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GRAM MATRICES OF REPRODUCING KERNEL HILBERT SPACES OVER GRAPHS II (GRAPH HOMOMORPHISMS AND DE BRANGES-ROVNYAK SPACES)

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ABSTRACT. We study graph homomorphisms over finite graphs from a viewpoint of reproducing kernel Hilbert space theory. In particular, introducing de Branges-Rovnyak theory into graph theory, the relation between injective graph homomorphisms and de Branges-Rovnyak spaces is discussed in detail.

1. Introduction

The purpose of this paper is to give a reproducing kernel Hilbert space framework dealing with graph homomorphisms as a sequel of [4]. First of all, we shall introduce our idea. Let G be a graph. All graphs appearing in this paper are assumed to be finite, non-directed and have neither loops nor multi-edges. The vertex set of G will be denoted by $V = V(G)$, the edge set by $E = E(G)$ and the adjacency matrix by $A = (A^G_{xy})_{x,y\in V}$.

Definition 1.1. Let G_1 and G_2 be graphs. A map φ from $V_1 = V(G_1)$ into $V_2 = V(G_2)$ is called a homomorphism of G_1 into G_2 if $A^{G_1}_{x_1y_1} \le A^{G_2}_{\varphi(x_1)\varphi(y_1)}$ for any x_1, y_1 in V_1. Further, G_1 and G_2 are said to be isomorphic if there exists a bijective map φ between V_1 and V_2 which preserves adjacency, that is, both φ and φ^{-1} are homomorphisms.

We shall explain correspondences between some problems on complex analysis and analysis on graphs. Let φ be a homomorphism from G_1 into G_2. Graphs will be identified with open sets in the complex plane. Then the inequality $A^{G_1}_{x_1y_1} \le A^{G_2}_{\varphi(x_1)\varphi(y_1)}$ $(x_1, y_1 \in V_1)$ can be seen as a discrete analogue of a fundamental principal in complex analysis that holomorphic maps preserve regions. Now, in complex

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analysis, there was once a famous problem called the Bieberbach conjecture. It was solved completely by L. de Branges in 1984[1]. Some of ingredients of his original proof are composition operators induced by injective holomorphic maps, generalized Dirichlet integrals and the theory developed by him and his collaborator J. Rovnyak in [2]. In graph theory, composition operators induced by graph homomorphisms can be defined easily, and Dirichlet integrals on graphs have already been introduced by many researchers. Therefore, under our identification, it is reasonable to expect that there would exist some interplay between graph theory and de Branges-Rovnyak theory. This is our basic idea.

This paper is divided into four sections. Section 1 is the introduction. In Section 2, we deal with Hilbert spaces constructed from adjacency matrices of graphs, which will be denoted by $\mathcal{H}_G$, and give some general properties of the composition operator C_φ induced by a homomorphism $\varphi : G_1 \to G_2$. In Section 3, we introduce de Branges-Rovnyak space $\mathcal{M}$ induced by the adjoint of C_φ. $\mathcal{M}$ is a Hilbert space consisting of vectors in the range of that operator with the inner product defined by the pullback operation. A certain condition that two graphs are isomorphic is given with the language of de Branges-Rovnyak theory. In Section 4, we study relations between those spaces and the growth of vertices and edges by an injective homomorphism.

2. Dirichlet spaces over graphs

Let G be a graph. Then $\mathcal{E}(\cdot,\cdot)$ will denote the discrete Dirichlet form on V defined as follows:

$$\mathcal{E}(u,v) = \frac{1}{2} \sum_{x,y \in V(G)} A^G_{xy}(u(x)-u(y))(v(x)-v(y)),$$

where u and v are real valued functions on V. Let δ_x denote the delta function at x, $\deg(x)$ denote the number of edges connected at x.

