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TRAPEZOIDAL TYPE INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL VIA ČEBYŠEV FUNCTIONAL WITH APPLICATIONS

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Abstract. Some new inequalities for the functional

$$\begin{aligned} & E_T(f,u) \\ & := f(b)\left(u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right)+f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right) \\ & -\int_a^b f(t)\,du(t), \end{aligned}$$

under various assumptions for the functions f and u are given. Applications for functions of selfadjoint operators and unitary operators on complex Hilbert spaces are also provided.

1. Introduction

For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1}$$

In 1935, Grüss [28] showed that

$$|C(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{2}$$

provided that there exists the real numbers m,M,n,N such that

$$m\le f(t)\le M \quad \text{and} \quad n\le g(t)\le N \quad \text{for a.e. } t\in[a,b]. \tag{3}$$

The constant $\frac{1}{4}$ is best possible in (1) in the sense that it cannot be replaced by a smaller quantity.

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Another, however less known result, even though it was obtained by Čebyšev in 1882, [5], states that

$$|C(f,g)| \leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a)^2, \tag{4}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_{\infty} = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (4) also holds if $f, g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_{\infty}[a,b]$ while $\|f'\|_{\infty} = ess\sup_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (2) and Čebyšev's one (4) is the following inequality obtained by Ostrowski in 1970, [39]:

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{5}$$

provided that f is *Lebesgue integrable* and satisfies (3) while g is absolutely continuous and $g' \in L_{\infty}[a,b]$. The constant $\frac{1}{8}$ is best possible in (5).

The case of *Euclidean norms* of the derivative was considered by A. Lupaş in [32] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [3] have proved the following results:

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}}, \tag{7}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 \cdot \frac{1}{b-a} ess\sup_{t\in[a,b]} \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|, \tag{8}$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; $p = 1$, $q = \infty$ or $p = \infty, q = 1$).

Notice that for $q = \infty, p = 1$ in (7) we obtain

$$\begin{aligned} |C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_{\infty} \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\ &\leq \|g\|_{\infty} \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \end{aligned} \tag{9}$$

and if g satisfies (3), then

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{10}$$
$$\leq \left\| g - \frac{n+N}{2} \right\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt$$
$$\leq \frac{1}{2}(N-n) \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt.$$

The inequality between the first and the last term in (10) has been obtained by Cheng and Sun in [6]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [4].

For other recent results on the Grüss inequality, see [30], [35] and [40] and the references therein.

For some recent inequalities for Riemann-Stieltjes integral see [7]-[12] and [31].

In this paper some bounds for the functional

$$\begin{aligned} &E_T(f,u) \\ &:= f(b)\left(u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right) + f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right) \\ &- \int_a^b f(t)\,du(t), \end{aligned}$$

under various assumptions for the functions f and u are obtained. Applications for functions of selfadjoint operators and unitary operators on complex Hilbert spaces are also provided.

2. Some Preliminary Results

We start with the following representation:

Lemma 2.1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function and $u : [a,b] \to \mathbb{C}$ a function of bounded variation. Then we have the equalities*

$$\frac{1}{b-a}\int_a^b \left[\frac{f(b)(t-a) + f(a)(b-t)}{b-a} - f(t)\right] du(t) \tag{1}$$
$$= \frac{1}{b-a} E_T(f,u) = C(f',u).$$

Proof. Integrating by parts, we have

$$\frac{1}{b-a}\int_a^b f'(t)\,u(t)\,dt-\frac{1}{b-a}\int_a^b f'(t)\,dt\frac{1}{b-a}\int_a^b u(t)\,dt$$
$$=\frac{1}{b-a}\left[f(t)\,u(t)|_a^b-\int_a^b f(t)\,u(t)\,dt\right]$$
$$-\frac{f(b)-f(a)}{b-a}\cdot\frac{1}{b-a}\int_a^b u(t)\,dt$$
$$=\frac{f(b)\,u(b)-f(a)\,u(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,u(t)\,dt$$
$$-\frac{f(b)-f(a)}{b-a}\cdot\frac{1}{b-a}\int_a^b u(t)\,dt$$
$$=\frac{1}{b-a}\left[f(b)\left(u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right)\right.$$
$$\left.+f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,du(t),$$

which proves the second equality in (1).

