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ON SOME GENERALIZED TRIANGLE INEQUALITIES AND ℓ_ψ-SPACES

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Abstract. In this paper, we consider a generalized triangle inequality of the following type:

$$\|a_1x_1+\cdots+a_1x_n\|^p \le \|x_1\|^p+\cdots+\|x_n\|^p \ (x_1,\dots,x_n\in X),$$

where $(X,\|\cdot\|)$ is a normed space, $(a_1,\dots,a_n)\in\mathbb{C}^n$ and $p>0$. By using generalized ℓ_p-spaces, we present a characterization of above inequality for infinite sequences $\{x_n\}_{n=1}^\infty \subset X$.

1. Introduction

The triangle inequality plays a fundamental role in establishing various properties of a normed linear space. In this paper, for a normed linear space $(X,\|\cdot\|)$, we consider a following generalized triangle inequality which is involved with the Euler-Lagrange type identity: for any fixed $n\in\mathbb{N}$ with $n\ge 2$ and fixed $p\in\mathbb{R}$ with $p>0$,

$$\frac{\|a_1x_1+\cdots+a_nx_n\|^p}{\lambda} \le \frac{\|x_1\|^p}{\mu_1}+\cdots+\frac{\|x_n\|^p}{\mu_n} \ (x_1,\dots,x_n\in X), \tag{1.1}$$

where $(a_1,\dots,a_n,\lambda,\mu_1,\dots,\mu_n)\in\mathbb{C}^n\times\mathbb{R}\times\mathbb{R}^n$. Several authors have been studying its characterizations (cf. [1], [2] and [10]). In [3], by using ψ-direct sums of Banach spaces (cf. [4]), we characterized all $(a_1,\dots,a_n)\in\mathbb{C}^n$ which satisfy a special case of (1.1):

$$\|a_1x_1+\cdots+a_nx_n\|^p \le \|x_1\|^p+\cdots+\|x_n\|^p \ (x_1,\dots,x_n\in X). \tag{1.2}$$

So we gave another approach to characterizations of all $(\mu_1,\dots,\mu_n)\in\mathbb{R}^n$ which satisfy the following inequality:

$$\|a_1x_1+\cdots+a_nx_n\|^p \le \frac{\|x_1\|^p}{\mu_1}+\cdots+\frac{\|x_n\|^p}{\mu_n} \ (x_1,\dots,x_n\in X).$$

In this paper, our aim is to present a characterization of (1.2) for infinite sequences $\{x_n\}_{n=1}^\infty\subset X$ by using generalized ℓ_p-spaces.

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2. Preliminaries

In this section, we summarize basic results of ℓ_ψ-spaces which is a generalization of ℓ_p-spaces by [6].

Let ℓ_0 denote the set of all infinite sequences of complex numbers with only finitely many non-zero elements. A norm $\|\cdot\|$ on ℓ_0 is called absolute if $\|\{z_n\}_{n=1}^\infty\| = \|\{|z_n|\}_{n=1}^\infty\|$ for all $\{z_n\}_{n=1}^\infty \in \ell_0$, and normalized if $\|e_n\| = 1$ for all $n = 1, 2, \ldots$, where $e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots) \in \ell_0$. We remark that every absolute normalized norm is monotone: if $|z_i| \le |w_i|$ for every $i = 1, 2, \cdots$, then $\|\{z_n\}_{n=1}^\infty\| \le \|\{w_n\}_{n=1}^\infty\|$, where $\{z_n\}_{n=1}^\infty,\ \{w_n\}_{n=1}^\infty \in \ell_0$.

Let AN_∞ be the family of all absolute normalized norms on ℓ_0, and put

$$\Delta_\infty = \left\{ t = \{t_n\}_{n=1}^\infty \in \ell_0 : t_n \ge 0, \sum_{n=1}^\infty t_n = 1 \right\}.$$

For every $\|\cdot\| \in AN_\infty$, we define the function on Δ_∞ such that

$$\psi(t) = \|t\| \ (t = \{t_n\}_{n=1}^\infty \in \Delta_\infty), \tag{2.1}$$

then ψ is a continuous convex function on Δ_∞ satisfying the following conditions:

$$\psi(e_n) = 1 \tag{$\mathrm{A_0}$}$$

$$\psi(t) \ge (1 - t_n)\psi\left(\frac{t_1}{1-t_n}, \ldots, \frac{t_{n-1}}{1-t_n}, 0, \frac{t_{n+1}}{1-t_n}, \ldots\right) \tag{$\mathrm{A_n}$}$$

for all $n = 1, 2, \ldots$ and every $t = \{t_n\}_{n=1}^\infty \in \Delta_\infty$ with $t_n \ne 1$, where $e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots) \in \ell_0$.

