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MULTIPLIERS OF A WANDERING SUBSPACE FOR A SHIFT INVARIANT SUBSPACE II

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Abstract. Let M be a shift invariant subspace in the two variable Hardy space $H^2(\Gamma_z \times \Gamma_w)$. We study $\mathcal{M}(M_z) = \{\phi \in H^\infty(\Gamma_z \times \Gamma_w) : \phi M_z \subseteq M_z\}$ where $M_z = M \ominus zM$. We give several sufficient conditions for $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$ where $H^\infty(\Gamma_w)$ is the one variable Hardy space.

1. Introduction

Let Γ^2 be the torus that is the Cartesian product of two unit circle Γ in $\mathbb{C}$. For $1 \le p \le \infty$, the usual Lebesgue spaces, with respect to the Lebesgue measure m on Γ^2, are denoted by $L^p = L^p(\Gamma^2)$, and $H^p = H^p(\Gamma^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j,\ell) = \int_{\Gamma^2} f(z,w)\bar{z}^j \bar{w}^\ell dm(z,w)$$

are zero as soon as at least one component of (j,ℓ) is negative. Then H^p is called a Hardy space. As $\Gamma^2 = \Gamma_z \times \Gamma_w$, $H^p(\Gamma_z)$ and $H^p(\Gamma_w)$ denote one variable Hardy spaces. $H^\infty(\Gamma_q)$ (or $L^\infty(\Gamma_q)$) is a weak $*$ closure of polynomials of q (or q and $\bar{q}$).

A closed subspace $M \subseteq H^2$ is said to be (shift) invariant if $zM \subseteq M$ and $wM \subseteq M$. Suppose $\zeta = \zeta(z,w)$ is in H^∞ and $T_\zeta f = \zeta f$ $(f \in H^2)$. Put $V_\zeta = T_\zeta \mid M$. Then $M = \overline{\mathrm{Ran}} V_\zeta \oplus \mathrm{Ker} V_\zeta^*$ where $\overline{\mathrm{Ran}} V_\zeta$ denotes the closure of the range of V_ζ and $\mathrm{Ker} V_\zeta^*$ = the kernel of V_ζ^*. We write $M_\zeta = \mathrm{Ker} V_\zeta^*$ and $[\zeta M] = \overline{\mathrm{Ran}} V_\zeta$. We call M_ζ a wandering subspace and $\mathcal{M}(M_\zeta) = \{f \in H^\infty : fM_\zeta \subseteq M_\zeta\}$ the set of multipliers of M_ζ. In this paper we assume $M_\zeta \neq \langle 0 \rangle$.

In the previous paper [5], we considered $\mathcal{M}(M_\zeta)$ when $\zeta = z$. Then we show $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$ when $\mathcal{M}(M_z) \cap H^\infty(\Gamma_w) \neq \mathbb{C}$. K. J. Izuchi pointed out me privately that the proof of Lemma 2 in the previous paper [5] has a gap. Lemma 2 can be proved only in a very special case. Hence Theorem in [5] has not shown yet in general. Therefore we would like to study the following problem for a nonzero invariant subspace M in H^2.

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I. If $\mathcal{M}(M_z)$ contains a nonconstant function then $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$.
II. It is true that $\mathcal{M}(M_z) \cap \mathcal{M}(M_w) = \mathbb{C}$.

Of course, I shows II.

In this paper, we use the following notations. Put $K_z = \{f \in M_z : w^\ell f \in M_z$ for $\ell = 1, 2, \cdots\}$. If $K_z = M_z$ then $\mathcal{M}(M_z)$ contains w. Z denotes the set of all integers and Z_+ denotes the set of nonnegative integers. If ϕ is a function in H^∞ with absolute value one, ϕ is called inner.

2. $\mathcal{M}(M_\zeta)$ for general ζ

In this section, $\mathcal{M}(M_\zeta)$ is studied for arbitrary $\zeta = \zeta(z, w)$ in H^∞.