Integrating again by parts, we have

$$u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt$$
$$=u(b)-\frac{1}{b-a}\left[u(t)\,t|_a^b-\int_a^b t\,du(t)\right]$$
$$=\frac{u(b)(b-a)-u(b)\,b+u(a)\,a+\int_a^b t\,du(t)}{b-a}$$
$$=\frac{\int_a^b t\,du(t)-a\,[u(b)-u(a)]}{b-a}=\frac{1}{b-a}\int_a^b (t-a)\,du(t)$$

and

$$\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)$$
$$=\frac{1}{b-a}\left[u(t)\,t|_a^b-\int_a^b t\,du(t)\right]-u(a)$$
$$=\frac{u(b)\,b-u(a)\,a-\int_a^b t\,du(t)-u(a)(b-a)}{b-a}$$
$$=\frac{b\,[u(b)-u(a)]-\int_a^b t\,du(t)}{b-a}=\frac{1}{b-a}\int_a^b (b-t)\,du(t).$$

Then

$$\frac{1}{b-a}\left[f(b)\left(u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right)\right.$$
$$\left.+f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,du(t)$$
$$=\frac{1}{b-a}\left[f(b)\frac{1}{b-a}\int_a^b (t-a)\,du(t)\right.$$
$$\left.+f(a)\frac{1}{b-a}\int_a^b (b-t)\,du(t)\right]-\frac{1}{b-a}\int_a^b f(t)\,du(t)$$
$$=\frac{1}{b-a}\int_a^b\left[\frac{f(b)(t-a)+f(a)(b-t)}{b-a}-f(t)\right]du(t)$$

and the first equality in (1) is also proved. □

Now, for $\gamma,\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0\ \text{ for each }\ t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\ \text{ for each }\ t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 2.1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma).\tag{2}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2-\left|z-\frac{\gamma+\Gamma}{2}\right|^2=\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (2) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2.1. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$,*we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma,\Gamma) = \{ f : [a,b] \to \mathbb{C} \mid \ & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \textit{ for each } t \in [a,b] \} . \end{aligned} \tag{3}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma,\Gamma) := \{ f : [a,b] \to \mathbb{C} \mid \ & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \ \text{ for each } \ t \in [a,b] \} . \end{aligned} \tag{4}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma) . \tag{5}$$

Lemma 2.2. *Let* $f, g : [a,b] \to \mathbb{C}$ *be Lebesgue measurable functions. Then*

$$\begin{aligned} & |C(f,g)| \\ & \leq \frac{1}{b-a} \begin{cases} \inf_{\gamma\in\mathbb{C}} \|g-\gamma\|_1 \cdot ess\sup_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| & \begin{array}{l} g \in L_1[a,b], \\ f \in L_\infty[a,b] \end{array} \\ \inf_{\gamma\in\mathbb{C}} \|g-\gamma\|_q \cdot \left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} & \begin{array}{l} g \in L_q[a,b], \\ f \in L_p[a,b], \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ \inf_{\gamma\in\mathbb{C}} \|g-\gamma\|_\infty \cdot \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt & \begin{array}{l} g \in L_\infty[a,b], \\ f \in L_1[a,b] \end{array} \end{cases} \end{aligned} \tag{6}$$

Proof. The assertion follows by the Sonin's identity for complex valued functions

$$C(f,g) = \frac{1}{b-a}\int_a^b (g(t)-\gamma)\left(f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right) dt$$

and by the integral Hölder inequality. □

Corollary 2.2. *Let $f, g : [a,b] \to \mathbb{C}$ be Lebesgue measurable functions. If $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, and $g \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, then*

$$|C(f,g)| \tag{7}$$
$$\leq \frac{1}{2}|\Gamma - \gamma| \begin{cases} ess\sup_{t\in[a,b]} \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| & f \in L_\infty[a,b] \\ \left(\frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & f \in L_p[a,b], \\ & p > 1, \\ \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt & f \in L_1[a,b]. \end{cases}$$

Another important corollary is as follows:

Corollary 2.3. *Let $f, g : [a,b] \to \mathbb{C}$ be Lebesgue measurable functions. If g is of bounded variation, then*

$$|C(f,g)| \tag{8}$$
$$\leq \frac{1}{2}\bigvee_a^b (g) \begin{cases} ess\sup_{t\in[a,b]} \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| & f \in L_\infty[a,b] \\ \left(\frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & f \in L_p[a,b], \\ & p > 1, \\ \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt & f \in L_1[a,b], \end{cases}$$

where $\bigvee_a^b (g)$ is the total variation of g on $[a,b]$.

