Nihonkai Math. J.
Vol.26(2015), 91–101

PERIODIC POINTS OF SOME DISCONTINUOUS MAPS

SATOMI MURAKAMI AND HIROKI OHWA

Abstract. The purpose of this paper is to investigate periodic points of discontinuous maps. For some discontinuous maps, we establish a characterization of periodic points.

1. Introduction

Let a and b be constants with $a < b$, and denote by I the closed and bounded interval defined by $I = [a, b]$. Moreover, let $f : I \to I$ be a (discontinuous) map and denote by f^n the n-th iteration defined by $f^0 = id$, where id is the identity map, and

$$f^n = f \circ f^{n-1} \quad \text{for each integer } n \geq 1.$$

In general, a point $p \in I$ is said to be a periodic point of f with period n if $f^n(p) = p$, $f^m(p) \neq p$ for any integer m with $1 \leq m < n$.

Let l be an integer with $l \geq 2$. Throughout this paper, we make the following assumptions.

(A) For any integer $n \geq 1$, the discontinuities of f^{2n-1} are upward jumps, i.e., f^{2n-1} is piecewise continuous and satisfies

$$f^{2n-1}(p-) \leq f^{2n-1}(p) \leq f^{2n-1}(p+) \quad \text{for each discontinuity point } p \in I,$$

where $f^{2n-1}(p-) := \lim_{x \to p-0} f^{2n-1}(x)$ and $f^{2n-1}(p+) := \lim_{x \to p+0} f^{2n-1}(x)$. Note that if the discontinuity is an endpoint of I, then the one-sided inequality is only defined.

(Al) There exist points $u < v < z$ ($u, v, z \in I$) such that $f^3(u) = z$, $f^{2l-1}(v) = v$ and $f(z) = z$. Moreover, the following inequality holds:

$$f^2(x) < x \quad \text{for any } x \in (u, v].$$

2010 *Mathematics Subject Classification.* Primary 26A65; Secondary 26A18.
Key words and phrases. Discontinuous maps, periodic points, iteration.

We remark that analogous results in this paper can be obtained even if assumption (Al) is replaced by the following assumption.

($\hat{\mathrm{A}}l$) There exist points $z < v < u$ ($z, v, u \in I$) such that $f(z) = z$, $f^{2l-1}(v) = v$ and $f^3(u) = z$. Moreover, the following inequality holds:

$$f^2(x) > x \quad \text{for any } x \in [v, u).$$

Although a lot of papers have been written on periodic points of continuous maps (e.g. [1], [2], [3], [4], [5], [6], [8], [9], [10] and others), few papers have been written on periodic points of discontinuous maps (e.g. [11]). In a recent paper [7], the authors investigate periodic points of some discontinuous maps whose discontinuities are downward jumps. In this paper, we improve the method of proof in [7] and investigate periodic points of some discontinuous maps whose discontinuities are upward jumps. We shall prove the following theorems on the existence of periodic points by applying an improved method of [7].

Theorem 1.1. *Let l be an integer with $l \geq 2$ and assume that* (A) *and* (Al) *are satisfied. Then, f has periodic points with period $2k-1$ for all integers $k \geq l$.*

Theorem 1.2. *For each integer $l \geq 2$, there exists a discontinuous map $f_l : I \to I$ satisfying assumptions* (A) *and* (Al) *such that f_l has a periodic point with period $2l-1$, but no periodic points with period $2k-1$ for any integers k with $1 < k < l$.*

Remark 1.1. Theorem 1.2 shows that the conclusion of Theorem 1.1 is sharp in the sense that for each integer $l \geq 2$, there exists a discontinuous map which has a periodic point with period $2l-1$, but no periodic points with period k for any integers $2k-1$ with $1 < k < l$.

Remark 1.2. For all discontinuous maps satisfying assumptions (A) and (Al), the same conclusion as in Sharkovsky theorem (cf. [8]) is not obtained. Indeed, there exists a discontinuous map satisfying assumptions (A) and (A2), which has a periodic point with period 3, but no periodic points with period 2 (see Example 3.1 below).