Conversely, we define the set Ψ_∞ of all continuous convex functions on Δ_∞ satisfying the conditions $(\mathrm{A_n})$ for all $n = 0, 1, 2, \ldots$. For any $\psi \in \Psi_\infty$, we define the mapping on ℓ_0:

$$\|\{z_n\}_{n=1}^\infty\|_\psi = \begin{cases} \left(\sum_{j=1}^\infty |z_j|\right) \psi\left(\frac{|z_1|}{\sum_{j=1}^\infty |z_j|}, \ldots, \frac{|z_n|}{\sum_{j=1}^\infty |z_j|}, \ldots\right) & (\{z_n\}_{n=1}^\infty \ne 0) \\ 0 & (\{z_n\}_{n=1}^\infty = 0), \end{cases}$$

then $\|\cdot\|_\psi \in AN_\infty$ and it satisfies (2.1).

In fact, AN_∞ and Ψ_∞ are in a one-to-one correspondence under the equation (2.1).

Using this, we introduce the ℓ_ψ-spaces. Let ℓ_∞ is the Banach space of all bounded infinite sequences of complex numbers. For any $\psi \in \Psi_\infty$, we define the space ℓ_ψ by

$$\ell_\psi = \left\{ \{z_n\}_{n=1}^\infty \in \ell_\infty : \lim_{n\to\infty} \|(z_1, \ldots, z_n, 0, 0, \ldots)\|_\psi < \infty \right\}. \tag{2.2}$$

Then ℓ_ψ is a Banach space with the norm

$$\|\{z_n\}_{n=1}^\infty\|_\psi = \lim_{n\to\infty} \|(z_1, \ldots, z_n, 0, 0, \ldots)\|_\psi.$$

Next, we consider the dual space of ℓ_ψ. Let $\psi \in \Psi_\infty$. For any $\{z_n\}_{n=1}^\infty \in \ell_0$, the dual norm of $\|\cdot\|_\psi$ is defined by following:

$$\|\{z_n\}\|_\psi^* = \sup\left\{\left|\sum_{n=1}^\infty z_n w_n\right| : w = \{w_n\}_{n=1}^\infty \in \ell_0, \|w\|_\psi = 1\right\}.$$

Then $\|\cdot\|_\psi^* \in AN_\infty$ and the corresponding convex function in Ψ_∞ is given by

$$\psi^*(s) = \sup_{t\in\Delta_\infty} \frac{\sum_{n=1}^\infty s_n t_n}{\psi(t)} \ (s = \{s_n\}_{n=1}^\infty \in \Delta_\infty),$$

and $\|\cdot\|_\psi^* = \|\cdot\|_{\psi^*}$. Then

$$\ell_{\psi^*} = \left\{\{w_n\}_{n=1}^\infty \in \ell_\infty : \lim_{n\to\infty} \|(w_1, \ldots, w_n, 0, 0, \ldots)\|_{\psi^*} < \infty\right\}$$

is also a Banach space with the norm

$$\|\{w_n\}_{n=1}^\infty\|_{\psi^*} = \lim_{n\to\infty} \|(w_1, \ldots, w_n, 0, 0, \ldots)\|_{\psi^*}.$$

Moreover we have the Generalized Hölder inequality:

$$\sum_{n=1}^\infty |z_n w_n| \le \|\{z_n\}_{n=1}^\infty\|_\psi \ \|\{w_n\}_{n=1}^\infty\|_{\psi^*} \tag{2.3}$$

for any $\{z_n\}_{n=1}^\infty \in \ell_\psi$ and any $\{w_n\}_{n=1}^\infty \in \ell_{\psi^*}$.