Proof. Since g is of bounded variation, then

$$\begin{aligned} \left|g(t) - \frac{g(a)+g(b)}{2}\right| &\leq \frac{1}{2}\left[|g(b) - g(t)| + |g(t) - g(a)|\right] \\ &\leq \frac{1}{2}\bigvee_a^b (g) \end{aligned} \tag{9}$$

for any $t \in [a,b]$.

We have

$$\begin{aligned} \left\|g(\cdot) - \frac{g(a)+g(b)}{2}\right\|_\infty &= ess\sup_{t\in[a,b]} \left|g(t) - \frac{g(a)+g(b)}{2}\right| \\ &\leq \frac{1}{2}\bigvee_a^b (g) \end{aligned}$$

and

$$\begin{aligned}\left\|g(\cdot)-\frac{g(a)+g(b)}{2}\right\|_q &= \left(\int_a^b \left|g(t)-\frac{g(a)+g(b)}{2}\right|^q dt\right)^{1/q} \\ &\le \frac{1}{2}\bigvee_a^b (g)\left(\int_a^b dt\right)^{1/q} = \frac{1}{2}(b-a)^{1/q}\bigvee_a^b (g)\end{aligned}$$

for $q \ge 1$.

Utilising (6) we get (8). □

For functions h that are *Lipschitzian in the middle point* with the constant $L_{\frac{a+b}{2}}$ and the exponent $s > 0$, i.e., satisfying the condition

$$\left|h(t)-h\left(\frac{a+b}{2}\right)\right| \le L_{\frac{a+b}{2}}\left|t-\frac{a+b}{2}\right|^s$$

for any $t \in [a,b]$, we have the following result as well.

Another important corollary is as follows:

Corollary 2.4. *Let $f, g : [a,b] \to \mathbb{C}$ be Lebesgue measurable functions. If g is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $s > 0$, then*

$$|C(f,g)| \le \frac{1}{2^s}L_{\frac{a+b}{2}} \times \begin{cases} \frac{(b-a)^s}{s+1}\cdot ess\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| & f \in L_\infty[a,b] \\ \frac{(b-a)^{s-\frac{1}{p}}}{(sq+1)^{1/q}}\cdot\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f\in L_p[a,b], \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1 \end{array} \\ (b-a)^{s-1}\cdot\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt & f\in L_1[a,b]. \end{cases} \tag{10}$$

Proof. We have, for $q \ge 1$, that

$$\begin{aligned}\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],q} &= \left(\int_a^b\left|g(t)-g\left(\frac{a+b}{2}\right)\right|^q dt\right)^{1/q} \\ &\le \left(\int_a^b L_{\frac{a+b}{2}}^q\left|t-\frac{a+b}{2}\right|^{sq} dt\right)^{1/q} \\ &= L_{\frac{a+b}{2}}\left(\int_a^b\left|t-\frac{a+b}{2}\right|^{sq} dt\right)^{1/q}.\end{aligned} \tag{11}$$

Observe that

$$\begin{aligned}
&\left(\int_a^b \left|t-\frac{a+b}{2}\right|^{sq} dt\right)^{1/q} \\
&= \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)^{sq} dt + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^{sq} dt\right)^{1/q} \\
&= \left(2\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^{sq} dt\right)^{1/q} = \left(2\left.\frac{\left(t-\frac{a+b}{2}\right)^{sq+1}}{sq+1}\right|_{\frac{a+b}{2}}^b\right)^{1/q} \\
&= \left(2\frac{\left(\frac{b-a}{2}\right)^{sq+1}}{sq+1}\right)^{1/q} = \left(\frac{(b-a)^{sq+1}}{2^{sq}(sq+1)}\right)^{1/q} = \frac{(b-a)^{s+1/q}}{2^s(sq+1)^{1/q}}.
\end{aligned}$$