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. This theorem follows from the following two lemmas.

Lemma 2.1. *Let n be an integer with $n \geq 1$ and assume that* (A) *is satisfied. If $f^{2n-1}(x_1) - x_1 > 0 > f^{2n-1}(x_2) - x_2$ for $x_1, x_2 \in I$ with $x_1 < x_2$, then there exists a point $p \in (x_1, x_2)$ such that $f^{2n-1}(p) = p$.*

Proof. Suppose that f^{2n-1} has m discontinuity points such that $x_1 < p_1 < p_2 < \cdots < p_m < x_2$. If $f^{2n-1}(p) \neq p$ for any $p \in (x_1, x_2)$, then, by the intermediate value theorem on (x_1, p_1), we see that $f^{2n-1}(p_1-) \geq p_1$. Therefore, by assumption (A), we have $f^{2n-1}(p_1) > p_1$. Moreover, by the intermediate value theorem on (p_1, p_2), we see that $f^{2n-1}(p_2-) \geq p_2$. Therefore, by assumption (A), we have $f^{2n-1}(p_2) > p_2$. By repeatedly using the arguments, we have $f^{2n-1}(p_m) > p_m$. Therefore, by the intermediate value theorem on (p_m, x_2), we see that $f^{2n-1}(x_2) \geq x_2$. But, this contradicts the assumption that $f^{2n-1}(x_2) < x_2$.

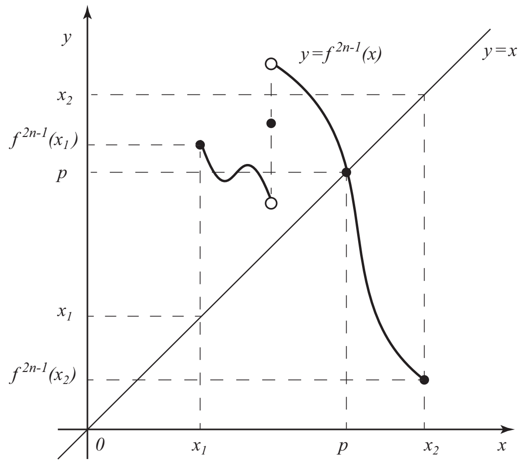

FIGURE 1. The figure where $y = f^{2n-1}(x)$ intersects $y = x$.

□

Lemma 2.2. *Let k and l be integers with $k \geq l \geq 2$ and assume that (A) and (Al) are satisfied. If $f^{2k-1}(p_k) = p_k$ for some point $p_k \in (u, v]$, then f has a periodic point $q_k \in (u, p_k]$ with period $2k - 1$.*

Proof. Setting

$$q_k = \inf \left\{ x : \ u < x \leq p_k, \ f^{2k-1}(x) = x \right\},$$

we obtain $f^{2k-1}(q_k) = q_k$. Indeed, if this equation does not hold, then, by assumption (A), we see that $f^{2k-1}(q_k) < q_k$. Since $f^{2k-1}(u) = f^{2k-4}(z) = z > u$, it follows from Lemma 2.1 that there exists a point $w_k \in (u, q_k)$ such that $f^{2k-1}(w_k) = w_k$. This contradicts the definition of q_k. Suppose now that q_k is a periodic point of f with period $2i - 1$ for some $i < k$. Then, by assumption (Al), we have $i \geq 2$. Since $f^{2i+1}(u) = f^{2i-2}(z) = z > u$ and $f^{2i+1}(q_k) = f^2(q_k) < q_k$, we have

$$f^{2i+1}(u) - u > 0 > f^{2i+1}(q_k) - q_k.$$

By Lemma 2.1, there exists a point $z_{i+1} \in (u, q_k)$ such that $f^{2i+1}(z_{i+1}) = z_{i+1}$. Moreover, since $f^{2i+3}(u) = f^{2i}(z) = z > u$ and $f^{2i+3}(z_{i+1}) = f^2(z_{i+1}) < z_{i+1}$, by