Now we note the ℓ_p-norm which is a good example of absolute normalized norms. For any $\{x_n\}_{n=1}^\infty \in \ell_0$, it is

$$\|\{z_n\}_{n=1}^\infty\|_p = \begin{cases} (\sum_{n=1}^\infty |z_n|^p)^{\frac{1}{p}} & (1 \le p < \infty) \\ \max_{1\le n<\infty} |z_n| & (p = \infty), \end{cases}$$

and also for every $\|\cdot\| \in AN_\infty$, we have $\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1$. In this case, $\psi = \psi_p \in \Psi_\infty$ is

$$\psi_p(t) = \begin{cases} (\sum_{n=1}^\infty t_n^p)^{\frac{1}{p}} & (1 \le p < \infty) \\ \max_{1\le n<\infty} t_n & (p = \infty) \end{cases}$$

for any $t = \{t_n\}_{n=1}^\infty \in \Delta_\infty$. For any $\{z_n\}_{n=1}^\infty \in \ell_\psi$, a norm $\|\cdot\|_p = \|\cdot\|_{\psi_p}$ is

$$\|\{z_n\}_{n=1}^\infty\|_p = \begin{cases} (\sum_{n=1}^\infty |z_n|^p)^{\frac{1}{p}} & (1 \le p < \infty) \\ \sup_{1\le n<\infty} |z_n| & (p = \infty), \end{cases}$$

and a dual norm $\|\cdot\|_p^* = \|\cdot\|_{\psi_p}^*$ is

$$\|\{z_n\}_{n=1}^\infty\|_p^* = \begin{cases} \left(\sum_{n=1}^\infty |z_n|^q\right)^{\frac{1}{q}} & (1 < p < \infty) \\ \\ \sup_{1\le n<\infty} |z_n| & (p = 1), \end{cases} \tag{2.4}$$

where $1/p + 1/q = 1$. Thus ℓ_ψ is a generalization of ℓ_p.

3. Main result and corollary I

Let $(X, \|\cdot\|)$ be a Banach space. For any $\psi \in \Psi_\infty$, we define the ψ-direct sums of X to be the space

$$\ell_\psi(X) = \left\{\{x_n\}_{n=1}^\infty \subset X : \{\|x_n\|\}_{n=1}^\infty \in \ell_\psi\right\},$$

where ℓ_ψ is (2.2). Then it is a Banach space with the norm $\|\{x_n\}_{n=1}^\infty\|_\psi = \|\{\|x_n\|\}_{n=1}^\infty\|_\psi$ (cf. [11]). We first prove the following result.

Theorem 3.1. *Let X be a Banach space, $\psi \in \Psi_\infty$ and $\{a_n\}_{n=1}^\infty \in \ell_\infty$. Then following conditions are equivalent* :

(i) *for all $\{x_n\}_{n=1}^\infty \in \ell_\psi(X)$, $\sum_{n=1}^\infty a_n x_n$ converges in X and satisfies $\|\sum_{n=1}^\infty a_n x_n\| \le \|\{x_n\}_{n=1}^\infty\|_\psi$;*

(ii) *$\{a_n\}_{n=1}^\infty \in \ell_{\psi^*}$ and satisfies $\|\{a_n\}_{n=1}^\infty\|_{\psi^*} \le 1$.*

Proof. If $\{a_n\}_{n=1}^\infty$ satisfies (ii), we remark that $\|(a_1, \ldots, a_n)\|_{\psi^*} \le 1$ for all $n \in \mathbb{N}$. As in the proof of [3, Theorem 3.1], from the Generalized Hölder inequality (2.3), we have

$$\sum_{j=1}^n \|a_j x_j\| \le \|(x_1, \ldots, x_n)\|_\psi$$

for all $x_1, \ldots, x_n \in X$. Then we have (i).

Conversely, assume that $\{a_n\}_{n=1}^\infty \in \ell_\infty$ satisfies (i). For all fixed $n \in \mathbb{N}$, put $x_{n+1} = x_{n+2} = \cdots = 0$, then we have

$$\left\|\sum_{j=1}^n a_j x_j\right\| \le \|(x_1, \ldots, x_n)\|_\psi.$$

From [3, Theorem 3.1], we have $\|(a_1, \ldots, a_n)\|_{\psi^*} \le 1$ for all $n \in \mathbb{N}$. Hence $\{a_n\}_{n=1}^\infty$ holds (ii). □

In this theorem, $\|\{a_n\}_{n=1}^\infty\|_{\psi^*} \le 1$ is an element in the unit ball of $(\ell_\psi(X))^* = \ell_{\psi^*}(X)$, where $(\ell_\psi(X))^*$ is a dual space of $\ell_\psi(X)$.

From this theorem, we have a following corollary by putting $\psi = \psi_p$ and using (2.4).