Then by (11) we have

$$\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],q} \le L_{\frac{a+b}{2}}\frac{(b-a)^{s+1/q}}{2^s(sq+1)^{1/q}}.$$

Also

$$\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty} \le L_{\frac{a+b}{2}}\frac{(b-a)^{s}}{2^s}.$$

By utilizing the inequality (6) we have

$$\begin{aligned}
&|C(f,g)| \\
&\le \frac{1}{b-a}\begin{cases}
L_{\frac{a+b}{2}}\frac{(b-a)^{s+1}}{2^s(s+1)}\cdot ess\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| & f\in L_\infty[a,b] \\
L_{\frac{a+b}{2}}\frac{(b-a)^{s+1/q}}{2^s(sq+1)^{1/q}}\cdot\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f\in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1 \end{array} \\
L_{\frac{a+b}{2}}\frac{(b-a)^{s}}{2^s}\cdot\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| dt & f\in L_1[a,b]
\end{cases} \\
&= \frac{1}{2^s}L_{\frac{a+b}{2}}\begin{cases}
\frac{(b-a)^{s}}{s+1}\cdot ess\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| & f\in L_\infty[a,b] \\
\frac{(b-a)^{s}}{(sq+1)^{1/q}}\cdot\left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f\in L_p[a,b], \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1 \end{array} \\
(b-a)^{s-1}\cdot\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| dt & f\in L_1[a,b]
\end{cases}
\end{aligned}$$

and the corollary is proved. □

Remark 2.1. *In the case when g is Lipschitzian with the constant $L > 0$, then*

$$|C(f,g)| \leq \frac{1}{2}L \times \begin{cases} \frac{1}{2}(b-a) \cdot ess\sup_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| & f \in L_\infty[a,b] \\ \frac{(b-a)^{1-\frac{1}{p}}}{(q+1)^{1/q}} \cdot \left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} & \begin{array}{l} f \in L_p[a,b], \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt & f \in L_1[a,b]. \end{cases} \tag{12}$$

3. Error Bounds for a Generalized Trapezoid Rule

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the generalized trapezoid formula

$$f(b)\left(u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right) + f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right)$$

we consider the error functional

$$\begin{aligned} & E_T(f,u) \\ & := f(b)\left(u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right) + f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right) \\ & - \int_a^b f(t)\,du(t). \end{aligned} \tag{1}$$

For some recent results concerning this functional see [24] and [36].

Theorem 3.1. *Let $f : [a,b] \to \mathbb{C}$ be absolutely continuous and $u : [a,b] \to \mathbb{C}$ of bounded variation.*

(i) *If* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *and* $u \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, *then*

$$|E_T(f,u)| \leq \frac{1}{2}|\Gamma - \gamma| \times \begin{cases} (b-a)\, ess\sup_{t\in[a,b]} \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| & f' \in L_\infty[a,b] \\ (b-a)^{\frac{1}{q}} \left(\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & f \in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| dt & f \in L_1[a,b]. \end{cases} \tag{2}$$

(ii) *If* $\varphi, \Phi \in \mathbb{C}$, $\varphi \neq \Phi$, *and* $f' \in \bar{\Delta}_{[a,b]}(\varphi, \Phi)$, *then*

$$|E_T(f,u)| \leq \frac{1}{2}|\Phi - \varphi| \times \begin{cases} (b-a)\, ess\sup_{t\in[a,b]} \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| & u \in L_\infty[a,b] \\ (b-a)^{\frac{1}{q}} \left(\int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & u \in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1 \\ \int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| dt & u \in L_1[a,b]. \end{cases} \tag{3}$$

Proof. From Lemma 2.1 we have the representation

$$E_T(f,u) = (b-a)\, C(f', u). \tag{4}$$

(i) If $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, and $u \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, then by Corollary 2.2 we have

$$|C(f',u)| \leq \frac{1}{2}|\Gamma - \gamma| \begin{cases} ess\sup_{t\in[a,b]} \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| & f' \in L_\infty[a,b] \\ \left(\frac{1}{b-a}\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & f \in L_p[a,b], \\ & p>1, \\ \frac{1}{b-a}\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| dt & f \in L_1[a,b], \end{cases}$$

which implies the desired result (2).