Lemma 2.1, there exists a point $z_{i+2} \in (u, z_{i+1})$ such that $f^{2i+3}(z_{i+2}) = z_{i+2}$. By repeatedly using the arguments, we inductively obtain points

$$u < \cdots < z_{k+1} < z_k < z_{k-1} < \cdots < z_{i+2} < z_{i+1} < q_k \leq p_k$$

such that $f^{2(i+j)-1}(z_{i+j}) = z_{i+j}$ $(j = 1, 2, 3, ...)$. In particular, we have $f^{2k-1}(z_k) = z_k < q_k$. This contradicts the definition of q_k. Thus, f has a periodic point $q_k \in (u, p_k]$ with period $2k-1$ and the proof of Lemma 2.2 is complete. $\square$

We now prove Theorem 1.1 by induction. By assumptions (A) and (Al) and by Lemma 2.2, f has a periodic point $q_l \in (u, v]$ with period $2l-1$. Suppose that $k \geq l$ is a positive integer such that f has a periodic point $q_k \in (u, v]$ with period $2k-1$. Then, since $f^{2k+1}(u) = f^{2k-2}(z) = z > u$ and $f^{2k+1}(q_k) = f^2(q_k) < q_k$, we have

$$f^{2k+1}(u) - u > 0 > f^{2k+1}(q_k) - q_k.$$

Therefore, by Lemma 2.1, there exists a point $p_{k+1} \in (u, q_k)$ such that $f^{2k+1}(p_{k+1}) = p_{k+1}$. Thus, by Lemma 2.2, f has a periodic point $q_{k+1} \in (u, p_{k+1}]$ with period $2k+1$. This completes the proof of Theorem 1.1.

3. Examples

In the following, we give a few examples which satisfy assumptions (A) and (Al).

Example 3.1. Consider the map f from $[0,1]$ into $[0,1]$ defined by

$$f(x) = \begin{cases} \frac{1}{2} & \left(x = 0, \frac{1}{2}, 1\right), \\ -2x+1 & \left(0 < x < \frac{1}{2}\right), \\ -2x+2 & \left(\frac{1}{2} < x < 1\right). \end{cases}$$

We show that f satisfies assumptions (A) and (A2). Let n be an integer with $n \geq 1$ and $p_{n,1} < p_{n,2} < \cdots < p_{n,m_n}$ discontinuity points of f^{2n-1}. Suppose that for each integer i with $1 \leq i \leq m_n - 1$, f^{2n-1} is decreasing continuous on $(p_{n,i}, p_{n,i+1})$ and satisfies $f^{2n-1}\big((p_{n,i}, p_{n,i+1})\big) = (0,1)$. Suppose furthermore that for each integer i with $1 \leq i \leq m_n$, $f^{2n-1}(p_{n,i}) = \frac{1}{2}$. Then, from the geometry of f^2 (see Figure 2), it follows that for each integer i with $1 \leq i \leq m_{n+1} - 1$, f^{2n+1} is decreasing continuous on $(p_{n+1,i}, p_{n+1,i+1})$ and satisfies $f^{2n+1}\big((p_{n+1,i}, p_{n+1,i+1})\big) = (0,1)$. For each integer i with $1 \leq i \leq m_{n+1}$, we additionally have $f^{2n+1}(p_{n+1,i}) = \frac{1}{2}$. Therefore, by induction, we see that assumption (A) is satisfied. Moreover, since

$$f^3\left(\frac{13}{24}\right) = \frac{2}{3}, \quad f^3\left(\frac{5}{9}\right) = \frac{5}{9}, \quad f\left(\frac{2}{3}\right) = \frac{2}{3},$$

and

$$f^2(x) < x \quad \text{for any } x \in \left(\frac{13}{24}, \frac{5}{9}\right],$$

assumption (A2) is satisfied with $u = \frac{13}{24}$, $v = \frac{5}{9}$ and $z = \frac{2}{3}$.