Lemma 2.1. *For any x and y in V,*

(i) $\mathcal{E}(\delta_x, \delta_y) = \begin{cases} \deg(x) & (x = y) \\ -A_{xy} & (x \neq y). \end{cases}$

(ii) *Let φ be a map from V_1 to V_2. Then*

$$\mathcal{E}(\delta_x \circ \varphi, \delta_x \circ \varphi) = |\{(w,z) \in (\varphi^{-1}(x) \times (\varphi^{-1}(x))^c) : \{w,z\} \in E\}|.$$

Proof. It is easy to see (i). We shall show (ii). Setting

$$I = \{(w,z) \in \varphi^{-1}(x) \times (\varphi^{-1}(x))^c : \{w,z\} \in E\},$$

[1]For its interesting history, see *"The Bieberbach Conjecture-Proceedings of the Symposium on the Occasion of the Proof"*, Math. Surveys Monogr., 21, Amer. Math. Soc., Providence, RI, 1986.

we have that

$$\begin{aligned}\mathcal{E}(\delta_x \circ \varphi, \delta_x \circ \varphi) &= \frac{1}{2}\sum_{w,z\in V} A_{wz}\{\delta_x(\varphi(w)) - \delta_x(\varphi(z))\}^2 \\ &= \sum_{\{w,z\}\in E} \{\delta_x(\varphi(w)) - \delta_x(\varphi(z))\}^2 \\ &= \sum_{(w,z)\in I} \{\delta_x(\varphi(w)) - \delta_x(\varphi(z))\}^2 \\ &= |I|.\end{aligned}$$

This concludes the proof. □

Let $\mathcal{H}_G$ denote the Hilbert space consisting of real valued functions on V with the following Sobolev norm:

$$\|u\|^2_{\mathcal{H}_G} = \|u\|^2_{\ell^2} + \mathcal{E}(u,u),$$

where $\|u\|_{\ell^2} = (\sum_{x\in V}|u(x)|^2)^{1/2}$. Then, since $\mathcal{H}_G$ is of finite dimension, $\mathcal{H}_G$ is a reproducing kernel Hilbert space. For every x in V, the reproducing kernel of $\mathcal{H}_G$ at x will be denoted by k^G_x, that is, k^G_x is the unique vector in $\mathcal{H}_G$ such that $\langle f, k^G_x\rangle_{\mathcal{H}_G} = f(x)$ for any f in $\mathcal{H}_G$. Let G_1 and G_2 be graphs, and let φ be a homomorphism from G_1 into G_2. For each function u in $\mathcal{H}_{G_2}$, $C_\varphi u = u \circ \varphi$ defines a linear operator C_φ from $\mathcal{H}_{G_2}$ into $\mathcal{H}_{G_1}$. We set $N_\varphi = \max_{x_2\in V_2}|\varphi^{-1}(x_2)|$.

Theorem 2.1. $\|C_\varphi u\|_{\mathcal{H}_{G_1}} \le N_\varphi \|u\|_{\mathcal{H}_{G_2}}$.

Proof. For any u in $\mathcal{H}_{G_2}$, we have that

$$\begin{aligned}\mathcal{E}_1(C_\varphi u, C_\varphi u) &= \mathcal{E}_1(u\circ\varphi, u\circ\varphi) \\ &= \frac{1}{2}\sum_{x_1,y_1\in V_1} A_{x_1y_1}|u\circ\varphi(x_1) - u\circ\varphi(y_1)|^2 \\ &\le \frac{1}{2}\sum_{x_1,y_1\in V_1} A_{\varphi(x_1)\varphi(y_1)}|u\circ\varphi(x_1) - u\circ\varphi(y_1)|^2 \\ &= \frac{1}{2}\sum_{x_2,y_2\in\varphi(V_1)} A_{x_2y_2}|u(x_2)-u(y_2)|^2|\varphi^{-1}(x_2)||\varphi^{-1}(y_2)| \\ &\le \frac{N_\varphi^2}{2}\sum_{x_2,y_2\in V_2} A_{x_2y_2}|u(x_2)-u(y_2)|^2 \\ &= N_\varphi^2\mathcal{E}_2(u,u)\end{aligned}$$

and

$$\sum_{x_1\in V_1}|u\circ\varphi(x_1)|^2 = \sum_{x_2\in\varphi(V_1)}|u(x_2)|^2|\varphi^{-1}(x_2)| \le N_\varphi\sum_{x_2\in V_2}|u(x_2)|^2.$$