(ii) If $\varphi, \Phi \in \mathbb{C}$, $\varphi \neq \Phi$, and $f' \in \bar{\Delta}_{[a,b]}(\varphi, \Phi)$, then by Corollary 2.2 we have

$$|C(f', u)| \leq \frac{1}{2}|\Phi - \varphi| \begin{cases} ess\sup_{t\in[a,b]} \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| & u \in L_\infty[a,b] \\ \left(\frac{1}{b-a}\int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & u \in L_p[a,b], \\ & p > 1, \\ \frac{1}{b-a}\int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| dt & u \in L_1[a,b], \end{cases}$$

which implies the desired result (3). □

The following result also holds:

Theorem 3.2. *Let $f : [a,b] \to \mathbb{C}$ be absolutely continuous and $u : [a,b] \to \mathbb{C}$ of bounded variation.*

(i) *We have*

$$|E_T(f,u)| \leq \frac{1}{2}\bigvee_a^b (u) \times \begin{cases} (b-a)\, ess\sup_{t\in[a,b]} \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| & f' \in L_\infty[a,b] \\ (b-a)^{\frac{1}{q}} \left(\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & f \in L_p[a,b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| dt & f \in L_1[a,b]. \end{cases} \tag{5}$$

(ii) *If f' is of bounded variation, then*

$$|E_T(f,u)| \leq \frac{1}{2}\bigvee_a^b (f') \times \begin{cases} (b-a)\, ess\sup_{t\in[a,b]} \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| & u \in L_\infty[a,b] \\ (b-a)^{\frac{1}{q}} \left(\int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & u \in L_p[a,b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| dt & u \in L_1[a,b]. \end{cases} \tag{6}$$

The proof follows by the identity (4) and from Corollary 2.3. We omit the details. The case of Lipschitzian functions is as follows:

Theorem 3.3. *Let $f : [a, b] \to \mathbb{C}$ be absolutely continuous and $u : [a, b] \to \mathbb{C}$ of bounded variation.*

(i) *If u is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $s > 0$, then*

$$|E_T(f,u)| \leq \frac{1}{2^s} L_{\frac{a+b}{2}} \times \begin{cases} \frac{(b-a)^{s+1}}{s+1} ess\sup_{t\in[a,b]} \left| f'(t) - \frac{f(b)-f(a)}{b-a} \right| & f' \in L_\infty[a,b] \\ \frac{(b-a)^{s+\frac{1}{q}}}{(sq+1)^{1/q}} \left(\int_a^b \left| f'(t) - \frac{f(b)-f(a)}{b-a} \right|^p dt \right)^{\frac{1}{p}} & \begin{array}{l} f \in L_p[a,b], \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\ (b-a)^s \int_a^b \left| f'(t) - \frac{f(b)-f(a)}{b-a} \right| dt & f \in L_1[a,b]. \end{cases} \tag{7}$$

(ii) *If f' is Lipschitzian in the middle point with the constant $K_{\frac{a+b}{2}}$ and the exponent $v > 0$, then*

$$|E_T(f,u)| \leq \frac{1}{2^v} K_{\frac{a+b}{2}} \times \begin{cases} \frac{(b-a)^{v+1}}{v+1} ess\sup_{t\in[a,b]} \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right| & u \in L_\infty[a,b] \\ \frac{(b-a)^{v+\frac{1}{q}}}{(vq+1)^{1/q}} \left(\int_a^b \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right|^p dt \right)^{\frac{1}{p}} & \begin{array}{l} u \in L_p[a,b], \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ (b-a)^v \int_a^b \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right| dt & u \in L_1[a,b]. \end{cases} \tag{8}$$

The proof follows by Corollary 2.4.

