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COMMUTING PAIRS OF NORMAL OPERATORS

OSAMU HATORI

Abstract. We give a condition on commutativity of a pair of normal operators with respect to the continuous functional calculus. We show a generalization of the theorem of Fuglede and Putnum with respect to the continuous functional calculus.

1. Introduction

Among the several conditions characterizing commutativity of C^*-algebras from various points of view [1, 2, 5, 6, 7, 8, 11, 12, 13] we mention Jeang and Ko [5] gave a characterization of commutativity of C^*-algebras with respect to the continuous functional calculus. The author gives a condition of commutativity of self-adjoint operators with respect to the continuous functional calculus in [4, Corollary 3.2].

In this paper we study conditions on commutativity for normal operators with respect to the continuous functional calculus. We also show a generalization of the famous commutativity theorem of Fuglede and Putnam.

Throughout the paper an operator means a bounded linear operator on a complex Hilbert space. The complex plane is denoted by $\mathbb{C}$ and $\mathbb{R}$ denotes the set of all real numbers. The spectrum of an operator a is denoted by $\sigma(a)$.

2. A condition of commutativity

Proposition 2.1. *Let Ω be a domain in $\mathbb{C}$. Suppose that a function $f : \Omega \to \mathbb{C}$ is continuous but not analytic. Then there exists $z_0 \in \Omega$, $\varepsilon_0 > 0$ and a positive integer n_0 such that*; 1) $\{z_0 + w + z : w \in \mathbb{C}$ *with* $|w| \leq 1/n_0$ *and* $z \in \mathbb{C}$ *with* $|z| \leq \varepsilon_0\} \subset \Omega$: 2) *there exists a sequence $\{s_m\}$ of complex linear combinations of constant functions and the functions of the form*

$$\{z \in \mathbb{C} : |z| \leq \varepsilon_0\} \ni z \mapsto f(z_0 + w + \alpha z)$$

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with $w \in \mathbb{C}$, $|w| \leq 1/n_0$ and $\alpha \in \mathbb{R} \cup i\mathbb{R}$, $|\alpha| < 1$ such that $s_m(z)$ uniformly converges to $\bar{z}$ on $\{z \in \mathbb{C} : |z| \leq \varepsilon_0\}$. Moreover, if u is a normal operator with $\|u\| \leq \varepsilon_0$, then $s_m(u)$ is well defined and $\lim s_m(u) = u^*$.

Proof. Since f is not analytic there exists a point $z_1 \in \Omega$ such that f is not analytic on any neighourhood of z_1. We can choose a sufficiently small $\varepsilon_1 > 0$ such that $\{z \in \mathbb{C} : |z - z_1| \leq 2\varepsilon_1\} \subset \Omega$. Let $\omega : \mathbb{C} \to \mathbb{R}$ be a continuously differentiable (with respect to $x = \operatorname{Re} w$ and $y = \operatorname{Im} w$ for $w \in \mathbb{C}$) function such that $\omega \geq 0$ on $\mathbb{C}$, $\omega(w) = 0$ for $|w| \geq 1$, and

$$\int_{|w|\leq 1} \omega(w) dx dy = 1.$$

For a positive integer n put

$$\omega_n(w) = n^2 \omega(nw), w \in \mathbb{C}.$$

Then ω_n is a continuously differentiable function such that $\omega_n \geq 0$ on $\mathbb{C}$, $\omega_n(w) = 0$ for $|w| \geq 1/n$ and

$$\int_{|w|\leq 1/n} \omega_n(w) dx dy = 1.$$

For a sufficiently large n

$$F_n(z) = \int_{|w|\leq 1/n} f(z - w)\omega_n(w) dx dy, \quad z \in \{z \in \mathbb{C} : |z - z_1| \leq 2\varepsilon_1\}$$

