes the cardinality of the set T.*

Proof. We assume that $W_k^1(C)$ is a finite set. The covering transformation group $\mathbb{Z}_m$ of π acts naturally on $W_k^1(C)$ via pull-back. Let H be the stabilizer subgroup at $g_k^1 \in W_k^1(C)$. Then we can find a member of g_k^1 which is fixed by H, because the action of a finite cyclic group on $\mathbb{P}^1$ necessarily has a fixed point. Being the degree of the invariant divisor, k must be a multiple of the order of H. If $\mathrm{Orb}(g_k^1)$ denotes the orbit of g_k^1 under the $\mathbb{Z}_m$-action, then $\#\mathrm{Orb}(g_k^1)$ coincides with the index of H in $\mathbb{Z}_m$. From these, we see that $k \times \#\mathrm{Orb}(g_k^1)$ is a multiple of m. Therefore, so is $k \times \#W_k^1(C)$. $\square$

Proposition 2.1. *Let $\pi : C \to D$ be an unramified m-sheeted cyclic covering. Put $\mathrm{gon}(C) = k$ and assume that $\#W_k^1(C)$ is smaller than the least prime factor of m. Then k is a multiple of m and the following hold:*

(1) *There exist surjective morphisms $\widetilde{\phi} : C \to \mathbb{P}^1$, $\phi : D \to \mathbb{P}^1$ of respective degree k and a cyclic covering $\varpi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree m such that $\phi \circ \pi = \varpi \circ \widetilde{\phi}$.*

$$\begin{array}{ccc} C & \xrightarrow{\pi} & D \\ {\scriptstyle\tilde{\phi}}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathbb{P}^1 & \xrightarrow[\varpi]{} & \mathbb{P}^1 \end{array}$$

(2) *There are two sets of points $\{P_1, \ldots, P_{k/m}\}$ and $\{Q_1, \ldots, Q_{k/m}\}$ on D such that $[\sum_{i=1}^{k/m} P_i - \sum_{j=1}^{k/m} Q_j]$ is the m-torsion bundle associated to π.*

Proof. By Lemma 2.2, our assumption on $\#W_k^1(C)$ is sufficient to imply that k is a multiple of m. Let us consider the action of the covering transformation group $\mathbb{Z}_m$ on $W_k^1(C)$. Then, every $g_k^1 \in W_k^1(C)$ is fixed by $\mathbb{Z}_m$, since $\#W_k^1(C)$ is smaller than any non-trivial divisor of m. We choose one g_k^1 and let $\widetilde{\phi} : C \to \mathbb{P}^1$ denote the morphism corresponding to the g_k^1. There are exactly two fixed points, say 0 and ∞, of the induced action of $\mathbb{Z}_m$ on the base curve $\mathbb{P}^1$. We let $\varpi : \mathbb{P}^1 \to \mathbb{P}^1/\mathbb{Z}_m \simeq \mathbb{P}^1$ be the quotient map. We may assume that ϖ is given by $w = z^m$ for suitable affine coordinates z and w. By the construction, we have a morphism $\phi : D \to \mathbb{P}^1$ such that $\phi \circ \pi = \varpi \circ \widetilde{\phi}$. Put $\widetilde{\phi}^* 0 = \tilde{P}_1 + \cdots + \tilde{P}_k \in g_k^1$ and $\widetilde{\phi}^* \infty = \tilde{Q}_1 + \cdots + \tilde{Q}_k \in g_k^1$. Now, $\mathbb{Z}_m$ acts on $\{\tilde{P}_1, \ldots, \tilde{P}_k\}$ and $\{\tilde{Q}_1, \ldots, \tilde{Q}_k\}$ without fixed points, and each of them decomposes into k/m orbits (i.e., fibers of π). Therefore, there are points $P_1, \ldots, P_{k/m}$; $Q_1, \ldots, Q_{k/m}$ on D such that $\pi_*(\tilde{P}_1 + \cdots + \tilde{P}_k) = m(P_1 + \cdots + P_{k/m})$ and $\pi_*(\tilde{Q}_1 + \cdots + \tilde{Q}_k) = m(Q_1 + \cdots + Q_{k/m})$, and ϕ is given by the linear system $|m(P_1 + \cdots + P_{k/m})| = |m(Q_1 + \cdots + Q_{k/m})|$. Now $[\sum_{i=1}^{k/m} P_i - \sum_{j=1}^{k/m} Q_j]$ is a torsion of order m and its pull-back to C by π is trivial. Hence we can assume that π is induced by $[\sum_{i=1}^{k/m} P_i - \sum_{j=1}^{k/m} Q_j]$. □

Since $\widetilde{F}$ admits a fixed-point-free action of $\mathbb{Z}_m$, it must be a rather special curve among those with the same gonality. Here we collect some features indicated by Lemma 2.2.

Corollary 2.1. *Let $F = mD$ be a smooth multiple fiber in a pencil of curves of genus $g \geq 3$. Let $\widetilde{F}$ be the reduction of F and put $\mathrm{gon}(\widetilde{F}) = k$. Then $\#W_k^1(\widetilde{F}) \geq m/k$. If $k < m$ and $(k, m) = 1$, then $\#W_k^1(\widetilde{F}) \geq m$.*

Proof. Immediately follows from Lemma 2.2. □

Corollary 2.2. *Assume that $h = 2$ and let $F = (g-1)D$ be a smooth multiple fiber in a pencil of curves of genus $g \geq 6$. Put $\mathrm{gon}(\widetilde{F}) = k$ for the reduction $\widetilde{F}$ of F. If $(k, g-1) = 1$, then either $W_k^1(\widetilde{F})$ is infinite or its cardinality is a multiple of $g-1$.*

Proof. It is known that the gonality of a curve of genus g is not bigger than $(g+3)/2$. Since $g \geq 6$, we see that k is smaller than $g-1$. Hence, the assertion follows from Lemma 2.2. □

Remark 2.2. If C is a tetragonal curve of genus 6 with several g_4^1, then it is either a plane quintic curve, a bielliptic curve or a plane sextic curve with 4 double points. In the former two cases, $W_4^1(C)$ is one dimensional. In the last case, which is the general case, C has at most five g_4^1's obtained by the projection from each of 4 double points and by conics through all 4 double points. We do not know, however, which type can be realized as an unramified five sheeted cyclic covering of a curve of genus two, or how to characterize such curves. For example, if C is a Fermat quintic $\{x^5 + y^5 + z^5 = 0\} \subset \mathbb{P}^2$, then the quotient of C by the action of the cyclic group of order 5 generated by the projective transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix} \qquad (\varepsilon = \exp(2\pi\sqrt{-1}/5))$$

gives us a smooth curve D of genus 2.