Remark 3.1. *If u is Lipschitzian with the constant $L > 0$, then*

$$|E_T(f,u)| \leq \frac{1}{2} L$$

$$\times\begin{cases} \frac{1}{2}(b-a)^2\, ess\sup_{t\in[a,b]}\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right| & f'\in L_\infty[a,b] \\ \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{1/q}}\left(\int_a^b\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & \begin{matrix} f\in L_p[a,b], \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \end{matrix} \\ (b-a)\int_a^b\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right| dt & f\in L_1[a,b]. \end{cases} \tag{9}$$

If f' is Lipschitzian with the constant $K>0$, then

$$|E_T(f,u)|\le\frac{1}{2}K$$

$$\times\begin{cases} \frac{1}{2}(b-a)^2\, ess\sup_{t\in[a,b]}\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right| & u\in L_\infty[a,b] \\ \frac{(b-a)^{1+\frac{1}{q}}}{(v+1)^{1/q}}\left(\int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{matrix} u\in L_p[a,b], \\ p>1, \frac{1}{p}+\frac{1}{q}=1 \end{matrix} \\ (b-a)\int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right| dt & u\in L_1[a,b]. \end{cases} \tag{10}$$

4. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases} 1, \text{ for } -\infty<s\le\lambda, \\ 0, \text{ for } \lambda<s<+\infty. \end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [29, p. 256]:

Theorem 4.1 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $M=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda\le E_{\lambda'}$ *for* $\lambda\le\lambda'$;

b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;

c) *We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ *defined on* $\mathbb{R}$ *there exists a unique operator* $\varphi(A) \in \mathcal{B}(H)$ *such that for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *satisfying the inequality*

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}} \right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda) dE_\lambda, \tag{2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 4.1. *With the assumptions of Theorem 4.1 for* A, E_λ *and* φ *we have the representations*

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda) dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)x, y \rangle = \int_{m-0}^{M} \varphi(\lambda) d\langle E_\lambda x, y \rangle \quad \text{for all } x, y \in H. \tag{3}$$

In particular,

$$\langle \varphi(A)x, x \rangle = \int_{m-0}^{M} \varphi(\lambda) d\langle E_\lambda x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2 d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y \rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [23].

Lemma 4.1. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any $x, y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \leq \langle (E_\beta - E_\alpha)\, x, x\rangle \,\langle (E_\beta - E_\alpha)\, y, y\rangle\,, \tag{4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x, y\right\rangle$ on $[\alpha, \beta]$.

Remark 4.1. *For $\alpha = m - \varepsilon$ with $\varepsilon > 0$ and $\beta = M$ we get from (4) the inequality*

$$\bigvee_{m-\varepsilon}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \langle (I - E_{m-\varepsilon})\, x, x\rangle^{1/2} \,\langle (I - E_{m-\varepsilon})\, y, y\rangle^{1/2} \tag{5}$$

for any $x, y \in H$.

This implies, for any $x, y \in H$, that

$$\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \|x\|\,\|y\|\,, \tag{6}$$

where $\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon\to 0+}\left[\bigvee_{m-\varepsilon}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 4.2. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda\,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda\,|\lambda \in Sp(A)\} =: \max Sp(A)$. If $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and $f : I \to \mathbb{C}$ is absolutely continuous on $[m, M] \subset \mathring{I}$ (the interior of I), then*

$$\begin{aligned}&\left|\left\langle\left[\frac{f(M)(A - m1_H) + f(m)(M1_H - A)}{M - m} - f(A)\right] x, y\right\rangle\right| \\ &\leq \frac{1}{2}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\begin{cases}(M - m)\, ess\sup_{t\in[m,M]}\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| \\ \\ (M - m)^{\frac{1}{q}}\left(\int_m^M\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right|^p dt\right)^{\frac{1}{p}} \\ \\ \int_m^M\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt\end{cases}\end{aligned}$$

$$\leq \frac{1}{2}\|x\|\,\|y\| \begin{cases} (M-m)\, ess\sup_{t\in[m,M]} \left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| \\ \\ (M-m)^{\frac{1}{q}} \left(\int_m^M \left|f'(t) - \frac{f(M)-f(m)}{M-m}\right|^p dt\right)^{\frac{1}{p}} \\ \\ \int_m^M \left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt \end{cases} \tag{7}$$

for any $x, y \in H$.