is well defined and continuously differentiable on $\{z \in \mathbb{C} : |z - z_1| < 2\varepsilon_1\}$. It is easy to see that F_n uniformly converges to f on $\{z \in \mathbb{C} : |z - z_1| \leq \varepsilon_1\}$ as n tends to infinity. Since f is not analytic on $\{z : |z - z_1| < \varepsilon_1\}$, there exists a positive integer n_0 such that F_{n_0} is not analytic on $\{z : |z - z_1| < \varepsilon_1\}$. Hence there exists a $z_0 \in \mathbb{C}$ with $|z_0 - z_1| < \varepsilon_1$ such that $\frac{\partial F_{n_0}}{\partial \bar{z}}(z_0) \neq 0$. Choose an arbitrary positive real number ε_0 with $2\varepsilon_0 < \varepsilon_1 - |z_1 - z_0|$. As F_{n_0} is continuously differentiable on $\{z : |z - z_1| < 2\varepsilon_1\}$ we have that $(F_{n_0}(z_0 + \delta z) - F_{n_0}(z_0))/\delta$ uniformly converges to $\frac{\partial F_{n_0}}{\partial x}(z_0) \operatorname{Re} z + \frac{\partial F_{n_0}}{\partial y}(z_0) \operatorname{Im} z$ on $\{z : |z| \leq \varepsilon_0\}$ as $1 > \delta \to 0$. We also have that $(F_{n_0}(z_0 + i\delta z) - F_{n_0}(z_0))/\delta$ uniformly converges to $\frac{\partial F_{n_0}}{\partial x}(z_0)(-\operatorname{Im} z) + \frac{\partial F_{n_0}}{\partial y}(z_0) \operatorname{Re} z$ on $\{z : |z| \leq \varepsilon_0\}$ as $1 > \delta \to 0$. We denote $\|g\| = \sup_{|z|\leq \varepsilon_0} |g(z)|$ for a complex-valued function $g(z)$ on $\{z : |z| \leq \varepsilon_0\}$. Let $\epsilon > 0$ be arbitrary. Then there exists $0 < \delta_\epsilon < 1$ such that

$$\left\| \frac{F_{n_0}(z_0 + \delta_\epsilon z) - F_{n_0}(z_0)}{\delta_\epsilon} + i \frac{F_{n_0}(z_0 + i\delta_\epsilon z) - F_{n_0}(z_0)}{\delta_\epsilon} - 2\frac{\partial F_{n_0}}{\partial \bar{z}}(z_0)\bar{z} \right\| < \epsilon,$$

as

$$\left(\frac{\partial F_{n_0}}{\partial x}(z_0) \operatorname{Re} z + \frac{\partial F_{n_0}}{\partial y}(z_0) \operatorname{Im} z \right) + i \left(\frac{\partial F_{n_0}}{\partial x}(z_0)(-\operatorname{Im} z) + \frac{\partial F_{n_0}}{\partial y}(z_0) \operatorname{Re} z \right)$$
$$= 2\frac{\partial F_{n_0}}{\partial \bar{z}}(z_0)\bar{z}.$$

By the definition of F_{n_0} the Riemann sum for the integral which defines F_{n_0} uniformly converges to F_{n_0} on $\{z : |z - z_1| \le \varepsilon_1\}$ hence on $\{z : |z - z_0| \le \varepsilon_0\}$. Therefore there exists a t_ϵ of the complex linear combination of a constant and the functions of the form

$$z \mapsto f(z_0 + w + z)$$

on $\{z : |z| \le \varepsilon_0\}$ for $|w| \le 1/n_0$ such that

$$\|F_{n_0}(z_0 + z) - t_\epsilon(z)\| < \delta_\epsilon \epsilon.$$

As $0 < \delta_\epsilon < 1$ we have

$$\begin{aligned}
&\left\| \frac{t_\epsilon(\delta_\epsilon z) - F_{n_0}(z_0)}{\delta_\epsilon} + i\frac{t_\epsilon(i\delta_\epsilon z) - F_{n_0}(z_0)}{\delta_\epsilon} - 2\frac{\partial F_{n_0}}{\partial \bar{z}}(z_0)\bar{z} \right\| \\
\le &\left\| \frac{F_{n_0}(z_0 + \delta_\epsilon z) - F_{n_0}(z_0)}{\delta_\epsilon} + i\frac{F_{n_0}(z_0 + i\delta_\epsilon z) - F_{n_0}(z_0)}{\delta_\epsilon} - 2\frac{\partial F_{n_0}}{\partial \bar{z}}(z_0)\bar{z} \right\| \\
&+ \left\| \frac{F_{n_0}(z_0 + \delta_\epsilon z) - t_\epsilon(\delta_\epsilon z)}{\delta_\epsilon} \right\| + \left\| \frac{F_{n_0}(z_0 + i\delta_\epsilon z) - t_\epsilon(i\delta_\epsilon z)}{\delta_\epsilon} \right\| \\
\le &\epsilon + 2\left\| \frac{F_{n_0}(z_0 + z) - t_\epsilon(z)}{\delta_\epsilon} \right\| < 3\epsilon.
\end{aligned}$$