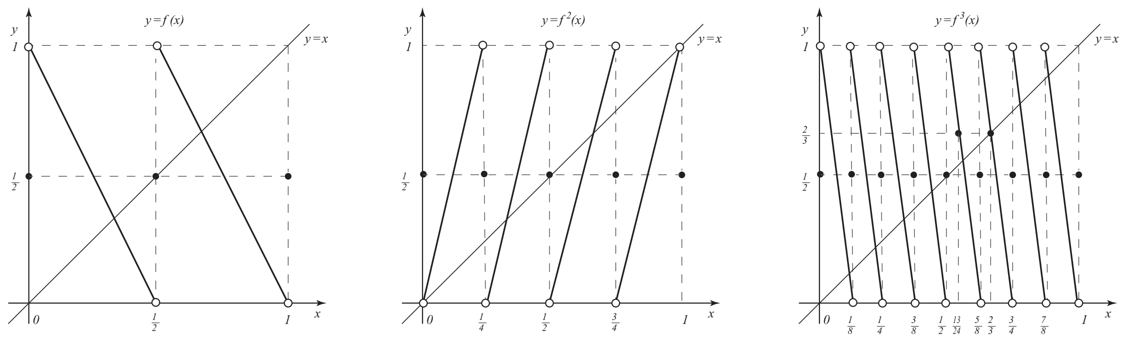

FIGURE 2. The geometries of f, f^2 and f^3.

Example 3.2. Consider the map f from $[0,1]$ into $[0,1]$ defined by

$$f(x) = \begin{cases} -2x+1 & \left(0 \le x \le \frac{2}{11}\right), \\ 1 & \left(\frac{2}{11} < x \le \frac{1}{2}\right), \\ -2x+2 & \left(\frac{1}{2} < x \le 1\right). \end{cases}$$

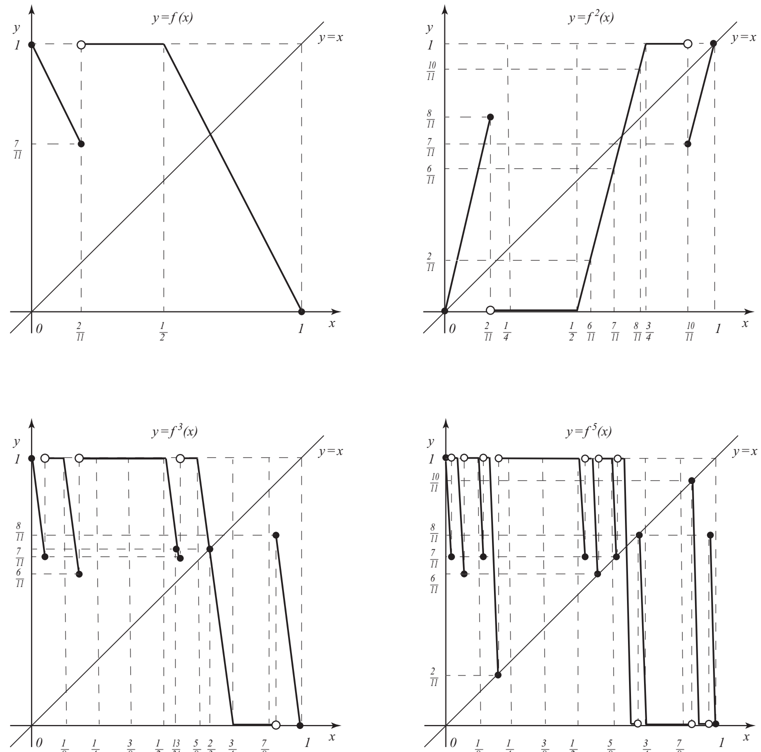

FIGURE 3. The geometries of f, f^2, f^3 and f^5.