Proposition 1. *Let ϕ and ψ be nonconstant functions in H^∞.*

(1) *If $V_\psi^* V_\phi = V_\phi V_\psi^*$ then $\phi \in \mathcal{M}(M_\psi)$ and $\psi \in \mathcal{M}(M_\phi)$.*
(2) *Suppose ϕ or ψ is inner. Then if $\phi \in \mathcal{M}(M_\psi)$ and $\psi \in \mathcal{M}(M_\phi)$ then $V_\psi^* V_\phi = V_\phi V_\psi^*$.*

Proof. (1) Since $V_\psi^* V_\phi = V_\phi V_\psi^*$ and $V_\phi^* V_\psi = V_\psi V_\phi^*$, it is clear because $\phi \mathrm{Ker} V_\psi^* \subseteq \mathrm{Ker} V_\psi^*$ and $\psi \mathrm{Ker} V_\phi^* \subseteq \mathrm{Ker} V_\phi^*$.

(2) Let ψ be inner. If ϕ is in $\mathcal{M}(M_\psi)$ then $V_\psi^* V_\phi = V_\phi V_\psi^*$ on $\mathrm{Ker} V_\psi^*$. While, $V_\psi^* V_\phi = V_\phi V_\psi^*$ holds clearly on ψM. This shows (2). □

Theorem 1. *Let $\zeta = \zeta(z, w)$ be a function in H^∞. Then $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w) = H^\infty(\Gamma_w)$ or $\mathbb{C}$.*

Proof. If ζ is a constant c then $M_\zeta = M$ or $M_\zeta = \{0\}$. Hence $\mathcal{M}(M_\zeta) = H^\infty$ and so $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w) = H^\infty(\Gamma_w)$. Suppose ζ is nonconstant and $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w) \neq \mathbb{C}$. If f is a nonconstant function in $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w)$ then $f(w) - f(0)$ belongs to $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w)$ because $f(0) \in \mathcal{M}(M_\zeta)$. Let $f(w) - f(0) = wh(w)$ and $h \in H^\infty(\Gamma_w)$. If g is a function in M_ζ then $whg \in M_\zeta$ and so $whg \perp w\zeta M$. This implies that $hg \perp \zeta M$ and so $hg \in M_\zeta$. Since g is arbitrary, $h \in \mathcal{M}(M_\zeta)$ and so $(f(w) - f(0))/w$ belongs to $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w)$. Since $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w)$ is a nonzero weak $*$ closed subalgebra in $H^\infty(\Gamma_w)$ which contains constants, by [1, Theorem 1] $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w) = H^\infty(\Gamma_w)$. □

Corollary 1. *Let $\zeta = \zeta(z, w)$ be a function in H^∞.*

(1) *If $\phi = \phi(w)$ is a nonconstant function in $\mathcal{M}(M_\zeta)$, then $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w) = H^\infty(\Gamma_w)$.*
(2) *If $\phi = \phi(w)$ is a nonconstant function and $\zeta = \zeta(z)$, then the inner part of ζ belongs to $\mathcal{M}(M_w)$ and w belongs to $\mathcal{M}(M_\zeta)$.*

Proof. (1) It is clear by Theorem 1.

(2) If $\zeta = \zeta(z)$ then we can write $\zeta = q(z)h(z)$ where q is inner and h is outer. Then $\mathcal{M}(M_\zeta) = \mathcal{M}(M_q)$ because h is outer. By (1) $w \in \mathcal{M}(M_q)$, and so by Proposition 1 $V_w^* V_q = V_q V_w^*$ and q belongs to $\mathcal{M}(M_w)$. □

3. One variable function and $\mathcal{M}(M_z)$

In this section, we study $\mathcal{M}(M_z)$ which contains nonconstant one variable functions in some sense. Corollary 2 is known in [3].