These inequalities conclude the proof. □

We set $T = C_\varphi^*/N_\varphi$, where we deal with C_φ as an operator from $\mathcal{H}_{G_2}$ into $\mathcal{H}_{G_1}$ Then T is a linear operator from $\mathcal{H}_{G_1}$ into $\mathcal{H}_{G_2}$, $\|T\| \leq 1$ by Theorem 2.1, and it is easy to see that $Tk_{x_1}^{G_1} = k_{\varphi(x_1)}^{G_2}/N_\varphi$ for every x_1 in V_1.

Theorem 2.2. *T is an onto isometry if and only if φ is an isomorphism.*

Proof. The if part is trivial. We shall show the only if part. First, by (ii) in Lemma 2.1, we have the following:

$$\begin{aligned}\|T^*\delta_{x_2}\|_{\mathcal{H}_{G_1}}^2 &= \frac{|\varphi^{-1}(x_2)| + |\ (w,z) \in \varphi^{-1}(x_2) \times (\varphi^{-1}(x_2))^c : \{w,z\} \in E_1\}|}{N_\varphi^2} \\ &\leq \frac{|\varphi^{-1}(x_2)|(1 + \deg_{G_2}(x_2))}{N_\varphi^2}. \qquad (2.1)\end{aligned}$$

Suppose that T is an onto isometry. Then we have that $|V_1| = |V_2|$ and $\|T^*\delta_{x_2}\|_{\mathcal{H}_{G_1}}^2 = \|\delta_{x_2}\|_{\mathcal{H}_{G_2}}^2$. It follows from (2.1) that

$$1 + \deg_{G_2}(x_2) = \|\delta_{x_2}\|_{\mathcal{H}_{G_2}}^2 = \|T^*\delta_{x_2}\|_{\mathcal{H}_{G_1}}^2 \leq \frac{|\varphi^{-1}(x_2)|}{N_\varphi^2}(1 + \deg_{G_2}(x_2)),$$

and which implies that $|\varphi^{-1}(x_2)| = 1$ for any x_2 in V_2, that is, φ is injective. Since $|V_1| = |V_2|$, φ is bijective. Furthermore, by (i) in Lemma 2.1, if $x_2 \neq y_2$ then we have that

$$\begin{aligned}-A_{x_2y_2}^{G_2} &= \langle \delta_{x_2}, \delta_{y_2} \rangle_{\mathcal{H}_{G_2}} \\ &= \langle T^*\delta_{x_2}, T^*\delta_{y_2} \rangle_{\mathcal{H}_{G_1}} \\ &= \langle \delta_{\varphi^{-1}(x_2)}, \delta_{\varphi^{-1}(y_2)} \rangle_{\mathcal{H}_{G_1}} \\ &= -A_{\varphi^{-1}(x_2)\varphi^{-1}(y_2)}^{G_1},\end{aligned}$$

that is, φ is an isomorphism. This concludes the proof. □

3. de Branges-Rovnyak spaces over graphs

In this section, we shall introduce the theory developed by de Branges and Rovnyak. This theory is well known to experts in Hilbert space operator theory. Standard references will be Ando [1], de Branges-Rovnyak [2], Sarason [3] and Vasyunin-Nikol'skiĭ [5]. We will refer to [3] for several results which we need in this paper.