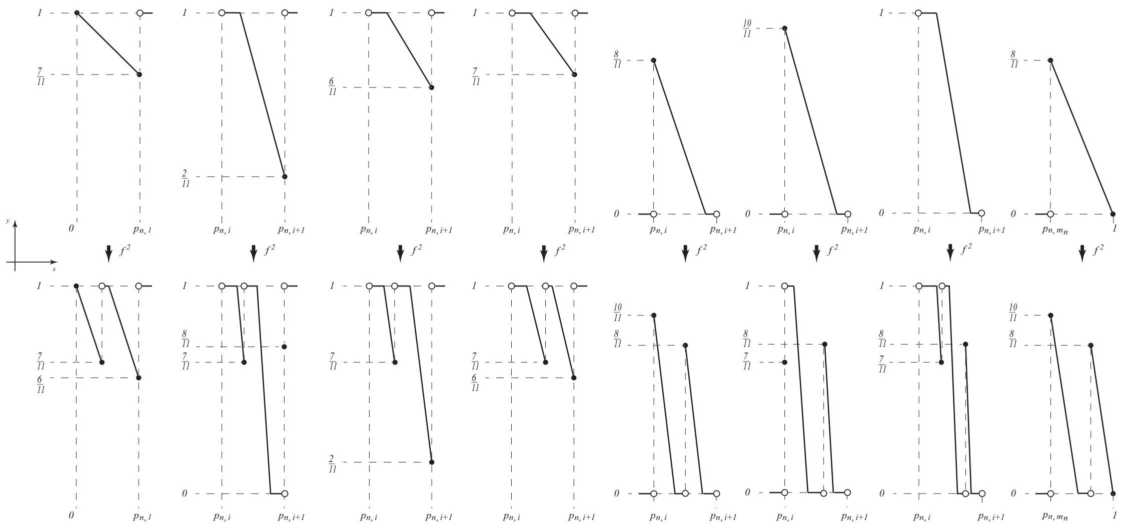

FIGURE 4. The images of f^2.

We show that f satisfies assumptions (A) and (A3). Let n be an integer with $n \geq 2$ and $p_{n,1} < p_{n,2} < \cdots < p_{n,m_n}$ discontinuity points of f^{2n-1}. For notational convenience, we set $p_{n,0} = 0$ and $p_{n,m_n+1} = 1$. Then, we clearly have $f^{2n-1}(p_{n,0}) = 1$ and $f^{2n-1}(p_{n,m_n+1}) = 0$. Let c, d and e be real numbers with $c \in \left\{\frac{2}{11}, \frac{6}{11}, \frac{7}{11}\right\}$, $d \in \left\{\frac{8}{11}, \frac{10}{11}\right\}$ and $e \in \left\{\frac{2}{11}, \frac{6}{11}, \frac{7}{11}, \frac{8}{11}, \frac{10}{11}\right\}$. Suppose that for each integer i with $0 \leq i \leq m_n$, f^{2n-1} is nonincreasing continuous on $(p_{n,i}, p_{n,i+1})$ and satisfies

$$f^{2n-1}\big((p_{n,i}, p_{n,i+1})\big) = \begin{cases} \left(\frac{7}{11}, 1\right) & (i = 0), \\ (c, 1],\ [0, d) \text{ or } [0, 1] & (1 \leq i \leq m_n - 1), \\ \left(0, \frac{8}{11}\right) & (i = m_n). \end{cases}$$

Suppose furthermore that for each integer i with $1 \leq i \leq m_n$, one of the following inequalities holds:

$$0 < f^{2n-1}(p_{n,i}-) = f^{2n-1}(p_{n,i}) < f^{2n-1}(p_{n,i}+) = 1,$$

$$0 = f^{2n-1}(p_{n,i}-) < f^{2n-1}(p_{n,i}) = f^{2n-1}(p_{n,i}+) < 1,$$

$$0 = f^{2n-1}(p_{n,i}-) < f^{2n-1}(p_{n,i}) < f^{2n-1}(p_{n,i}+) = 1,$$

with $f^{2n-1}(p_{n,i}) = e$. Then, from the geometry of f^2 (see Figures 3 and 4), it follows that for each integer i with $0 \leq i \leq m_{n+1}$, f^{2n+1} is nonincreasing continuous on

$(p_{n+1,i}, p_{n+1,i+1})$ and satisfies

$$f^{2n+1}\big((p_{n+1,i}, p_{n+1,i+1})\big) = \begin{cases} \big(\frac{7}{11}, 1\big) & (i = 0), \\ (c, 1],\ [0, d) \text{ or } [0, 1] & (1 \le i \le m_{n+1} - 1), \\ \big(0, \frac{8}{11}\big) & (i = m_{n+1}). \end{cases}$$