We close the section by an apparent remark:

Lemma 2.3. *Assume that a general fiber of $f : S \to \Delta$ is a k-gonal curve. Then $\mathrm{gon}(\widetilde{F}) \leq k$.*

Proof. Let $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ be the semi-stable reduction. In the present case, it is nothing more than a deformation family of smooth algebraic curves. Since the gonality is lower semi-continuous with respect to the moduli parameters by Namba [9, §5.3], the assertion follows. □

3. Base points on multiple fibers

Let $F = mD$ be a multiple fiber in a relatively minimal fibration $f : S \to B$ of genus g, where D is a numerically 1-connected curve of arithmetic genus $h \geq 2$. Here, an effective divisor E is called *numerically k-connected* if either E is irreducible or $E_1E_2 \geq k$ holds for any effective decomposition $E = E_1 + E_2$, $0 \prec E_1, E_2$.

As we showed in [8], the restriction map $H^0(F, K_F) \to H^0(iD, K_F)$ is surjective for any integer i with $1 \leq i < m$. Hence the canonical linear system $|K_F|$ on F has a base point if and only if so does $|K_F \otimes \mathcal{O}_D|$. When $h = 2$, we have $\deg K_F|_D = 2$ and $h^0(D, K_F) = 1$ implying that $|K_F|$ has a base point. In general, however, it is difficult to determine whether $|K_F|$ has a base point or not, even when D is a non-singular hyperelliptic curve.

The following may be well-known for experts, but we include the proof for the sake of completeness.

Lemma 3.1. *Let $F = mD$ be a multiple fiber and assume that D is a numerically 2-connected curve of arithmetic genus $h > 0$. Then, for a non-singular point P of D, the following three conditions are equivalent.*

(1) *P is a base point of $|K_F|$.*
(2) *$K_F|_D = K_D + P - Q$ for another non-singular point $Q \in D$, $Q \neq P$.*
(3) *$[D]|_D = [Q - P]$ for another non-singular point $Q \in D$, $Q \neq P$.*

Proof. Since $K_F|_D = K_S|_D$ and $K_D = (K_S + [D])|_D$ by the adjunction formula, the equivalence between (2) and (3) is obvious.

We put $L := K_F|_D$ for simplicity, and show (1) $\Rightarrow$ (2). Assume that a non-singular point $P \in D$ is a base point of $|L|$ and consider the exact sequence

$$0 \to \mathcal{O}_D(L - P) \to \mathcal{O}_D(L) \to \mathbb{C}_P \to 0.$$

Since P is a base point, the restriction map $H^0(D, L) \to \mathbb{C}_P$ is the zero map and, thus, we have $0 \neq H^1(D, L-P) \simeq H^0(D, K_D-L+P)^*$ by the Serre duality theorem. Since $\mathcal{O}_D(K_D - L + P)$ is a nef invertible sheaf of degree 1 on a 2-connected curve D, we get a point Q as the zero of any non-zero element of $H^0(D, K_D - L + P)$ by [5, (A.5)]. Then $L = K_D + P - Q$.

It remains to show that (2) $\Rightarrow$ (1). Assume that $K_F|_D = K_D + P - Q$. Since $P \neq Q$ and $h > 0$, we have $h^0(D, Q - P) = 0$. Hence $h^0(D, K_D + P - Q) = h - 1$. Since $|K_D|$ is free from base points if D is a 2-connected curve of positive arithmetic genus ([5, (A.7)]), one has $h^0(D, K_D - Q) = h - 1$. These together with the fact $|K_D - Q| + P \subseteq |K_D + P - Q|$ imply $P \in \mathrm{Bs}|K_D + P - Q|$. □

We restrict ourselves to the case of smooth multiple fibers. As to the number of possible base points, we have the following:

Lemma 3.2. *Let $F = mD$ be a smooth multiple fiber in a pencil of curves of genus $g \geq 3$ and assume that $|K_F|$ has a base point. If D is non-hyperelliptic, then* $\mathrm{Bs}|K_F|$ *consists of one point. If D is hyperelliptic, then* $\mathrm{Bs}|K_F|$ *consists of two points which may be infinitely near.*

Proof. Let P be a base point of $|K_F|$ and write $K_F|_D = K_D + P - Q$ with another point Q. Then we have $h^0(D, K_D - Q) = h - 1$ by $h^0(D, K_F) = h - 1$. Let us consider the exact sequence

$$0 \to H^0(D, K_D - Q) \to H^0(D, K_D + P - Q) \to \mathbb{C}_P$$

If D is hyperelliptic, then we clearly have $\mathrm{Bs}|K_D - Q| = \{Q'\}$, where Q' denotes the conjugate of Q (that is, the point Q' satisfying $Q + Q' \in g^1_2$) and it follows that $\mathrm{Bs}|K_F| = \{P,\ Q'\}$. In this case, we have $[D]|_D = [Q - P] = [P' - Q']$, where P' denotes the conjugate of P. It may be possible that $Q = P'$. If D is non-hyperelliptic, then $|K_D - Q|$ is free from base points, since K_D is very ample. It follows $\mathrm{Bs}|K_F| = \{P\}$. □

The presence of a base point affects the gonality.

Lemma 3.3. *Let $f : S \to \Delta$ be a relatively minimal fibration of genus $g \geq 3$, $F = mD$ a smooth multiple central fiber and $\widetilde{F}$ the reduction of F. Assume that $|K_F|$ has a base point. Then $\mathrm{gon}(D) \leq m$ and $\mathrm{gon}(\widetilde{F}) \leq m$. Furthermore, there are morphisms $\phi : D \to \mathbb{P}^1$, $\widetilde{\phi} : \widetilde{F} \to \mathbb{P}^1$ of respective degree m and an m-sheeted cyclic covering $\varpi : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\phi \circ \pi = \varpi \circ \widetilde{\phi}$, where $\pi : \widetilde{F} \to D$ denotes the natural covering map.*