Let $P_{(\ker T)^\perp}$ and $P_{(\ker T^*)^\perp}$ denote the orthogonal projections onto the orthogonal complements of $\ker T$ and $\ker T^*$ in $\mathcal{H}_{G_1}$ and $\mathcal{H}_{G_2}$, respectively. Now, we introduce new inner products on linear spaces $T\mathcal{H}_{G_1}$ and $T^*\mathcal{H}_{G_2}$ defined as follows:

$$\begin{aligned}\langle Tu_1, Tv_1 \rangle_T &= \langle P_{(\ker T)^\perp}u_1, P_{(\ker T)^\perp}v_1 \rangle_{\mathcal{H}_{G_1}} \quad (u_1, v_1 \in \mathcal{H}_{G_1}), \\ \langle T^*u_2, T^*v_2 \rangle_{T^*} &= \langle P_{(\ker T^*)^\perp}u_2, P_{(\ker T^*)^\perp}v_2 \rangle_{\mathcal{H}_{G_2}} \quad (u_2, v_2 \in \mathcal{H}_{G_2}).\end{aligned}$$

We are interested in Hilbert spaces $\mathcal{M}(T) = (T\mathcal{H}_{G_1}, \|\cdot\|_T)$ and $\mathcal{M}(T^*) = (T^*\mathcal{H}_{G_2}, \|\cdot\|_{T^*})$ rather than $T\mathcal{H}_{G_1}$ and $T^*\mathcal{H}_{G_2}$ as usual Hilbert subspaces.

It is easy to see that $\mathcal{M}(T) = \mathcal{H}_{G_2}$ as Hilbert spaces if and only if T is an onto isometry, that is, φ is an isomorphism by Theorem 2.2. Since $\|T\| \leq 1$, we have the following quasi-orthogonal decomposition of $\mathcal{H}_{G_1}$ and $\mathcal{H}_{G_2}$ by (I-12) in [3]:

$$\mathcal{H}_{G_2} = \mathcal{M}(T) + \mathcal{H}(T), \tag{3.1}$$

$$\mathcal{H}_{G_1} = \mathcal{M}(T^*) + \mathcal{H}(T^*), \tag{3.2}$$

where $\mathcal{H}(T) = \mathcal{M}(\sqrt{I_{\mathcal{H}_{G_2}} - TT^*})$ (resp. $\mathcal{H}(T^*) = \mathcal{M}(\sqrt{I_{\mathcal{H}_{G_1}} - T^*T})$), and will be called the de Branges-Rovnyak complement of $\mathcal{M}(T)$ (resp. $\mathcal{M}(T^*)$).

Remark 3.1. In our framework, injective homomorphisms will be essential. Because, first, by the construction of $\mathcal{M}(T)$, when φ is injective, the inner product of $\mathcal{M}(T)$ is inherited from that of $\mathcal{H}_{G_1}$, secondly, the structure of $\mathcal{H}_G$ is equivalent to that of G in general. Hence, it can be expected that the data of $\varphi : G_1 \to G_2$ will be encoded into the contractive embedding $\mathcal{M}(T) \hookrightarrow \mathcal{H}_{G_2}$. Then de Branges-Rovnyak complements will replace not only as orthogonal complements but also as quotient spaces.

We note that $\mathcal{M}(T) = \mathcal{M}(|T^*|)$ and $\mathcal{M}(T^*) = \mathcal{M}(|T|)$ by (ii) of (I-5) in [3]. In general, the intersection of $\mathcal{M}(T)$ and $\mathcal{H}(T)$, which is called the overlapping space with respect to T, is non-trivial. In fact, by (I-9) in [3], TT^* (resp. T^*T) is an orthogonal projection if and only if (3.1) (resp. (3.2)) is the usual orthogonal direct sum. By the formula in (I-3) in [3], $\mathcal{M}(T)$, $\mathcal{H}(T)$, $\mathcal{M}(T^*)$ and $\mathcal{H}(T^*)$ are reproducing kernel Hilbert spaces, and their reproducing kernels are

$$TT^*k_{x_2}^{G_2}, \quad (I_{\mathcal{H}_{G_2}} - TT^*)k_{x_2}^{G_2}, \quad T^*Tk_{x_1}^{G_1} \quad \text{and} \quad (I_{\mathcal{H}_{G_1}} - T^*T)k_{x_1}^{G_1},$$