Corollary 3.1. *Let X be a Banach space, $p \in \mathbb{R}$ with $p > 1$, and $\{a_n\}_{n=1}^{\infty} \in \ell_{\infty}$. Then following conditions are equivalent :*

(i) *for all $\{x_n\}_{n=1}^{\infty} \in \ell_p(X)$, $\sum_{n=1}^{\infty} a_n x_n$ converges in X and satisfies $\| \sum_{n=1}^{\infty} a_n x_n \|^p \le \|\{x_n\}_{n=1}^{\infty}\|^p$;*

(ii) *$\{a_n\}_{n=1}^{\infty} \in \ell_q$ and satisfies $\|\{a_n\}_{n=1}^{\infty}\|_q \le 1$, where $1/p + 1/q = 1$.*

4. A set $\widetilde{\Psi}_{\infty}$ of concave functions

In this section, we generalize the result of a set $\widetilde{\Psi}$ of concave functions which gave by [7], and introduce the $\ell_{\tilde{\psi}}$-space.

For each $n \in \mathbb{N}$ with $n \ge 2$, put

$$\Delta_n = \left\{ (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n : t_1, t_2, \ldots, t_n \ge 0, \sum_{j=1}^{n} t_j = 1 \right\}.$$

Let $\widetilde{\Psi}_n$ denote the family of all continuous concave functions for $\tilde{\psi}$ on Δ_n with

$$\tilde{\psi}(1, 0, \ldots, 0) = \tilde{\psi}(0, 1, 0, \ldots, 0) = \cdots = \tilde{\psi}(0, \ldots, 0, 1) = 1.$$

Let us define the mapping $\| \cdot \|_{\tilde{\psi}}$ on $\mathbb{C}^n$ by

$$\|(z_1, \ldots, z_n)\|_{\tilde{\psi}} = \begin{cases} (|z_1| + \cdots + |z_n|)\tilde{\psi}\left(\frac{|z_1|}{|z_1|+\cdots+|z_n|}, \ldots, \frac{|z_n|}{|z_1|+\cdots+|z_n|}\right) \\ \qquad\qquad ((z_1, \ldots, z_n) \ne (0, \ldots, 0)) \\ 0 \qquad\qquad ((z_1, \ldots, z_n) = (0, \ldots, 0)). \end{cases}$$

This mapping is monotone since the following proposition holds.

Proposition 4.1. *For any $(p_1, \ldots, p_n), (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that $0 \le p_i \le a_i$ $(i = 1, \ldots, n)$, we have that*

$$\|(p_1, \ldots, p_n)\|_{\tilde{\psi}} \le \|(a_1, \ldots, a_n)\|_{\tilde{\psi}}. \tag{4.1}$$

Proof. We first show that, if $0 \le p_1 < a_1$, then

$$\|(p_1, p_2, \ldots, p_n)\|_{\tilde{\psi}} \le \|(a_1, p_2 \ldots, a_n)\|_{\tilde{\psi}}. \tag{4.2}$$

This is, we show that if $0 \le p_1 < a_1$

$$(p_1 + p_2 + \cdots + p_n)\tilde{\psi}\left(\frac{p_1}{p_1 + p_2 + \cdots + p_n}, \ldots, \frac{p_n}{p_1 + p_2 + \cdots + p_n}\right)$$
$$\le (a_1 + p_2 + \cdots + p_n)\tilde{\psi}\left(\frac{a_1}{a_1 + p_2 + \cdots + p_n}, \ldots, \frac{p_n}{a_1 + p_2 + \cdots + p_n}\right).$$

Take any $(s_1, \ldots, s_n) \in \Delta_n$ such that $s_1 + \cdots + s_n = 1$, and consider the line segment

$$\left[(1, 0, \ldots, 0), \left(0, \frac{s_2}{1 - s_1}, \ldots, \frac{s_n}{1 - s_1}\right)\right]$$

in Δ_n. For any real number λ such that $1 < \lambda \leq 1/(1-s_1)$, we put

$$(s_1', s_2', \ldots, s_n') = (1, 0, \ldots, 0) + \lambda\{(s_1, s_2, \ldots, s_n) - (1, 0, \ldots, 0)\}.$$

Then we have

$$(s_1, s_2, \ldots, s_n) = \frac{1}{\lambda}(s_1', s_2', \ldots, s_n') + \left(1 - \frac{1}{\lambda}\right)(1, 0, \ldots, 0).$$