Proof. Let P be a base point of $|K_F|$ and write $K_F|_D = K_D + P - Q$ with another point Q. Then the m-torsion bundle $[D]|_D$ is given by $[Q - P]$ (cf. Lemma 3.1). It follows that we have $mP \sim mQ$ but $iP \not\sim iQ$ for $0 < i < m$. In particular, we see that $|mP|$ is free from base points and $\mathrm{gon}(D) \leq m$. Let $\phi : D \to \mathbb{P}^1$ be the morphism of degree m corresponding to the pencil spanned by mP and mQ. Let $\widetilde{F}$ be the reduction of F and $\pi : \widetilde{F} \to D$ the natural unramified cyclic covering map of degree m. Since $\pi^*[Q - P]$ is trivial on $\widetilde{F}$, the pencil Λ spanned by π^*P and π^*Q gives us a morphism $\widetilde{\phi} : \widetilde{F} \to \mathbb{P}^1$ of degree m. Hence, $\mathrm{gon}(\widetilde{F}) \leq m$. Note that Λ is invariant under the action of the Galois group $\mathrm{Gal}(\widetilde{F}/F) = \mathbb{Z}_m$. Hence, $\mathbb{Z}_m$ acts on the base curve $\mathbb{P}^1$ of $\widetilde{\phi}$ and we have a cyclic m-sheeted branched covering $\varpi : \mathbb{P}^1 \to \mathbb{P}^1 = \mathbb{P}^1/\mathbb{Z}_m$ such that $\phi \circ \pi = \varpi \circ \widetilde{\phi}$. □

Remark 3.1. Let the notation be as in the proof of Lemma 3.2. Assume that D is a hyperelliptic curve and $Q \neq P'$. If $m \geq 3$, then we have two g^1_m's on D as in the proof of Lemma 3.3: one coming from (P, Q) and the other from (P', Q'). Hence there are two g^1_m's on $\widetilde{F}$ in this case.

The following shows that a high multiple of a hyperelliptic curve merely appears.

Lemma 3.4. *Let $F = mD$ be a multiple fiber in a pencil of curves. Assume that D is a hyperelliptic curve of genus h, and $P \in \mathrm{Bs}|K_F|$. If mP is a special divisor on D, then $m = 2$. In particular, if $h \geq m$, then $m = 2$.*

Proof. We write $K_F|_D = K_D + P - Q$. By the proof of Lemma 3.3, $|mP|$ is free from base points. Since $|mP|$ is a special linear system on a hyperelliptic curve, it must be a multiple of g^1_2. In particular, we see that m is even.

We assume that P is not a Weierstrass point. If P' is the conjugate of P, we have $(m/2)P \sim (m/2)P'$ from $mP \sim (m/2)g^1_2 \sim (m/2)(P + P')$. This implies that $[(m/2)P]$ is free from base points and, again, it is a multiple of g^1_2 being a special divisor. In particular, $m/2$ is even and we have $(m/2^2)P \sim (m/2^2)P'$. Continuing such a process, we finally see that $[2P]$ is free from base points. But this implies that P must be a Weierstrass point, a contradiction. Hence, P is a Weierstrass point. Similarly, we can show that Q is a Weierstrass point, because $mQ \in |mP| = (m/2)g^1_2$. Since $[Q - P]$ is a m-torsion, this implies that $m = 2$.

As we have seen in the proof of Lemma 3.3, we have $h^0(D, mP) \geq 2$. By the Riemann-Roch theorem, $h^0(D, mP) - h^1(D, mP) = m + 1 - h$. Hence $h^1(D, mP) \neq 0$ provided that $h \geq m$. In other words, mP is special when $h \geq m$. $\square$

The following is our main result.

Theorem 3.1. *Let $F = mD$ be a smooth multiple fiber in a pencil of curves of genus g, and $\widetilde{F}$ the reduction of F. Let p be a prime number satisfying $g > (p-1)^2$. Then, $\widetilde{F}$ is a p-gonal curve if and only if $m = p$ and $|K_F|$ has a base point.*

Proof. Assume that $\widetilde{F}$ is a p-gonal curve. If $\widetilde{F}$ were have distinct two g^1_p's, then we would have a morphism $\widetilde{F} \to \mathbb{P}^1 \times \mathbb{P}^1$ that is birational onto the image, and the Castelnuovo-Severi inequality would imply $g \leq (p-1)^2$, a contradiction. Hence, $W^1_p(\widetilde{F})$ is one point. Then, by Proposition 2.1, we get $m = p$ and $[D]|_D$ is of the form $[Q - P]$. This last is equivalent to saying that $|K_F|$ has a base point by Lemma 3.1.

Conversely, assume that $m = p$ and $\mathrm{Bs}|K_F| \neq \emptyset$. Then, by Lemma 3.3, we have $\mathrm{gon}(\widetilde{F}) \leq p$ and there is a base point free g^1_p on $\widetilde{F}$. If $\mathrm{gon}(\widetilde{F}) = k < p$, then the Castelnuovo-Severi inequality shows $g \leq (k-1)(p-1)$, which is absurd. Hence $\widetilde{F}$ is a p-gonal curve. $\square$

It would be worth stating a more accurate result for $p = 2$, 3.

Proposition 3.1. *Let $F = mD$ be a smooth multiple fiber in a pencil of curves of genus $g \geq 3$ and $\widetilde{F}$ the reduction of F.*

(1) *$\widetilde{F}$ is a hyperelliptic curve if and only if $m = 2$ and $|K_F|$ has two base points which are Weierstrass points on the hyperelliptic curve D.*

(2) *When $g \geq 5$, $\widetilde{F}$ is a trigonal curve if and only if $m = 3$ and $|K_F|$ has a unique base point on a trigonal curve D.*

Proof. By Theorem 3.1, we only have to show the assertions for D and $\mathrm{Bs}|K_F|$. (1) If $\widetilde{F}$ is hyperelliptic, we can apply Lemma 2.1 (and Proposition 2.1) to see that $m = 2$ and $|K_F|$ has two base points which are Weierstrass points. (2) Assume that $\widetilde{F}$ is trigonal. Since $g \geq 5$, $W^1_3(\widetilde{F})$ is one point. Hence we can apply Proposition 2.1 to see that $m = 3$ and $\mathrm{Bs}|K_F| \neq \emptyset$. Then we have $\mathrm{gon}(D) \leq 3$ by Lemma 3.3 and there is a base point free g^1_3 on D. If D were a hyperelliptic curve, then it would follow from Lemma 3.4 that $h < m = 3$, which is impossible when $g \geq 5$. Therefore, D is not hyperelliptic. Then Lemma 3.2 implies that $|K_F|$ has only one base point. $\square$

For hyperelliptic fibrations, we have the following:

Proposition 3.2. *Let $f : S \to \Delta$ be a hyperelliptic fibration of genus $g \geq 3$. Then the multiplicity of a multiple fiber is at most two. If $F = 2D$ is a smooth double fiber, then $|K_F|$ has two base points which are Weierstrass points on the hyperelliptic curve D. Furthermore, $\mathrm{Bs}|K_F|$ coincides with the set of isolated fixed points of the hyperelliptic involution of $f : S \to \Delta$.*

Proof. Let $\rho : \widehat{S} \to S$ be the minimal succession of blowing-ups which eliminates all the isolated fixed points of the hyperelliptic involution of $f : S \to \Delta$. Then we have a finite double covering $\theta : \widehat{S} \to \widehat{W}$ over Δ, where $\widehat{W}$ is a non-singular surface and the induced morphism $\widehat{W} \to \Delta$ is a $\mathbb{P}^1$ fiber space. Then the fiber over 0 of $\widehat{W} \to \Delta$ has a component Γ of multiplicity one corresponding to the fiber of one of its relatively minimal models. Since $\widehat{S} \to \widehat{W}$ is a finite double covering, the multiplicity of the pull-back to $\widehat{S}$ of Γ is at most 2. Therefore, so is the multiplicity of the central fiber.