respectively. Then, it is easy to see that

$$\langle TT^*k_{x_2}^{G_2}, TT^*k_{y_2}^{G_2}\rangle_{\mathcal{M}(T)} = \langle T^*k_{x_2}^{G_2}, T^*k_{y_2}^{G_2}\rangle_{\mathcal{H}_{G_1}} = \frac{1}{N_\varphi^2}\langle k_{x_2}^{G_2} \circ \varphi, k_{y_2}^{G_2} \circ \varphi\rangle_{\mathcal{H}_{G_1}}$$

and

$$\langle T^*Tk_{x_1}^{G_1}, T^*Tk_{y_1}^{G_1}\rangle_{\mathcal{M}(T^*)} = \langle Tk_{x_1}^{G_1}, Tk_{y_1}^{G_1}\rangle_{\mathcal{H}_{G_2}} = \frac{1}{N_\varphi^2}\langle k_{\varphi(x_1)}^{G_2}, k_{\varphi(y_1)}^{G_2}\rangle_{\mathcal{H}_{G_2}}.$$

In general, those reproducing kernels might not be linearly independent. Two matrices

$$K(\mathcal{M}(T)) = (\langle TT^*k_{x_2}^{G_2}, TT^*k_{y_2}^{G_2}\rangle_{\mathcal{M}(T)})_{x_2, y_2 \in V_2}$$

and

$$K(\mathcal{H}(T)) = (\langle (I - TT^*)k_{x_2}^{G_2}, (I - TT^*)k_{y_2}^{G_2}\rangle_{\mathcal{H}(T)})_{x_2, y_2 \in V_2}$$

will be called Gram matrices of $\mathcal{M}(T)$ and $\mathcal{H}(T)$, respectively. Since entries of those matrices are values of corresponding reproducing kernels, Gram matrices are essentially equal to reproducing kernels. Similarly, Gram matrices $K(\mathcal{M}(T^*))$ and $K(\mathcal{H}(T^*))$ are defined.

Let H_1 and H_2 be graphs, and let ψ be a homomorphism from H_1 into H_2. We set $S = C_\psi^*/N_\psi$.

Definition 3.1. T is said to be compatible with S if there exists a bijective map Ψ from $V(G_2)$ onto $V(H_2)$ such that the following three conditions hold:

(i) $\langle TT^*k_{x_2}^{G_2}, TT^*k_{y_2}^{G_2}\rangle_{\mathcal{M}(T)} = \langle SS^*k_{\Psi(x_2)}^{H_2}, SS^*k_{\Psi(y_2)}^{H_2}\rangle_{\mathcal{M}(S)}$,

(ii) $\langle (I-TT^*)k_{x_2}^{G_2}, (I-TT^*)k_{y_2}^{G_2}\rangle_{\mathcal{H}(T)} = \langle (I-SS^*)k_{\Psi(x_2)}^{H_2}, (I-SS^*)k_{\Psi(y_2)}^{H_2}\rangle_{\mathcal{H}(S)}$,

(iii) the following two linear relations are mutually equivalent:

$$\sum_{x_2\in V_2} a_{x_2}TT^*k_{x_2}^{G_2} = \sum_{x_2\in V_2} b_{x_2}(I-TT^*)k_{x_2}^{G_2},$$

$$\sum_{x_2\in V_2} a_{x_2}SS^*k_{\Psi(x_2)}^{H_2} = \sum_{x_2\in V_2} b_{x_2}(I-SS^*)k_{\Psi(x_2)}^{H_2}.$$

In other words, T and S are said to be compatible if there exists a bijective map Ψ from $V(G_2)$ onto $V(H_2)$ such that the following three conditions hold:

(i) $K(\mathcal{M}(T)) \cong K(\mathcal{M}(S))$ up to the permutation induced by Ψ,

(ii) $K(\mathcal{H}(T)) \cong K(\mathcal{H}(S))$ up to the permutation induced by Ψ,

(iii) $K(\mathcal{M}(T))\mathbf{a} = K(\mathcal{H}(T))\mathbf{b}$ if and only if $K(\mathcal{M}(S))\mathbf{a} = K(\mathcal{H}(S))\mathbf{b}$ under the identification in (i) and (ii), where $\mathbf{a}$ and $\mathbf{b}$ denote vectors in $\mathbb{R}^{|V(G_2)|}$.