By the concavity of $\tilde{\psi}$,

$$\begin{aligned} \tilde{\psi}(s_1, s_2, \ldots, s_n) &\geq \frac{1}{\lambda}\tilde{\psi}(s_1', s_2', \ldots, s_n') + \left(1 - \frac{1}{\lambda}\right)\tilde{\psi}(1, 0, \ldots, 0) \\ &\geq \frac{1}{\lambda}\tilde{\psi}(s_1', s_2', \ldots, s_n') \\ &= \frac{1-s_1}{1-s_1'}\tilde{\psi}(s_1', s_2', \ldots, s_n'). \end{aligned}$$

Thus, we have

$$\frac{\tilde{\psi}(s_1, s_2, \ldots, s_n)}{1-s_1} \geq \frac{\tilde{\psi}(s_1', s_2', \ldots, s_n')}{1-s_1'}. \tag{4.3}$$

Since $0 \leq p_1 < a_1$, we put

$$\begin{aligned} &(s_1, s_2, \ldots, s_n) \\ &= \left(\frac{a_1}{a_1 + p_2 + \cdots + p_n}, \frac{p_2}{a_1 + p_2 + \cdots + p_n}, \ldots, \frac{p_n}{a_1 + p_2 + \cdots + p_n}\right), \\ &(s_1', s_2', \ldots, s_n') \\ &= \left(\frac{p_1}{p_1 + p_2 + \cdots + p_n}, \frac{p_2}{p_1 + p_2 + \cdots + p_n}, \ldots, \frac{p_n}{p_1 + p_2 + \cdots + p_n}\right), \\ &\lambda = \frac{a_1 + p_2 + \cdots + p_n}{p_1 + p_2 + \cdots + p_n} > 1 \end{aligned}$$

respectively in (4.3). Thus, we have

$$\begin{aligned} &\frac{\tilde{\psi}\left(\frac{a_1}{a_1+p_2+\cdots+p_n}, \frac{p_2}{a_1+p_2+\cdots+p_n}, \ldots, \frac{p_n}{a_1+p_2+\cdots+p_n}\right)}{1 - \frac{a_1}{a_1+p_2+\cdots+p_n}} \\ &\geq \frac{\tilde{\psi}\left(\frac{p_1}{p_1+p_2+\cdots+p_n}, \frac{p_2}{p_1+p_2+\cdots+p_n}, \ldots, \frac{p_n}{p_1+p_2+\cdots+p_n}\right)}{1 - \frac{p_1}{p_1+p_2+\cdots+p_n}}. \end{aligned}$$

This implies (4.2). Similarly, we can show that for $2 \leq i \leq n$,

$$\|(p_1, \ldots, p_{i-1}, p_i, \ldots, p_n)\|_{\tilde{\psi}} \leq \|(a_1, \ldots, p_{i-1}, a_i, \ldots, a_n)\|_{\tilde{\psi}}.$$

Therefore, we have (4.1). □

We define the set $\widetilde{\Psi}_\infty$ of all continuous concave functions on Δ_∞ satisfying the following conditions: if $\tilde{\psi} \in \widetilde{\Psi}_\infty$, then $\tilde{\psi}(e_n) = 1$ for all $n = 1, 2, \ldots$, where $e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots) \in \ell_0$. For any $\tilde{\psi} \in \widetilde{\Psi}_\infty$, we define a mapping $\| \cdot \|_{\tilde{\psi}}$ on ℓ_0 such that

$$\|\{z_n\}_{n=1}^\infty\|_{\tilde{\psi}} = \begin{cases} (\sum_{j=1}^\infty |z_j|)\tilde{\psi}\left(\frac{|z_1|}{\sum_{j=1}^\infty |z_j|}, \ldots, \frac{|z_n|}{\sum_{j=1}^\infty |z_j|}, \ldots\right) & (\{z_n\}_{n=1}^\infty \neq 0) \\ 0 & (\{z_n\}_{n=1}^\infty = 0). \end{cases}$$

Using this, we introduce the $\ell_{\tilde{\psi}}$-spaces. Let ℓ_∞ is the Banach space of all bounded infinite sequences of complex numbers. For any $\tilde{\psi} \in \widetilde{\Psi}_\infty$, we define the space $\ell_{\tilde{\psi}}$ by

$$\ell_{\tilde{\psi}} = \left\{ \{z_n\}_{n=1}^\infty \in \ell_\infty : \lim_{n\to\infty} \|(z_1, \ldots, z_n, 0, 0, \ldots)\|_{\tilde{\psi}} < \infty \right\}.$$

For any $\{z_n\}_{n=1}^\infty \in \ell_{\tilde{\psi}}$, we define the mapping

$$\|\{z_n\}_{n=1}^\infty\|_{\tilde{\psi}} = \lim_{n\to\infty} \|(z_1, \ldots, z_n, 0, 0, \ldots)\|_{\tilde{\psi}}.$$