If $f : S \to \Delta$ is a hyperelliptic pencil, then the semi-stable reduction $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ is also hyperelliptic and, hence, $\widetilde{F}$ must be a hyperelliptic curve of genus g. The rest follows from Proposition 3.1 (1) and Lemma 2.1. $\square$

Remark 3.2. The above consideration suggests us a canonical way to obtain a smooth double fiber. Let $g \geq 3$ be an odd integer and $f : S \to \Delta$ a hyperelliptic fibration of genus g. Assume that the central fiber F is a smooth double fiber and put $F = 2D$. Let σ be the hyperelliptic involution of $f : S \to \Delta$. Then, it is clear from the proof of Lemma 2.1 that there are two distinct isolated fixed points of $\langle\sigma\rangle$ on F, equivalently, the base points of $|K_F|$. Let $\rho : \widehat{S} \to S$ be the blowing-up with center $\mathrm{Bs}|K_F|$. Then, the involution $\widehat{\sigma}$ induced by σ has no isolated fixed points and the quotient $\widehat{W} = \widehat{S}/\langle\widehat{\sigma}\rangle$ is non-singular. Two exceptional (-1)-curves E_1, E_2 for ρ are among the 1-dimensional fixed locus of $\widehat{\sigma}$, that is, the ramification divisor for $\theta : \widehat{S} \to \widehat{W}$. If we denote by $\widehat{D}$ the proper transform of D, then $E_i \cap \widehat{D}$ is a

Weierstrass point of $\widehat{D}$ for $i=1,2$. The central fiber of the induced fibration $\widehat{S} \to \Delta$ is $\widehat{F} = 2\widehat{D} + 2E_1 + 2E_2$. Note that $\theta|_{\widehat{D}}$ ramifies at $2h+2(=g+3)$ points in total, two of which come from the E_i's. Hence, the ramification divisor has $g+1$ horizontal local components each of which meets $\widehat{D}$ at a distinct Weierstrass point. Since E_i is a (-1)-curve in the ramification divisor, $\theta(E_i)$ is a (-2)-curve. Then $\theta(\widehat{D}) \simeq \mathbb{P}^1$ and its self-intersection number is -1, because $\widehat{D}^2 = -2$. The central fiber of $\widehat{W} \to \Delta$ is of the form $2\theta(\widehat{D}) + \theta(E_1) + \theta(E_2)$. Contracting the (-1)-curve $\theta(\widehat{D})$ to a point ξ, we have the fiber consisting of two (-1)-curves meeting simply at ξ. The branch locus has ordinary $(g+3)$-ple point at ξ. Contracting one of two (-1)-curves, we get a relatively minimal model $W \simeq \mathbb{P}^1 \times \Delta$ of $\widehat{W}$. Here the branch locus contains the fiber of $W \to \Delta$ and has one so-called $(g+2 \to g+2)$-point on it.

Conversely, put $W = \mathbb{P}^1 \times \Delta$ and let $p : W \to \Delta$ be the natural projection. Take a reduced curve B_0 on W such that B_0 meets $p^{-1}(t)$ at distinct $2g+2$ points when $t \neq 0$, and it has a 2-fold $(g+1)$-ple point on $\Gamma = p^{-1}(0)$. Put $B_1 = B_0 + \Gamma$ and let S' be the double covering of W branched along B_1. Take the canonical resolution of singularities of S' and let S be the relatively minimal model. The central fiber of $S \to \Delta$ is then a double fiber. A sample equation of B_1 is $x\{(x+y^2)^{g+1} - 2x^{g+1}\}$, where x is the parameter on Δ and y the inhomogeneous coordinate on $\mathbb{P}^1$.

For trigonal fibrations, we have the following:

Theorem 3.2. *Let $f : S \to \Delta$ be a relatively minimal trigonal fibration of genus $g \geq 3$ and assume that the central fiber $F = mD$ is a smooth multiple fiber. Then $m \leq 3$. Furthermore, the following hold.*

(1) *$m = 3$ if and only if $\widetilde{F}$ is trigonal. When $m = 3$ and $g \neq 4$, D is a trigonal curve and $|K_F|$ has a unique base point.*

(2) *$m = 2$ if and only if $\widetilde{F}$ is hyperelliptic, and $|K_F|$ has two base points which are Weierstrass points on D.*

Proof. Since f is a trigonal fibration, its semi-stable reduction $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ is also trigonal. It follows that $\mathrm{gon}(\widetilde{F})$ is at most three. When $g \leq 4$, we automatically have $m \leq 3$. If $g \geq 5$, then W^1_k, $k = \mathrm{gon}(\widetilde{F})$, consists of one point and it follows from Lemma 2.2 that m must divide $\mathrm{gon}(\widetilde{F})$. Therefore, we have $m = 2$ or 3 according to whether $\mathrm{gon}(\widetilde{F}) = 2$ or 3. □

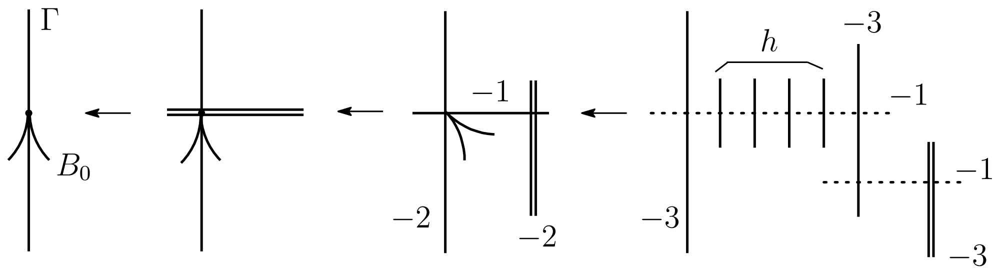
Remark 3.3. Let D be as in (1) of Theorem 3.2. As the proof of Lemma 3.3 shows, the trigonal structure of D is given by a triple covering $\phi : D \to \mathbb{P}^1$ that has at least two totally ramified points P, Q. Hence, D cannot be an arbitrary trigonal curve.