Theorem 2. *Let M be a nonzero invariant subspace.*

(1) *$\mathcal{M}(M_z)$ does not contain any nonconstant function f with $f = f(z)$.*

(2) *If $\mathcal{M}(M_z)$ contains a nonconstant function f with $f = f(w)$ then $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$.*

Proof. (1) Suppose f is a nonconstant function in $\mathcal{M}(M_z) \cap L^\infty(\Gamma_z)$. If $g \in M_z$ then $|g|^2 \perp zk\overline{f^\ell}$ for $k \in H^\infty(\Gamma_z)$ and $\ell \geq 0$. Hence for any $\ell, t \geq 0$ and any $k, h \in H^\infty(\Gamma_z)$, $\bar{z}|g|^2$ is orthogonal to $k\overline{f^\ell} + h\overline{f^t}$ and $(k\overline{f^\ell})(h\overline{f^t})$. Therefore $\bar{z}|g|^2$ is orthogonal to the weak $*$ closed algebra generated by $H^\infty(\Gamma_z)$ and $\bar{f}$. Wermer's maximality theorem in [2] shows such an algebra is just $L^\infty(\Gamma_z)$. Thus $\bar{z}|g|^2$ is orthogonal to $L^\infty(\Gamma_z)$ and so $g \equiv 0$. This contradiction shows (1).

(2) By Corollary 1, $\mathcal{M}(M_z) \cap H^\infty(\Gamma_w) = H^\infty(\Gamma_w)$. By [4, Theorem 5] $M = QH^2$ for some inner Q and so $M_z = QH^2(\Gamma_w)$. Thus $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$. □

Corollary 2. *Let $\zeta = \zeta(z)$ be in $H^\infty(\Gamma_z)$. If $\mathcal{M}(M_\zeta) \cap H^\infty(\Gamma_w) \neq \mathbb{C}$ then $\mathcal{M}(M_\zeta) = H^\infty(\Gamma_w)$.*

Proof. This is a result of Corollary 1 and (2) of Theorem 2. □

Corollary 3. *Let M be a nonzero invariant subspace. If $\phi(z,w) = \phi_1(z,w)\phi_2(w)$ and ϕ_2 is nonconstant, ϕ_1 is inner and ϕ is in $\mathcal{M}(M_z)$ then $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$.*

Proof. If $f \in M_z$ then $\phi_1\phi_2 f \in M_z$ and so $\phi_1\phi_2 f \perp z\phi_1 M$. Hence $\phi_2 f \in M_z$ and so ϕ_2 belongs to $\mathcal{M}(M_z)$. (2) of Theorem 2 implies the corollary. □

Theorem 3. *Let M be a nonzero invariant subspace. If $\bar{w}$ is contained in the weak $*$ closed subalgebra generated by the complex conjugate of $\mathcal{M}(M_z)$ and H^∞, then $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$.*

Proof. If $k \in M_z$ and $f \in \mathcal{M}(M_z)$ then $f^\ell k \in M_z$ for any $\ell \geq 0$. Hence $|k|^2$ is orthogonal to $\bar{f}^\ell z H^\infty$ for $\ell \geq 0$. By the hypothesis, $\bar{w}^t$ can be approximated by $\sum_{t,\ell \geq 0} \bar{f}_t^\ell g_{\ell t}$ where $f_t \in \mathcal{M}(M_z)$ and $g_{\ell t} \in H^\infty$. Therefore $|k|^2$ is orthogonal to $\bar{w}^t z H^\infty$ for $t \geq 0$. Hence $|k(z,w)|^2 = u(w)$ for some $u \in L^1(\Gamma_w)$ and so there exists an outer function $h_1 = h_1(w)$ in $H^2(\Gamma_w)$ such that $|k(z,w)|^2 = |h_1(w)|^2$. Since $\phi k \in \mathcal{M}_z$ for any $\phi \in \mathcal{M}(M_z)$, by the proof above $|\phi(z,w)k(z,w)|^2 = |h_2(w)|$ for some outer function $h_2 = h_2(w)$. Hence $k(z,w) = q_1(z,w)h_1(w)$ and $\phi(z,w)k(z,w) = q_2(z,w)h_2(w)$ where q_j is inner in H^∞ $(j = 1,2)$. Therefore $\phi = q_2h_2/q_1h_1$. If $q = q_2\bar{q}_1$, $h = h_2/h_1$ and $\phi = qh$ then q is inner and h is outer in $H^2(\Gamma_w)$. Thus $qh \in \mathcal{M}(M_z)$ and so $h \in \mathcal{M}(M_z)$. If h is nonconstant, by (2) of Theorem 2 $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$. Now we may assume that any nonzero functions in $\mathcal{M}(M_z)$ are scalar multiples of inner functions. Thus $\mathcal{M}(M_\zeta) = \langle q \rangle$. Since $\mathcal{M}(M_\zeta)$ is an algebra, this shows $\mathcal{M}(M_\zeta) = \mathbb{C}$. □