Similarly, the compatibility of T^* and S^* is defined.

Theorem 3.1. *If there exist isomorphisms Φ and Ψ such that the following diagram commutes:*

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Psi} \\ H_1 & \xrightarrow[\psi]{} & H_2, \end{array}$$

then T and T^* are compatible with S and S^*, respectively.

Proof. We set $U_1 = C_\Phi^*$ and $U_2 = C_\Psi^*$. Since the Sobolev norm is invariant under isomorphisms, U_1 and U_2 are onto isometries such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_{G_1} & \xrightarrow{T} & \mathcal{H}_{G_2} \\ {\scriptstyle U_1}\big\downarrow & & \big\downarrow{\scriptstyle U_2} \\ \mathcal{H}_{H_1} & \xrightarrow[S]{} & \mathcal{H}_{H_2}. \end{array}$$

Then, trivially, we have that $SS^* = U_2TT^*U_2^*$ and $S^*S = U_1T^*TU_1^*$. Further, by (ii) of (I-5) in [3], it suffices to show the statement for $\mathcal{M}(|T^*|)$ and $\mathcal{M}(|S^*|)$. We shall see that

(i) $C_\Psi^* \oplus C_\Psi^*$ is an isometry from $\mathcal{M}(|T^*|) \oplus \mathcal{H}(|T^*|)$ onto $\mathcal{M}(|S^*|) \oplus \mathcal{H}(|S^*|)$,
(ii) $C_\Psi^* TT^* k_x^{G_2} = SS^* k_{\Psi(x)}^{H_2}$,
(iii) $C_\Psi^* (I - TT^*) k_x^{G_2} = (I - SS^*) k_{\Psi(x)}^{H_2}$,
(iv) the following two linear relations are mutually equivalent:

$$\sum_{x_2 \in V_2} a_{x_2} TT^* k_{x_2}^{G_2} = \sum_{x_2 \in V_2} b_{x_2} (I - TT^*) k_{x_2}^{G_2},$$

$$\sum_{x_2 \in V_2} a_{x_2} SS^* k_{\Psi(x_2)}^{H_2} = \sum_{x_2 \in V_2} b_{x_2} (I - SS^*) k_{\Psi(x_2)}^{H_2}.$$

First, it is trivial that $U_2 \mathcal{M}(|T^*|) = \mathcal{M}(|S^*|)$ as linear spaces by $SS^* = U_2 TT^* U_2^*$. Furthermore, for any u_2 in $\mathcal{H}_{G_2}$, we have that

$$\begin{aligned} \|U_2 |T^*| u_2\|_{\mathcal{M}(|S^*|)} &= \||S^*| U_2 u_2\|_{\mathcal{M}(|S^*|)} \\ &= \|P_{(\ker |S^*|)^\perp} U_2 u_2\|_{\mathcal{H}_{H_2}} \\ &= \|P_{(\ker U_2 |T^*| U_2^*)^\perp} U_2 u_2\|_{\mathcal{H}_{H_2}} \\ &= \|U_2 P_{(\ker |T^*|)^\perp} U_2^* U_2 u_2\|_{\mathcal{H}_{H_2}} \\ &= \|P_{(\ker |T^*|)^\perp} u_2\|_{\mathcal{H}_{G_2}} \\ &= \||T^*| u_2\|_{\mathcal{M}(|T^*|)}. \end{aligned}$$