Example 3.1. Here, we give an example of a smooth triple fiber. Put $W = \mathbb{P}^1 \times \Delta$ and $\Gamma = \mathbb{P}^1 \times \{0\}$. Take a positive integer h and let B_0 be the reduced curve on W

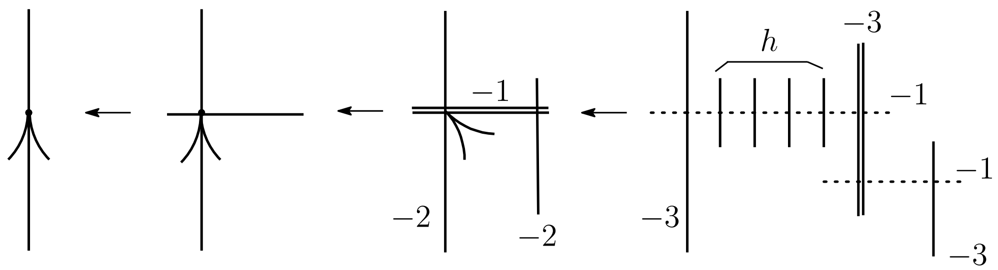
defined by

$$b_0(x,y) = (x+y^3)^h - 2x^h,$$

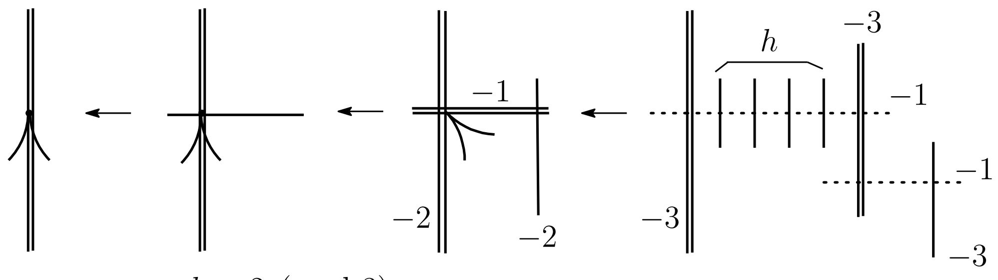
where x denotes the coordinate on Δ and y is the inhomogeneous coordinate on $\mathbb{P}^1$. Then B_0 meets $\mathbb{P}^1 \times \{x\}$ at distinct $3h$ points, when $x \neq 0$, while it has a 3-fold h-ple point at $(0,0) \in \Gamma$. Put $B_1 = \epsilon\Gamma + B_0$, where $\epsilon = 2$ when $h \equiv 2 \pmod 3$, otherwise $\epsilon = 1$. Let S' be the cyclic triple covering of W branched along B_1 defined by $w^3 = x^\epsilon b_0(x,y)$. We resolve the singular point of S' by the method described in [3]. This can be done by 4 times of blowing-ups at singular points of the branch loci as in the picture below, where the dotted, solid and double lines respectively mean the multiplicity in the branch locus of the corresponding component is zero, one and two (see [3, Example 3.7]).



$h \equiv 1 \pmod 3$



$h \equiv 0 \pmod 3$



$h \equiv 2 \pmod 3$

Taking cyclic triple covering with the resolved branch locus and performing the normalization along its compound cusps, we get a non-singular surface $S^\sharp$ with the natural birational morphism $S^\sharp \to S'$. Then, $S^\sharp$ has three (-1)-curves originating from (-3)-curves in the branch locus. After contracting them all, we find another (-1)-curve resulting from a (-3)-curve on $S^\sharp$. Then contracting it, we get a trigonal fibration $S \to \Delta$ of genus $g = 3h - 2$ as the relatively minimal model of $S^\sharp \to S' \to W \to \Delta$. Its central fiber is the triple fiber $F = 3D$, where D is the non-singular curve of genus h obtained by resolving cusps of the cyclic triple covering $D^\sharp$ of $\mathbb{P}^1$ (the (-1)-curve expressed as a dotted line). The number of $(2,3)$-cusps on $D^\sharp$ is one, zero or two according to $h \equiv 0, 1, 2 \pmod 3$.

In light of the hyperelliptic case, it would be worth remarking that, in our example, the base point of $|K_F|$ is the isolated fixed point of the natural $\mathbb{Z}_3$-action around which it is normalized as $(z_1, z_2) \mapsto (\omega z_1, \omega^2 z_2)$, where $\omega = \exp(2\pi\sqrt{-1}/3)$.

We summarize here some properties of smooth multiple fibers for g small.

The case $g = 3$: If $F = 2D$ is a smooth double fiber in a pencil of curves of genus 3, then $|K_F|$ has exactly two base points which are Weierstrass points on D. Furthermore, the reduction $\widetilde{F}$ of F is a hyperelliptic curve of genus 3.

The case $g = 4$: If $F = 3D$ is a smooth triple fiber in a pencil $f : S \to \Delta$ of curves of genus 4, then D is of genus 2 but f cannot be a hyperelliptic fibration. Then $|K_F|$ has two base points which may be infinitely near. Assume that $\mathrm{Bs}|K_F|$ consists of two distinct points. Then, as we already noted in Remark 3.1, there are two distinct g^1_3's on the reduction $\widetilde{F}$. It follows that $\widetilde{F}$, identified with its canonical image, lies on a quadric of rank 4. Hence so does a general fiber of $f : S \to \Delta$. In particular, if a general fiber of f lies on a quadric of rank 3, then $\mathrm{Bs}|K_F|$ consists of one point of multiplicity two.

The case $g = 5$: We have $m = 2$ or $m = 4$. Hence $\widetilde{F}$ is not a trigonal curve by Proposition 3.1. If $\widetilde{F}$ is hyperelliptic, then $m = 2$, D is also hyperelliptic and $|K_F|$ has two base points. When $f : S \to \Delta$ is a hyperelliptic or trigonal fibration, we have only such a case. Assume that $f : S \to \Delta$ is tetragonal and that $\widetilde{F}$ is a tetragonal curve. If $m = 2$, then $\mathrm{Bs}|K_F| = \emptyset$ since otherwise $\widetilde{F}$ would be hyperelliptic. If $m = 4$, then $\mathrm{Bs}|K_F|$ consists of two points. When $\mathrm{Bs}|K_F|$ consists of two distinct points, $\widetilde{F}$ has at least two g^1_4's.