For each integer i with $1 \le i \le m_{n+1}$, we additionally have one of the following inequalities:

$$0 < f^{2n+1}(p_{n+1,i}-) = f^{2n+1}(p_{n+1,i}) < f^{2n+1}(p_{n+1,i}+) = 1,$$
$$0 = f^{2n+1}(p_{n+1,i}-) < f^{2n+1}(p_{n+1,i}) = f^{2n+1}(p_{n+1,i}+) < 1,$$
$$0 = f^{2n+1}(p_{n+1,i}-) < f^{2n+1}(p_{n+1,i}) < f^{2n+1}(p_{n+1,i}+) = 1,$$

with $f^{2n+1}(p_{n+1,i}) = e$. Therefore, noting that

$$f^{2n+1}(p_{n+1,0}) = 1 \text{ and } f^{2n+1}(p_{n,m_{n+1}+1}) = 0,$$

by induction, we see that assumption (A) is satisfied. Moreover, since

$$f^3\Big(\frac{13}{24}\Big) = \frac{2}{3}, \quad f^5\Big(\frac{7}{11}\Big) = \frac{7}{11}, \quad f\Big(\frac{2}{3}\Big) = \frac{2}{3},$$

and

$$f^2(x) < x \quad \text{for any } x \in \Big(\frac{13}{24}, \frac{7}{11}\Big],$$

assumption (A3) is satisfied with $u = \frac{13}{24}$, $v = \frac{7}{11}$ and $z = \frac{2}{3}$.

4. Proof of Theorem 1.2

We here prove Theorem 1.2. By changing variables, it is sufficient to prove Theorem 1.2 for $I = [0, 1]$. Consider now the map f_l from $[0, 1]$ into $[0, 1]$ defined by

$$f_l(x) = \begin{cases} -2x + 1 & \big(0 \le x \le \frac{4^{l-1}+2}{6\cdot 4^{l-1}+3}\big), \\ 1 & \big(\frac{4^{l-1}+2}{6\cdot 4^{l-1}+3} < x \le \frac{1}{2}\big), \\ -2x + 2 & \big(\frac{1}{2} < x \le 1\big). \end{cases}$$

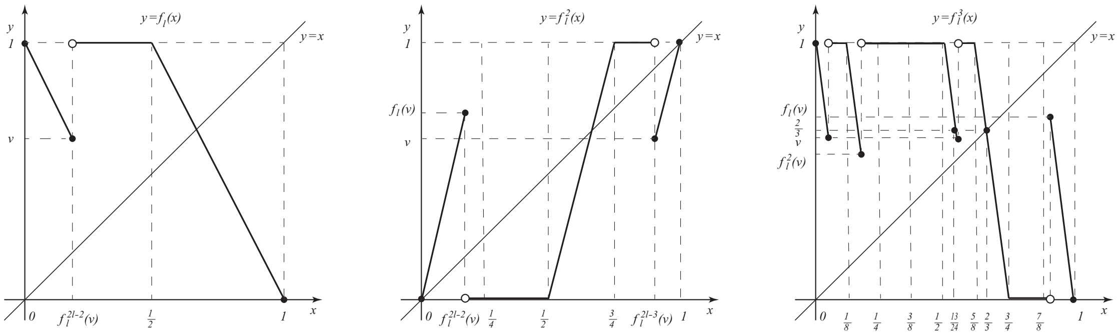

FIGURE 5. The geometries of f_l, f_l^2 and f_l^3.