As $\frac{\partial F_{n_0}}{\partial \bar{z}}(z_0) \ne 0$, we obtain that there is a sequence $\{s_m(z)\}$ of complex linear combinations of constant functions and the functions of the form

$$z \mapsto f(z_0 + w + \alpha z)$$

for $|w| \le 1/n_0$, $\alpha \in \mathbb{R} \cup i\mathbb{R}$ with $|\alpha| < 1$ such that $s_m(z)$ uniformly converges to $\bar{z}$ on $\{z : |z| \le \varepsilon_0\}$ as $m \to \infty$. In fact, for a positive integer m we may choose $s_m(z)$ as

$$\frac{1}{2\delta_{\frac{1}{m}} \frac{\partial F_{n_0}}{\partial \bar{z}}(z_0)} \left\{ (t_{\frac{1}{m}}(\delta_{\frac{1}{m}} z) - F_{n_0}(z_0)) + i(t_{\frac{1}{m}}(i\delta_{\frac{1}{m}} z) - F_{n_0}(z_0)) \right\}.$$

Suppose that u is a normal operator with $\|u\| \le \varepsilon_0$. Then $\sigma(u) \subset \{z : |z| \le \varepsilon_0\}$. Hence $s_m(u)$ is well defined. As $s_m(z)$ uniformly converges to $\bar{z}$ on $\{z : |z| \le \varepsilon_0\}$ as $m \to \infty$ we have $\lim s_m(u) = u^*$. □

Theorem 2.2. *Let Ω_j be a domain in $\mathbb{C}$ and $f_j : \Omega_j \to \mathbb{C}$ a non-constant continuous function for $j = 1, 2$. Let u_j be a normal operator for $j = 1, 2$. Suppose that*

$$f_1(\lambda_1 + \mu_1 u_1) f_2(\lambda_2 + \mu_2 u_2) = f_2(\lambda_2 + \mu_2 u_2) f_1(\lambda_1 + \mu_1 u_1) \tag{1}$$

holds for every complex number λ_j and $\mu_j \in \mathbb{R} \cup i\mathbb{R}$ with $\sigma(\lambda_j + \mu_j u_j) \subset \Omega_j$ for $j = 1, 2$. Then $u_1 u_2 = u_2 u_1$.

Proof. Suppose that both of f_1 and f_2 are analytic. Since f_j is not constant, there is a $z_j \in \Omega_j$ with $f_j'(z_j) \neq 0$. For a sufficiently small $\delta > 0$, $\sigma(z_j + \delta u_j) \subset \Omega_j$, hence $f_j(z_j + \delta u_j)$ is well defined. By (1) we infer that

$$\begin{aligned}((f_1(z_1 + \delta u_1) - f_1(z_1))/\delta)&((f_2(z_2 + \delta u_2) - f_2(z_2))/\delta) \\ &= ((f_2(z_2 + \delta u_2) - f_2(z_2))/\delta)((f_1(z_1 + \delta u_1) - f_1(z_1))/\delta).\end{aligned}$$

Letting $\delta \to 0$ we have

$$f_1'(z_1)u_1 f_2'(z_2)u_2 = f_2'(z_2)u_2 f_1'(z_1)u_1,$$

and $u_1u_2 = u_2u_1$ since $f_1'(z_1) \neq 0$ and $f_2'(z_2) \neq 0$.

Suppose that neither f_1 nor f_2 is analytic. By Proposition 2.1 there exists $z_j \in \Omega_j$, $\varepsilon_j > 0$ and a positive integer n_j such that; 1) $\{z_j + w + z : w \in \mathbb{C}$ with $|w| \leq 1/n_j$ and $z \in \mathbb{C}$ with $|z| \leq \varepsilon_j\} \subset \Omega_j$: 2) there exists a sequence $\{s_m^{(j)}\}$ of complex linear combinations of constant functions and the functions of the form

$$\{z \in \mathbb{C} : |z| \leq \varepsilon_j\} \ni z \mapsto f_j(z_j + w + \alpha z) \tag{2}$$

with $w \in \mathbb{C}$, $|w| \leq 1/n_j$ and $\alpha \in \mathbb{R} \cup i\mathbb{R}$, $|\alpha| < 1$ such that $s_m^{(j)}(z)$ uniformly converges to $\bar{z}$ on $\{z \in \mathbb{C} : |z| \leq \varepsilon_j\}$ for $j = 1, 2$. Choose a $\delta > 0$ so that $\|\delta u_j\| \leq \varepsilon_j$ for $j = 1, 2$. Then $s_m^{(j)}(\delta u_j)$ is well defined and converges to δu_j^* for $j = 1, 2$. As $s_m^{(j)}$ is a linear combination of the functions of the form $f_j(z_j + w + \alpha\cdot)$ with $|w| \leq 1/n_j$, we infer that

$$s_m^{(1)}(\delta u_1)s_m^{(2)}(\delta u_2) = s_m^{(2)}(\delta u_2)s_m^{(1)}(\delta u_1).$$