The case $g = 6$: We have $m = 5$ and $h = 2$. Since $g = 6$, $\mathrm{gon}(\widetilde{F})$ is at most 4. By Corollary 2.1 or Proposition 3.1, we get $\mathrm{gon}(\widetilde{F}) = 4$. Therefore, $f : S \to \Delta$ must be a tetragonal fibration. $\widetilde{F}$ is one of the curves described in Remark 2.2 by Lemma 2.2.

4. Local invariants

By using the geometric information obtained in the previous sections, we shall compute local invariants of smooth multiple fibers (cf. [4]).

The notion of fiber germs is defined in a natural way. Namely, two relatively minimal fibrations $f_i : S_i \to \Delta_i$ ($i = 1, 2$) of curves are equivalent if and only if there exist a open disk $0 \in \Delta'_i \subset \Delta_i$ for $i = 1, 2$ and bi-holomorphic maps between $S'_i := f_i^{-1}(\Delta'_i)$, Δ'_i ($i = 1, 2$) such that the diagram

$$\begin{tikzcd} S'_1 \arrow[r, "\simeq"] \arrow[d, "f_1"'] & S'_2 \arrow[d, "f_2"] \\ \Delta'_1 \arrow[r, "\simeq"] & \Delta'_2 \end{tikzcd}$$

commutes. Then a fiber germ is one of the equivalence classes.

Let g be a positive odd integer ≥ 3 and $\mathcal{A}$ the set of all fiber germs of $(g+3)/2$-gonal fibrations of genus g. Recall from [7] and [4] that, we have a well-defined function $\mathrm{Ind} : \mathcal{A} \to \mathbb{Q}_{\geq 0}$ such that, for any $(g+3)/2$-gonal fibration $f : S \to B$ of genus g, $\mathrm{Ind}(f^{-1}P) = 0$ holds whenever $P \in B$ is general and we have

$$K_{S/B}^2 = \frac{6(g-1)}{g+1}\chi_f + \sum_{P\in B} \mathrm{Ind}(f^{-1}P),$$

where $K_{S/B} = K_S - f^*K_B$ and $\chi_f = \deg f_*\mathcal{O}_S(K_{S/B})$. $\mathrm{Ind}(f^{-1}P)$ is called the *Horikawa index* of the fiber germ $f^{-1}P$. Furthermore, if we put

$$\sigma(f^{-1}P) := \frac{2(g+1)}{3(g+3)}\mathrm{Ind}(f^{-1}P) - \frac{g+7}{3(g+3)}e_f(f^{-1}P),$$

where $e_f(f^{-1}P) = e(f^{-1}P) - (2-2g)$ is the relative Euler number, then the signature of S can be written as

$$\mathrm{Sign}(S) = \sum_{P\in B} \sigma(f^{-1}P)$$

(see, [4]). We call $\sigma(f^{-1}P)$ the *local signature* of $f^{-1}P$.

Since the Horikawa index is defined as an alternating sum of the length of certain $\mathcal{O}_{B,P}$-modules, it is hard to compute it in general. So, we “linearize” the situation to simplify the problem. Let F be the central fiber of a $(g+3)/2$-gonal fibration $f : S \to \Delta$. We denote by $K_{p,q}(F, K_F)$ the (co)homology group at the middle term of the Koszul complex (cf. [6])

$$\bigwedge^{p+1} H^0(K_F) \otimes H^0((q-1)K_F) \to \bigwedge^{p} H^0(K_F) \otimes H^0(qK_F)$$
$$\to \bigwedge^{p-1} H^0(K_F) \otimes H^0((q+1)K_F).$$

Then, the (linearized) Horikawa index can be defined by

$$\mathrm{Ind}_0(F) = \binom{g-3}{(g-3)/2}^{-1} \sum_{i=2}^{(g+1)/2} (-1)^i \dim K_{(g+1)/2-i,i}(F, K_F).$$

We also put

$$\sigma_0(F) := \frac{2(g+1)}{3(g+3)}\mathrm{Ind}_0(F) - \frac{g+7}{3(g+3)}e_f(F).$$

We in fact have $\mathrm{Ind}_0(F) = \mathrm{Ind}(F)$ if the dependency on the moduli parameter is simple. For example, for a non-hyperelliptic fibration of genus 3, the Horikawa index is defined as the length of the cokernel of the multiplication map

$$\mathrm{Sym}^2(f_*\omega_{S/\Delta}) \to f_*(\omega_{S/\Delta}^{\otimes 2})$$

which is surjective off $0 \in \Delta$ by the Max Noether theorem. On the other hand, we have $\mathrm{Ind}_0(F) = \dim \mathrm{Coker}\{H^0(K_F) \otimes H^0(K_F) \to H^0(2K_F)\}$.

Now, let $F = mD$ be a smooth multiple fiber and $\zeta \in H^0(S, [D])$ the section defining D. Then $\zeta^m = 0$ on F.

4.1. Non-hyperelliptic fibrations of genus three

Let $f : S \to \Delta$ be a (relatively minimal) non-hyperelliptic fibration of genus 3. If $F = 2D$ is a smooth double fiber, then we have $\mathrm{Ind}_0(F) = 3$. This can be seen as follows. Let x_0, x_1 be sections of $K_S + [D]$ which induce a basis for $H^0(D, K_D)$. We take a section y of $K_S + [F]$ restricted to a non-zero element in $H^0(D, K_F|_D)$. Then the restrictions to F of ζx_0, ζx_1 and y form a basis for $H^0(F, K_F)$. Since $\zeta^2 = 0$ on F, we see that their products give us only three independent elements $\zeta x_0 y$, $\zeta x_1 y$ and y^2 in the six dimensional vector space $H^0(F, 2K_F)$. Hence, $\mathrm{Ind}_0(F) = 6 - 3 = 3$. Since $e_f(F) = 2$, we get $\sigma_0(F) = 2/9$. Similarly, we have $\mathrm{Ind}_0(\widetilde{F}) = 1$, since the reduction $\widetilde{F}$ is hyperelliptic: the map $\mathrm{Sym}^2 H^0(K_{\widetilde{F}}) \to H^0(2K_{\widetilde{F}})$ between 6-dimensional vector spaces has one dimensional kernel defining the plane conic curve (= the canonical image of $\widetilde{F}$). Thus, we get $\sigma_0(\widetilde{F}) = 4/9 = 2\sigma_0(F)$ by $e_{\widetilde{f}}(\widetilde{F}) = 0$.