We remark that if $l = 3$, then f_l becomes the map of Example 3.2. Setting $v = \frac{4^l-1}{6\cdot 4^{l-1}+3}$, since

$$\frac{1}{2} < f_l^i(v) < \frac{3}{4} \quad (i = 0, 1, 2, ..., 2l-4)$$

and

$$\frac{3}{4} < f_l^{2l-3}(v) < 1,$$

we have

$$f_l^i(v) = \frac{4^l - 3\cdot(-2)^i + 2}{6\cdot 4^{l-1}+3} \quad (i = 0, 1, 2, ..., 2l-2).$$

Therefore, we have

$$f_l^{2l-1}(v) = f_l\big(f_l^{2l-2}(v)\big) = f_l\Big(\frac{4^{l-1}+2}{6\cdot 4^{l-1}+3}\Big) = \frac{4^l-1}{6\cdot 4^{l-1}+3} = v.$$

Let n be an integer with $n \geq 2$ and $p_{n,1} < p_{n,2} < \cdots < p_{n,m_n}$ discontinuity points of f_l^{2n-1}. For notational convenience, we set $p_{n,0} = 0$ and $p_{n,m_n+1} = 1$. Then, we clearly have $f_l^{2n-1}(p_{n,0}) = 1$ and $f_l^{2n-1}(p_{n,m_n+1}) = 0$. Let c, d and e be real numbers with $c \in \big\{f_l^i(v) : i = 0, 2, 4, ..., 2l-2\big\}$, $d \in \big\{f_l^i(v) : i = 1, 3, 5, ..., 2l-3\big\}$ and $e \in \big\{f_l^i(v) : i = 0, 1, 2, ..., 2l-2\big\}$. Suppose that for each integer i with $0 \leq i \leq m_n$, f_l^{2n-1} is nonincreasing continuous on $(p_{n,i}, p_{n,i+1})$ and satisfies

$$f_l^{2n-1}\big((p_{n,i}, p_{n,i+1})\big) = \begin{cases} \big(f_l^2(v), 1\big) & (i = 0), \\ (c, 1],\ [0, d) \text{ or } [0, 1] & (1 \leq i \leq m_n - 1), \\ \big(0, f_l(v)\big) & (i = m_n). \end{cases}$$

Suppose furthermore that for each integer i with $1 \leq i \leq m_n$, one of the following inequalities holds:

$$0 < f_l^{2n-1}(p_{n,i}-) = f_l^{2n-1}(p_{n,i}) < f_l^{2n-1}(p_{n,i}+) = 1,$$

$$0 = f_l^{2n-1}(p_{n,i}-) < f_l^{2n-1}(p_{n,i}) = f_l^{2n-1}(p_{n,i}+) < 1,$$

$$0 = f_l^{2n-1}(p_{n,i}-) < f_l^{2n-1}(p_{n,i}) < f_l^{2n-1}(p_{n,i}+) = 1,$$

with $f_l^{2n-1}(p_{n,i}) = e$. Then, from the geometry of f_l^2 (see Figure 5) (cf. Figure 4), it follows that for each integer i with $0 \leq i \leq m_{n+1}$, f_l^{2n+1} is nonincreasing continuous on $(p_{n+1,i}, p_{n+1,i+1})$ and satisfies

$$f_l^{2n+1}\big((p_{n+1,i}, p_{n+1,i+1})\big) = \begin{cases} \big(f_l^2(v), 1\big) & (i = 0), \\ (c, 1],\ [0, d) \text{ or } [0, 1] & (1 \leq i \leq m_{n+1} - 1), \\ \big(0, f_l(v)\big) & (i = m_{n+1}). \end{cases}$$

For each integer i with $1 \leq i \leq m_{n+1}$, we additionally have one of the following inequalities:

$$0 < f_l^{2n+1}(p_{n+1,i}-) = f_l^{2n+1}(p_{n+1,i}) < f_l^{2n+1}(p_{n+1,i}+) = 1,$$

$$0 = f_l^{2n+1}(p_{n+1,i}-) < f_l^{2n+1}(p_{n+1,i}) = f_l^{2n+1}(p_{n+1,i}+) < 1,$$

$$0 = f_l^{2n+1}(p_{n+1,i}-) < f_l^{2n+1}(p_{n+1,i}) < f_l^{2n+1}(p_{n+1,i}+) = 1,$$

with $f_l^{2n+1}(p_{n+1,i}) = e$. Therefore, noting that

$$f_l^{2n+1}(p_{n+1,0}) = 1 \text{ and } f_l^{2n+1}(p_{n,m_{n+1}+1}) = 0,$$

by induction, we see that assumption (A) is satisfied. Also, for each integer k with $1 < k < l$, we have