This mapping is not a norm, however, we have the generalized inverse Minkowski inequality:

$$\|\{|z_n| + |w_n|\}_{n=1}^\infty\|_{\tilde{\psi}} \geq \|\{|z_n|\}_{n=1}^\infty\|_{\tilde{\psi}} + \|\{|w_n|\}_{n=1}^\infty\|_{\tilde{\psi}} \tag{4.4}$$

for any $\{z_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty \in \ell_{\tilde{\psi}}$. For all $p \in \mathbb{R}$ with $0 < p \leq 1$,

$$\tilde{\psi}_p(t) = \left(\sum_{n=1}^\infty t_n^{\,p}\right)^{\frac{1}{p}}$$

is an element of $\widetilde{\Psi}_\infty$ and $\|\{z_n\}_{n=1}^\infty\|_{\tilde{\psi}_p} = \|\{z_n\}_{n=1}^\infty\|_p = (\sum_{n=1}^\infty |z_n|^p)^{\frac{1}{p}}$.

5. Main result and corollary II

Let $(X, \| \cdot \|)$ be a Banach space. For any $\tilde{\psi} \in \widetilde{\Psi}_\infty$, we define the $\tilde{\psi}$-direct sums of X to be the space

$$\ell_{\tilde{\psi}}(X) = \left\{ \{x_n\}_{n=1}^\infty \subset X : \{\|x_n\|\}_{n=1}^\infty \in \ell_{\tilde{\psi}} \right\},$$

with the mapping $\|\{x_n\}_{n=1}^\infty\|_{\tilde{\psi}} = \|\{\|x_n\|\}_{n=1}^\infty\|_{\tilde{\psi}}$. We have a following result.

Theorem 5.1. *Let X be a Banach space, $\tilde{\psi} \in \widetilde{\Psi}_\infty$ and $\{a_n\}_{n=1}^\infty \in \ell_\infty$. Then following conditions are equivalent :*

(i) *for all $\{x_n\}_{n=1}^\infty \in \ell_{\tilde{\psi}}(X)$, $\sum_{n=1}^\infty a_n x_n$ converges in X and satisfies $\| \sum_{n=1}^\infty a_n x_n\| \leq \|\{x_n\}_{n=1}^\infty\|_{\tilde{\psi}}$;*

(ii) $\sup_{1\leq n<\infty} |a_n| \leq 1$.

Proof. If $\{a_n\}_{n=1}^{\infty}$ satisfies (ii), we remark that $\max\{|a_1|, \dots, |a_n|\} \le 1$ for all $n \in \mathbb{N}$. As in the proof of [3, Theorem 3.2], from the generalized inverse Minkowski inequality (4.4), we have

$$\sum_{j=1}^{n} \|a_j x_j\| \le \|(x_1, \dots, x_n)\|_{\tilde{\psi}}$$

for all $x_1, \dots, x_n \in X$. Then we have (i).

Conversely, assume that $\{a_n\}_{n=1}^{\infty} \in \ell_\infty$ satisfies (i). For all fixed $n \in \mathbb{N}$, put $x_{n+1} = x_{n+2} = \dots = 0$, then we have

$$\left\| \sum_{j=1}^{n} a_j x_j \right\| \le \|(x_1, \dots, x_n)\|_{\tilde{\psi}}.$$

From [3, Theorem 3.2], we have $\max\{|a_1|, \dots, |a_n|\} \le 1$ for all $n \in \mathbb{N}$. Hence $\{a_n\}_{n=1}^{\infty}$ holds (ii). □

From this theorem, we have a following corollary by putting $\tilde{\psi} = \tilde{\psi}_p$, where $0 < p \le 1$.

Corollary 5.1. *Let X be a Banach space, $p \in \mathbb{R}$ with $0 < p \le 1$ and $\{a_n\}_{n=1}^{\infty} \in \ell_\infty$. Then following conditions are equivalent :*

(i) *for all $\{x_n\}_{n=1}^{\infty} \in \ell_p(X)$, $\sum_{n=1}^{\infty} a_n x_n$ converges in X and satisfies $\| \sum_{n=1}^{\infty} a_n x_n\|^p \le \|\{x_n\}_{n=1}^{\infty}\|^p$;*

(ii) $\sup_{1 \le n < \infty} |a_n| \le 1$.

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