4.2. Tetragonal fibrations of genus five

Let $f : S \to \Delta$ be a tetragonal fibration of genus 5. We put $V = H^0(F, K_F)$ and let $S(V)$ be the symmetric algebra over V. We denote by S_n the homogeneous part of degree n. Put $R_n = H^0(F, nK_F)$ and let $R = \oplus_{n \geq 0} R_n$ be the canonical ring of F.

(I) Firstly, we consider the case $m = 2$. As we have already seen, there are two possibilities: (i) $\mathrm{Bs}|K_F| = \emptyset$, (ii) $\mathrm{Bs}|K_F| \neq \emptyset$. Let x_0, x_1, x_2 be sections of $K_S + [D]$ which give us a basis for $H^0(D, K_D)$. If y_0, y_1 are sections of $K_S + [F]$ inducing a basis for $H^0(D, K_F|_D)$, then $\{e_i\}_{i=0}^4$ forms a basis for V, where $e_i = \zeta x_i$ $(0 \leq i \leq 2)$, $e_3 = y_0$ and $e_4 = y_1$.

(i) We assume that $\mathrm{Bs}|K_F| = \emptyset$. Then $\widetilde{F}$ is a tetragonal curve as we saw in the previous section. We shall show that $\mathrm{Ind}_0(F) = 4$. We have $K_{0,3}(F, K_F) = 0$, because $\mathrm{Bs}|K_F| = \emptyset$ implies the surjectivity of $R_1 \otimes R_2 \to R_3$ (see, [8]). So it suffices to compute $\dim K_{1,2}(F, K_F)$. We let W be the 3-dimensional subspace of V generated by e_0, e_1, e_2. Since $K_F|_D = K_D + [D]|_D$ induces a free pencil on D, the evaluation map gives us an exact sequence

$$0 \to \mathcal{O}_D(-D) \to H^0(D, K_F) \otimes \mathcal{O}_D(K_D) \to \mathcal{O}_D(K_F|_D + K_D) \to 0.$$

We have $h^0(D, -[D]|_D) = 0$. Hence the multiplication map $H^0(K_F|_D) \otimes H^0(K_D) \to H^0(K_F|_D + K_D)$ is an isomorphism between 6-dimensional vector spaces. Similarly, $\mathrm{Sym}^2 H^0(K_F|_D) \to H^0(2K_F|_D)$ is injective. It follows that the 9 products $\zeta x_i y_j$, $y_i y_j$ are independent in R_2. Then the kernel I_2 of the restriction $\mu_2 : S_2 \to R_2$ is 6-dimensional and, in fact, we have $I_2 = \mathrm{Sym}^2(W)$. Now, consider the commutative diagram of Koszul complexes:

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^3 R_1 \arrow[r] & \bigwedge^2 R_1 \otimes R_1 \arrow[r] & R_1 \otimes R_2 \arrow[r] & R_3 \arrow[r] & 0 \\
0 \arrow[r] & \bigwedge^3 S_1 \arrow[r] \arrow[u, equal] & \bigwedge^2 S_1 \otimes S_1 \arrow[r] \arrow[u, equal] & S_1 \otimes S_2 \arrow[r] \arrow[u, "1\otimes\mu_2"] & S_3 \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

where the bottom sequence is exact.

We claim that

$$\mathrm{Im}\left(\bigwedge^2 S_1 \otimes S_1\right) \cap S_1 \otimes I_2 = \mathrm{Im}\left(\bigwedge^2 W \otimes W\right)$$

and it is of dimension 8. To see this, we argue as follows. We take an element

$$\alpha = \frac{1}{2}\sum_{i,j} a_{ij} e_i \wedge e_j \in \bigwedge^2 S_1 \otimes S_1, \tag{1}$$

where the a_{ij}'s are linear forms in $e_0, \ldots, e_4$ satisfying $a_{ji} = -a_{ij}$. Then its image in $S_1 \otimes S_2$ under the differential is given by

$$\sum_i \left(\sum_j a_{ij} e_j\right) \otimes e_i. \tag{2}$$

Assume that it is contained in $S_1 \otimes I_2$. Recall that $I_2 = \mathrm{Sym}^2(W)$. Then we have $a_{ij} = 0$ when $e_j \notin W$, and $a_{ij} \in W$ when $e_j \in W$. Hence $\alpha = (1/2)\sum_{0 \leq i,j \leq 2} a_{ij} e_i \wedge e_j \in \wedge^2 W \otimes W$. Furthermore, we have $\dim \mathrm{Im}(\wedge^2 W \otimes W) = \dim(\wedge^2 W \otimes W) - \dim(\wedge^3 W) = 9 - 1 = 8$. This shows the claim. It follows that $\dim K_{1,2}(F, K_F) = 8$

and, hence, $\mathrm{Ind}_0(F) = 4$. Since $e_f(F) = 4$, we get $\sigma_0(F) = 0$. We also have $\mathrm{Ind}_0(\widetilde{F}) = 0$, $e_{\tilde{f}}(\widetilde{F}) = 0$ and $\sigma_0(\widetilde{F}) = 0$ for the reduction $\widetilde{F}$ of F.

(ii) Assume that $\mathrm{Bs}|K_F| \neq \emptyset$. Then $\widetilde{F}$ and D are both hyperelliptic. Since $|K_F|$ has two base points, we get $\dim K_{0,3}(F, K_F) = 2$. This can be seen as follows. Put $K_F|_D = K_D + P - Q$. Then P, Q are Weierstrass points and $K_F|_D - P - Q$ gives us g_2^1. Therefore,

$$0 \to \mathcal{O}_D(-K_F + P + Q) \to H^0(D, K_F - P - Q) \otimes \mathcal{O}_D \to \mathcal{O}_D(K_F - P - Q) \to 0$$

is exact. The exact sequence obtained by tensoring $2K_F|_D$ shows that $H^0(D, K_F - P - Q) \otimes H^0(D, 2K_F) \to H^0(D, 3K_F - P - Q)$ is surjective. Hence the image of $H^0(D, K_F) \otimes H^0(D, 2K_F) \to H^0(D, 3K_F)$ is the subspace of codimension 2 consisting of sections vanishing at P, Q. Furthermore, $H^0(D, K_D) \otimes H^0(D, 2K_F) \to H^0(D, 3K_F - D)$ is surjective, since the canonical ring of D is generated in degrees ≤ 2. These together show that $\dim K_{0,3}(F, K_F) = 2$.