Proof. Utilising the representation (1) and the inequality (5) we have

$$\left|\int_{m-\varepsilon}^{M} \left[\frac{f(M)(t-m+\varepsilon) + f(m-\varepsilon)(M-t)}{M-m+\varepsilon} - f(t)\right] d\langle E_t x, y\rangle\right|$$

$$\leq \frac{1}{2} \bigvee_{m-\varepsilon}^{M} \left(\langle E_{(\cdot)} x, y\rangle\right) \begin{cases} (M-m+\varepsilon)\, ess\sup_{t\in[m-\varepsilon,M]} \left|f'(t) - \frac{f(M)-f(m-\varepsilon)}{M-m+\varepsilon}\right| \\ \\ (M-m+\varepsilon)^{\frac{1}{q}} \left(\int_{m-\varepsilon}^M \left|f'(t) - \frac{f(M)-f(m-\varepsilon)}{M-m+\varepsilon}\right|^p dt\right)^{\frac{1}{p}} \\ \\ \int_{m-\varepsilon}^M \left|f'(t) - \frac{f(M)-f(m-\varepsilon)}{M-m+\varepsilon}\right| dt \end{cases}$$

for small $\varepsilon > 0$ and for any $x, y \in H$.

Taking the limit over $\varepsilon \to 0+$ and using the continuity of f and the Spectral Representation Theorem, we deduce the desired result (7). □

For recent results concerning inequalities for functions of selfadjoint operators, see [1], [14], [15], [16], [17], [18], [19], [23], [33], [37], [38], [41] and the books [21], [22] and [27].

5. Applications for Unitary Operators

A unitary operator is a bounded linear operator $U : H \to H$ on a Hilbert space H satisfying

$$U^*U = UU^* = 1_H$$

where U^* is the adjoint of U, and $1_H : H \to H$ is the identity operator. This property is equivalent to the following:

(i) U preserves the inner product $\langle\cdot,\cdot\rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux, Uy\rangle = \langle x, y\rangle$ and

(ii) U is surjective.

The following result is well known [29, pp. 275–276]:

Theorem 5.1 (Spectral Representation Theorem). *Let U be a unitary operator on the Hilbert space H. Then there exists a family of projections $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, called the spectral family of U, with the following properties*

a) $P_\lambda \le P_{\lambda'}$ *for* $\lambda \le \lambda'$;

b) $P_0 = 0, P_{2\pi} = I$ *and* $P_{\lambda+0} = P_\lambda$ *for all* $\lambda \in [0, 2\pi)$;

c) *We have the representation*

$$U = \int_0^{2\pi} \exp(i\lambda)\, dP_\lambda.$$

More generally, for every continuous complex-valued function φ defined on the unit circle $\mathcal{C}(0,1)$ there exists a unique operator $\varphi(U) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(U) - \sum_{k=1}^{n} \varphi(\exp(i\lambda_k'))\left[P_{\lambda_k} - P_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(U) = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda, \tag{1}$$

where the integral is of Riemann-Stieltjes type.

Corollary 5.1. *With the assumptions of Theorem 5.1 for U, P_λ and φ we have the representations*

$$\varphi(U)x = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(U)x, y\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{2}$$

In particular,

$$\langle \varphi(U)x, x\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(U)x\|^2 = \int_0^{2\pi} |\varphi(\exp(i\lambda))|^2\, d\|P_\lambda x\|^2 \quad \text{for all } x \in H.$$

The following result holds:

Theorem 5.2. *Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ the spectral family of U. Let f be a differentiable complex-valued function defined on an open disk containing the unit circle $\mathcal{C}(0,1)$. Then we have*

$$\begin{aligned} & |\langle [2\pi f(1) - f(U)]\, x, y\rangle| \qquad (3)\\ & \le \frac{1}{2}\bigvee_0^{2\pi}\left(\langle P_{(\cdot)}x,y\rangle\right)\begin{cases} 2\pi ess\sup_{t\in[0,2\pi]}\left|f'(e^{it})\right|; \\ (2\pi)^{\frac{1}{q}}\left(\int_0^{2\pi}\left|f'(e^{it})\right|^p dt\right)^{\frac{1}{p}}; \\ \int_0^{2\pi}\left|f'(e^{it})\right| dt; \end{cases} \\ & \le \frac{1}{2}\|x\|\,\|y\|\begin{cases} 2\pi ess\sup_{t\in[0,2\pi]}\left|f'(e^{it})\right|; \\ (2\pi)^{\frac{1}{q}}\left(\int_0^{2\pi}\left|f'(e^{it})\right|^p dt\right)^{\frac{1}{p}}; \\ \int_0^{2\pi}\left|f'(e^{it})\right| dt, \end{cases} \end{aligned}$$

for all $x, y \in H$.