Corollary 4. *Let M be a nonzero invariant subspace. If $\phi(z,w) = f(zw)$ and f is nonconstant in $H^\infty(\Gamma)$, and ϕ is in $\mathcal{M}(M_z)$ then $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$.*

Proof. The weak * closed algebra $[\overline{f(zw)}, H^\infty]$ generated by $\overline{f(zw)}$ and H^∞ contains $[\overline{f(zw)}, H^\infty(\Gamma_{zw})]$. Since $H^\infty(\Gamma_{zw})$ is maximal in $L^\infty(\Gamma_{zw})$ as a weak star closed subalgebra by [2], $[\overline{f(zw)}, H^\infty(\Gamma_{zw})]$ contains $\overline{zw}$. Hence $[\bar{f}, H^\infty]$ contains $\bar{w}$. Now Theorem 2 shows the corollary. □

4. K_z and $\mathcal{M}(M_z)$

If $K_z = M_z$ then $\mathcal{M}(M_z)$ contains w and so (2) of Theorem 2 shows $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$. Hence we are interested in when $K_z \neq M_z$.

Theorem 4. *Let M be a nonzero invariant subspace in H^2. If $K_z \neq \{0\}$ and $\mathcal{M}(M_z) \neq \mathbb{C}$ then $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$.*

Proof. Since $K_z \neq \{0\}$, there exists $f \in M_z$ such that $w^\ell f$ belongs to M_z for $\ell \in Z_+$. Then $w^\ell f \perp f w^m z^t$ for $m \in Z_+$ and $t \in Z_+ \setminus \{0\}$. Thus

$$\int |f|^2 w^s z^t dm = 0 \quad (s \in Z, t \in Z_+ \setminus \{0\}).$$

Therefore $F = |f|^2 \in L^1(\Gamma_w)$ and $\log F \in L^1(\Gamma_w)$. Hence $F = |h|^2$ for some outer $h \in H^2(\Gamma_w)$. Then $f = qh$ and q is an inner function in H^∞.

Since $w^\ell(qh) \in M_z$, $qH^2(\Gamma_w) \subset M_z$. Let ϕ be in $\mathcal{M}(M_z)$. Then $\phi q H^2(\Gamma_w)$ is orthogonal to $z^t \phi q H^2(\Gamma_w)$ for $t \in Z_+ \setminus \{0\}$. Therefore $|\phi|^2 \perp z^t L^1(\Gamma_w)$ for $t \in Z_+ \setminus \{0\}$ and so $|\phi|^2 \in L^\infty(\Gamma_w)$. There exists an outer function k in $H^\infty(\Gamma_w)$ such that $\phi = Qk$ and Q is an inner function in H^∞. By Corollary 3 $k = k(w)$ belongs to $\mathcal{M}(M_z)$ and so (2) of Theorem 2 shows $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$. □

5. Intersection of $\mathcal{M}(M_z)$ and $\mathcal{M}(M_w)$

If $M = qH^2$ and $q = q(z,w)$ is inner, then $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$ and $\mathcal{M}(M_w) = H^\infty(\Gamma_z)$. Hence $\mathcal{M}(M_z) \cap \mathcal{M}(M_w) = \mathbb{C}$. If $M = q_1 H^2 + q_2 H^2$, and $q_1 = q_1(z)$ and

$q_2 = q_2(w)$ are inner, then $\mathcal{M}(M_z) = \mathbb{C}$ and $\mathcal{M}(M_w) = \mathbb{C}$ by [4, Example 3]. Hence $\mathcal{M}(M_z) \cap \mathcal{M}(M_w) = \mathbb{C}$.