Take two elements t_0, t_1 of $H^0(D, K_F - P - Q)$ such that the zero divisors satisfy $(t_0) = 2P$ and $(t_1) = 2Q$. We may assume that $x_0 = t_0^2$, $x_1 = t_0 t_1$ and $x_2 = t_1^2$ on D. Furthermore, $y_i = s t_i$ on D, where s is the section of $[P + Q]$ defining $P + Q$, i.e., $(s) = P + Q$. Hence we may assume $s^2 = t_0 t_1 = x_1$. Then

$$I_2 = \langle e_0^2, e_0e_1, e_0e_2, e_1e_2, e_1^2, e_2^2, e_1e_3 - e_0e_4, e_2e_3 - e_1e_4 \rangle.$$

We find 14 linear sygyzies among the generators of I_2. Since it equals to the dimension of $(S_1 \otimes I_2) \cap \mathrm{Ker}(S_1 \otimes S_2 \to S_3)$, we get $\dim K_{1,2}(F, K_F) = 14$. It follows $\mathrm{Ind}_0(F) = (14 - 2)/2 = 6$, $e_f(F) = 4$ and $\sigma_0(F) = 1$.

Let us compute $\mathrm{Ind}_0(\widetilde{F})$. We have $K_{0,3}(\widetilde{F}, K_{\widetilde{F}}) = 0$, since the canonical ring of $\widetilde{F}$ is generated in degrees ≤ 2. If $\{\xi_i\}_{i=0}^4$ is a basis for $H^0(\widetilde{F}, K_{\widetilde{F}})$, then $K_{1,2}(\widetilde{F}, K_{\widetilde{F}})$ is isomorphic to the space of linear sygyzies among 6 relations given by 2×2 minors of

$$\begin{pmatrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{pmatrix}.$$

It is easy to see that there are 8 such sygyzies. Hence, $\mathrm{Ind}_0(\widetilde{F}) = 4$ and $\sigma_0(\widetilde{F}) = 2 = 2\sigma_0(F)$.

(II) Lastly, we consider the case $m = 4$. Then D is of genus 2 and we have $e_f(F) = 6$. Let $\{x_0, x_1\}$ be sections of $K_S + [D]$ inducing a basis for $H^0(D, K_D)$ and y, z, w sections of $K_S + 2[D]$, $K_S + 3[D]$ and $K_S + 4[D]$, respectively, giving non-zero elements of $H^0(D, K_F|_D - i[D])$ for $i = 2, 1, 0$. We write $K_F|_D = K_D + P - Q$ as in Lemma 3.1. Then we have $(w|_D) = P + Q'$, where Q' is the conjugate of Q. It is easy to see that $(y|_D) = P' + Q$, where P' is the conjugate of P. By using the fact that $[D]|_D$ is torsion of order 4, one can show that $(z|_D) = P'' + Q''$ with two points $P'', Q'' \notin \{P, Q, P', Q'\}$. Then, it is not so hard to see that the image of the multiplication map $H^0(F, K_F) \otimes H^0(F, 2K_F) \to H^0(F, 3K_F)$ is the subspace of codimension 2 consisting of elements vanishing at P, Q', by considering it for each "eigen-space":

$$\bigoplus_{i=0}^{j} H^0(K_F|_D - iD) \otimes H^0(2K_F|_D - (j-i)D) \to H^0(3K_F|_D - jD)$$

for $j \in \{0, 1, 2, 3\}$, if one note that $2K_F|_D - (j-i)[D]|_D$ is free from base points. In fact, the above map fails to be surjective only when $j = 0$. Therefore, we have $K_{0,3}(F, K_F) = 2$.

Now $\{e_0 := \zeta^3 x_0, e_1 := \zeta^3 x_1, e_2 := \zeta^2 y, e_3 := \zeta z, e_4 := w\}$ forms a basis for $H^0(F, K_F)$. We have the obvious relations in degree 2 from $\zeta^4 = 0$: $e_0^2 = e_1^2 = e_2^2 = e_0e_1 = e_1e_2 = e_2e_0 = e_0e_3 = e_1e_3 = 0$. Then one sees

$$I_2 = \langle e_0^2, e_1^2, e_2^2, e_0e_1, e_1e_2, e_2e_0, e_0e_3, e_1e_3 \rangle.$$

Let $\alpha \in \wedge^2 S_1 \otimes S_1$ be as in (1). Then its image in $S_1 \otimes S_2$ is as in (2). If it is in $S_1 \otimes I_2$, then one sees the following: $a_{i4} = 0$ for any i. If $j = 0, 1$, then $a_{ij} \in \langle e_0, e_1, e_2, e_3 \rangle$. If $j = 2$, then $a_{i2} \in \langle e_0, e_1, e_2 \rangle$. If $j = 3$, then $a_{i3} \in \langle e_0, e_1 \rangle$. Since $a_{ji} = -a_{ij}$, we see that $a_{01} \in \langle e_0, e_1, e_2, e_3 \rangle$; $a_{02}, a_{12} \in \langle e_0, e_1, e_2 \rangle$; $a_{03}, a_{13}, a_{23} \in \langle e_0, e_1 \rangle$, and

$$\alpha = a_{01} e_0 \wedge e_1 + a_{02} e_0 \wedge e_2 + a_{03} e_0 \wedge e_3 + a_{12} e_1 \wedge e_2 + a_{13} e_1 \wedge e_3 + a_{23} e_2 \wedge e_3.$$

Modulo the image of $\wedge^3 S_1$, such α's form a 14-dimensional vector space. Hence $\dim K_{1,2}(F, K_F) = 14$. Of course, this can be computed as the number of linear syzygies among generators of I_2. Then, $\mathrm{Ind}_0(F) = (14-2)/2 = 6$ and $\sigma_0(F) = 6/2 - 6/2 = 0$. Since $\widetilde{F}$ is tetragonal, $\mathrm{Ind}_0(\widetilde{F}) = 0$ and $\sigma_0(\widetilde{F}) = 0$.

Remark 4.1. (1) In all cases above, we have $\sigma_0(\widetilde{F}) = m\sigma_0(F)$. This fact follows from Ashikaga's formula in [2, Theorem 5.2.1] for the *signature defect*. Using it, $\mathrm{Ind}_0(F)$ can be computed as

$$\mathrm{Ind}_0(F) = \frac{1}{m}\mathrm{Ind}_0(\widetilde{F}) + \left(1 - \frac{1}{m}\right)\frac{g+7}{g+1}(g-1)$$

once we know $\mathrm{Ind}(\widetilde{F})$.

(2) If $F = 2D$ is a smooth double fiber in a hyperelliptic fibration of odd genus g, then it is known

$$\mathrm{Ind}_H(F) = g - \frac{1}{g}, \quad \sigma_H(F) = 0,$$

where Ind_H and σ_H are the corresponding local invariants (see, [1], [4]).

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