$$f_l^{2k-1}\left(\left[0, \frac{1}{2}\right]\right) \subset \left(\frac{1}{2}, 1\right]$$

so that

$$f_l^{2k-1}(x) > x \quad \text{for any } x \in \left[0, \frac{1}{2}\right].$$

Therefore, if $f_l^{2k-1}(p) = p$ for some $p \in [0, 1] \backslash \{\frac{2}{3}\}$, then we see that

$$f_l^i(p) \in \left(\frac{1}{2}, 1\right] \setminus \left\{\frac{2}{3}\right\} \quad \text{for all integers } i \geq 0.$$

However, we then have

$$f_l^i(p) = \left(p - \frac{2}{3}\right)(-2)^i + \frac{2}{3}$$

so that

$$\left|f_l^i(p)\right| \to \infty \quad \text{as } i \to \infty.$$

This is a contradiction. Hence, for all integers k with $1 < k < l$, we obtain $f_l^k(x) \neq x$ for any $x \in [0,1] \backslash \{\frac{2}{3}\}$, which means that f_l has no periodic points with period $2k-1$. Moreover, since

$$f_l^3\Big(\frac{13}{24}\Big) = \frac{2}{3}, \quad f_l^{2l-1}\Big(\frac{4^l - 1}{6 \cdot 4^{l-1} + 3}\Big) = \frac{4^l - 1}{6 \cdot 4^{l-1} + 3}, \quad f_l\Big(\frac{2}{3}\Big) = \frac{2}{3},$$

and

$$f_l^2(x) < x \quad \text{for any } x \in \Big(\frac{13}{24}, \frac{4^l - 1}{6 \cdot 4^{l-1} + 3}\Big],$$

assumption (Al) is satisfied with $u = \frac{13}{24}$, $v = \frac{4^l-1}{6 \cdot 4^{l-1}+3}$ and $z = \frac{2}{3}$. Thus, it follows from Theorem 1.1 that f_l has a periodic point with period $2l - 1$. This completes the proof of Theorem 1.2.

References

[1] L. Block and W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics **1513**, Springer, 1992.

[2] K. Burns and B. Hasselblatt, *The Sharkovsky theorem: A natural direct proof*, Amer. Math. Monthly **118** (2011), 229–244.

[3] U. Burkhart, *Interval mapping graphs and periodic points of continuous functions*, J. Combin. Theory Ser. B **32** (1982), 57–68.

[4] B.-S. Du, *A simple proof of Sharkovsky's theorem*, Amer. Math. Monthly **111** (2004), 595–599.

[5] B.-S. Du, *A simple proof of Sharkovsky's theorem revisited*, Amer. Math. Monthly **114** (2007), 152–155.

[6] C. W. Ho and C. Morris, *A graph-theoretic proof of Sharkovsky's theorem on the periodic points of continuous functions*, Pacific J. Math. **96** (1981), 361–370.

[7] S. Murakami and H. Ohwa, *Periodic points of some discontinuous mappings*, Amer. Math. Monthly **122** (2015), 581–587.

[8] A. N. Sharkovsky, *Co-existence of cycles of a continuous map of the line into itself*, Ukrain. Mat. Zh. **16** (1964) 61–71.

[9] P. Štefan, *A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. **54** (1977), 237–248.

[10] P. D. Straffin Jr., *Periodic points of continuous functions*, Math. Mag. **51** (1978), 99–105.

[11] P. Szuca, *Sharkovskii's holds for some discontinuous functions*, Fund. Math. **179** (2002), 27–41.

(Satomi Murakami) Department of Mathematics, School of Education, Waseda University, Nishi-Waseda, Shinjuku-ku, Tokyo, 169-8050, Japan

E-mail address: jenniferhigewitt@asagi.waseda.jp

(Hiroki Ohwa) Graduate School of Science and Technology, Niigata University, 8050, Ikarashi 2-no-cho, Nishi-ku, Niigata, 950-2181, Japan
E-mail address: hiroohwa@math.sc.niigata-u.ac.jp

Received June 22, 2015
Revised September 18, 2015