Proof. Utilising the representation (1), the inequality (5) and the fact that f is differentiable as a complex function, we have

$$\begin{aligned} & \left|\int_0^{2\pi}\left[\frac{f(e^{i2\pi})(t-0)+f(e^0)(2\pi-t)}{2\pi}-f\left(e^{it}\right)\right] d\langle P_\lambda x,y\rangle\right| \\ & \le \frac{1}{2}\bigvee_0^{2\pi}\left(\langle P_{(\cdot)}x,y\rangle\right)\begin{cases} 2\pi ess\sup_{t\in[0,2\pi]}\left|ie^{it}f'(e^{it})-\frac{f\left(e^{i2\pi}\right)-f\left(e^0\right)}{2\pi}\right| \\ (2\pi)^{\frac{1}{q}}\left(\int_0^{2\pi}\left|ie^{it}f'(e^{it})-\frac{f\left(e^{i2\pi}\right)-f\left(e^0\right)}{2\pi}\right|^p dt\right)^{\frac{1}{p}} \\ \int_0^{2\pi}\left|ie^{it}f'(e^{it})-\frac{f\left(e^{i2\pi}\right)-f\left(e^0\right)}{2\pi}\right| dt \end{cases} \qquad (4) \end{aligned}$$

for all $x, y \in H$.

The inequality (4) is equivalent with

$$\left|\int_0^{2\pi}\left[f(1)-f\left(e^{it}\right)\right] d\langle P_\lambda x,y\rangle\right|$$

$$\leq \frac{1}{2}\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\begin{cases} 2\pi ess\sup_{t\in[0,2\pi]}\left|f'(e^{it})\right| \\ \\ (2\pi)^{\frac{1}{q}}\left(\int_0^{2\pi}\left|f'(e^{it})\right|^p dt\right)^{\frac{1}{p}} \\ \\ \int_0^{2\pi}\left|f'(e^{it})\right| dt \end{cases}$$

and the desired result (3) is proved. □

Remark 5.1. *Consider the exponential function* $f:\mathbb{C}\to\mathbb{C}$, $f(z)=\exp z := \sum_{n=0}^{\infty}\frac{1}{n!}z^n$. *Then* $f'(z)=\exp z$ *and*

$$\begin{aligned} \left|f'(e^{it})\right| &= \left|\exp(\cos t + i\sin t)\right| = \exp(\cos t)\left|\exp(i\sin t)\right| \\ &= \exp(\cos t) \end{aligned}$$

for $t\in[0,2\pi]$.

Observe that

$$\sup_{t\in[0,2\pi]}\left|f'(e^{it})\right| = e$$

and for $p\geq 1$

$$\left(\int_0^{2\pi}\left|f'(e^{it})\right|^p dt\right)^{\frac{1}{p}} = \left(\int_0^{2\pi}\exp(p\cos t)\,dt\right)^{\frac{1}{p}} = \left[2\pi I_0(p)\right]^{1/p}$$

where I_0 *is the modified Bessel function of the first kind, i.e., we recall that*

$$I_0(z) := \sum_{m=0}^{\infty}\frac{1}{(m!)^2}\left(\frac{z}{2}\right)^{2m},\ z\in\mathbb{C}.$$

Let U *be a unitary operator on the Hilbert space* H *and* $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ *the spectral family of* U. *Then we have by (3)*

$$\left|\left\langle\left[2\pi e - \exp(U)\right]x,y\right\rangle\right| \tag{5}$$

$$\leq \pi\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\begin{cases} e; \\ \\ (I_0(p))^{\frac{1}{p}};\quad p>1 \\ \\ I_0(1); \end{cases}$$

$$\leq \pi\left\|x\right\|\left\|y\right\|\begin{cases} e; \\ \\ (I_0(p))^{\frac{1}{p}};\quad p>1 \\ \\ I_0(1), \end{cases}$$

for all $x,y\in H$.

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