Letting $m \to \infty$ we have $\delta u_1^* \delta u_2^* = \delta u_2^* \delta u_1^*$. Hence we have $u_1u_2 = u_2u_1$.

Suppose that f_1 (resp. f_2) is analytic and f_2 (resp. f_1) is not. In a way similar to the above we have

$$u_1u_2^* = u_2^*u_1 \text{ (resp. } u_1^*u_2 = u_2u_1^*).$$

By the theorem of Fuglede and Putnam we have that $u_1u_2 = u_2u_1$. □

3. A generalization of the theorem of Fuglede and Putnam

The famous theorem of Fuglede and Putnam [3, 9] states that if $u^*a = av^*$ holds for normal operators u and v and an operator a, then $ua = av$ (cf. [10]). The theorem can be restated as follows; if $f(u)a = af(v)$, then $ua = av$, where f denotes the complex conjugation; $f(z) = \bar{z}$. Instead of $f(z) = \bar{z}$, for examples, $f(z) = |z|$, $f(z) = \bar{z}^n$ $(n \geq 2)$, $f(z) = \exp \bar{z}$ do not induce the conclusion $ua = av$ even if $f(u)a = af(v)$ holds. But applying the theorem of Fuglede and Putnam it is not difficult to prove by simple calculations that $f(\lambda + \mu u)a = af(\lambda + \mu v)$ for every pair of complex numbers λ and μ induces that $ua = av$ for the case where $f(z) = |z|$

or $f(z) = \bar{z}^n$ $(n \geq 2)$ or $f(z) = \exp \bar{z}$. Applying Proposition 2.1 we prove that it is the case for any non-constant continuous function f.

Theorem 3.1. *Let f be a non-constant complex-valued continuous function on a domain Ω in the complex plane $\mathbb{C}$. Let a be an operator and u and v normal operators. Suppose that*

$$f(\lambda + \mu u)a = af(\lambda + \mu v) \tag{3}$$

holds for every $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{R} \cup i\mathbb{R}$ such that $\sigma(\lambda + \mu u)$, $\sigma(\lambda + \mu v) \subset \Omega$. Then we have

$$ua = av.$$

Proof. Suppose that f is analytic. As f is not constant, there is $z_0 \in \Omega$ with $f'(z_0) \neq 0$. In a way similar to the proof of Theorem 2.2 we have that $f'(z_0)ua = af'(z_0)v$. Hence we infer that $ua = av$.

Suppose that f is not analytic. By Proposition 2.1 there exists a positive integer n_0, $z_0 \in \Omega$, an $\varepsilon_0 > 0$, and a sequence $\{s_m(z)\}$ of complex linear combinations of constant functions and the functions on $\{z \in \mathbb{C} : |z| \leq \varepsilon_0\}$ of the form

$$\{z \in \mathbb{C} : |z| \leq \varepsilon_0\} \ni z \mapsto f(z_0 + w + \alpha z)$$

with $w \in \mathbb{C}$, $|w| \leq 1/n_0$ and $\alpha \in \mathbb{R} \cup i\mathbb{R}$, $|\alpha| < 1$ such that $\lim s_m(b) = b^*$ for any normal operator b with $\|b\| \leq \varepsilon_0$. For the given normal operators u and v, choose a small positive δ with $\|\delta u\| \leq \varepsilon_0$ and $\|\delta v\| \leq \varepsilon_0$. Then we have by (3) that $s_m(\delta u)a = as_m(\delta v)$. Letting $m \to \infty$ we have $\delta u^* a = a\delta v^*$, so that $u^* a = av^*$. By the theorem of Fuglede and Putnam we conclude that $ua = av$. □

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Department of Mathematics, Faculty of Science, Niigata University, 950-2181 Niigata, Japan
E-mail address: `hatori@math.sc.niigata-u.ac.jp`

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