Lemma 1. *Suppose M is a nonzero invariant subspace in H^2. If M is orthogonal to an invariant subspace M' in H^2 then $M' = \{0\}$.*

Proof. It is easy to see. □

Theorem 5. *Let M be a nonzero invariant subspace. If ϕ is a nonzero function in $\mathcal{M}(M_z) \cap \mathcal{M}(M_w)$ then $[\phi M] = M$. Hence $[\phi M_z] = M_z$ and $[\phi M_w] = M_w$.*

Proof. If $\phi \in \mathcal{M}(M_z)$ then by Proposition 1 $V_\phi V_z^* = V_z^* V_\phi$ and so $V_\phi^* V_z = V_z V_\phi^*$. This shows $zM_\phi \subseteq M_\phi$. Similarly $\phi \in \mathcal{M}(M_w)$ shows $wM_\phi \subseteq M_\phi$. Hence M_ϕ and $[\phi M]$ are invariant subspaces in H^2, and M_ϕ is orthogonal to $[\phi M]$. Therefore Lemma 1 shows $[\phi M] = M$. Since $\phi M_z \subset M_z$ and $\phi zM \subset zM$, $\phi M = \phi M_z \oplus \phi zM$. This shows $[\phi M_z] = M_z$ because $[\phi M] = M$ and so $[\phi zM] = zM$. □

Corollary 5. *Suppose ϕ is a nonzero function in $\mathcal{M}(M_z) \cap \mathcal{M}(M_w)$. If ϕ has an inner factor then its part is constant.*

Proof. Since $\phi \neq 0$, by Theorem 5 $[\phi M] = M$. This shows the corollary. □

Theorem 6. *Let M be a nonzero invariant subspace. If $M_z \cap H^2(\Gamma_w) \neq \{0\}$ and $M_w \cap H^2(\Gamma_z) \neq \{0\}$ then $\mathcal{M}(M_z) \cap \mathcal{M}(M_w) = \mathbb{C}$.*

Proof. If f is a nonzero function in $\mathcal{M}_z \cap H^2(\Gamma_w)$ then for any $n \geq 0$ $w^n f \in M$ and $w^n f \perp zM$ because $\bar{z} w^n f \perp H^2$. Hence $w^n f \in M_z$ for any $n \geq 0$ and so $K_z \neq \{0\}$. If ϕ is a nonconstant function in $\mathcal{M}(M_z)$ then by Theorem 4 $\mathcal{M}(M_z) = H^\infty(\Gamma_w)$. Similarly if f is a nonzero function in $M_w \cap H^2(\Gamma_z)$ and ϕ is a nonconstant function in $\mathcal{M}(M_w)$ then $\mathcal{M}(M_w) = H^\infty(\Gamma_z)$. Thus $\mathcal{M}(M_z) \cap \mathcal{M}(M_w) = \mathbb{C}$. □

If M is of finite codimension in H^2 then $\mathcal{M}(M_z) \cap \mathcal{M}(M_w) = \mathbb{C}$. This is a corollary of Theorem 6. In fact, if M is of finite codimension then by [4, (3) of Theorem 6] $M \supseteq q_z H^2 + q_w H^2$ where q_z and q_z are one variable inner functions. Since $zM \perp q_w H^2(\Gamma_w)$, M satisfies the condition in Theorem 6. When M is a nonzero invariant subspace and $M' = FM$ where F is a unimodular function in L^∞, it is easy to see $\mathcal{M}(M_z) = \mathcal{M}(M'_z)$ and $\mathcal{M}(M_w) = \mathcal{M}(M'_w)$. Therefore Theorem 6 can be applied to a lot of examples.

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