Hence $\mathcal{M}(|T^*|)$ is isomorphic to $\mathcal{M}(|S^*|)$. Similarly, it is shown that $\mathcal{H}(|T^*|)$ is isomorphic to $\mathcal{H}(|S^*|)$ by U_2. Thus we have (i). Since $U_2 TT^* = SS^* U_2$, we have that

$$U_2 TT^* k_x^{G_2} = SS^* U_2 k_x^{G_2} = SS^* k_{\Psi(x)}^{H_2}.$$

This concludes (ii) and (iii). It is easy to see that (iv) follows from (ii) and (iii). $\square$

Next, we shall show the following:

Theorem 3.2. *Let $\varphi : G_1 \to G_2$ and $\psi : H_1 \to H_2$ be homomorphisms. Then*
(i) *G_2 and H_2 are isomorphic if T and S are compatible,*
(ii) *G_1 and H_1 are isomorphic if T^* and S^* are compatible.*

In order to prove this theorem, we need some lemmas.

Lemma 3.1. *If T is compatible with S then*

$$U_1 : \mathcal{M}(T) \to \mathcal{M}(S), \quad TT^* k_x^{G_2} \mapsto SS^* k_{\Psi(x)}^{H_2}$$

and

$$U_2 : \mathcal{H}(T) \to \mathcal{H}(S), \quad (I - TT^*) k_x^{G_2} \mapsto (I - SS^*) k_{\Psi(x)}^{H_2}$$

are isometries.

Proof. Since

$$\|\sum_{x_2\in V_2} c_{x_2} TT^* k^{G_2}_{x_2}\|^2_{\mathcal{M}(T)} = \|\sum_{x_2\in V_2} c_{x_2} SS^* k^{G_2}_{\Psi(x_2)}\|^2_{\mathcal{M}(S)}$$

by (i) in Definitoin 3.1, U_1 is well defined and isometric. □

We set $\mathbb{U} = U_1 \oplus U_2$ for U_1 and U_2 in Lemma 3.1. Then $\mathbb{U}$ is an onto isometry from $\mathcal{M}(|T^*|) \oplus \mathcal{H}(|T^*|)$ onto $\mathcal{M}(|S^*|) \oplus \mathcal{H}(|S^*|)$ if T is compatible with S by Lemma 3.1. Further, we set

$$\mathbb{T} : \mathcal{M}(|T^*|) \oplus \mathcal{H}(|T^*|) \to \mathcal{H}_{G_2}, \quad u \oplus v \mapsto u + v,$$
$$\mathbb{S} : \mathcal{M}(|S^*|) \oplus \mathcal{H}(|S^*|) \to \mathcal{H}_{H_2}, \quad u \oplus v \mapsto u + v.$$

Lemma 3.2. *$\mathcal{M}(\mathbb{T}) = \mathcal{H}_{G_2}$ as Hilbert spaces.*

Proof. This proof is taken from Ando [1]. Since $x = TT^*x + (I - TT^*)x$, we have that $\mathcal{M}(\mathbb{T}) = \mathcal{H}_{G_2}$ as linear spaces. We shall show that $\|x\|_{\mathcal{M}(\mathbb{T})} = \|x\|_{\mathcal{H}_{G_2}}$ for any x in $\mathcal{H}_{G_2}$. First, since

$$\|x\|^2_{\mathcal{M}(\mathbb{T})} = \|\mathbb{T}(TT^*x, (I-TT^*)x)\|_{\mathcal{M}(\mathbb{T})} \le \|TT^*x\|^2_{\mathcal{M}(T)} + \|(I-TT^*)x\|^2_{\mathcal{H}(T)} = \|x\|^2_{\mathcal{H}_{G_2}},$$

we have that $\|x\|_{\mathcal{M}(\mathbb{T})} \le \|x\|_{\mathcal{H}_{G_2}}$. Next, let $x = \mathbb{T}(Ta_1, (I - TT^*)^{1/2}a_2)$ where $(Ta_1, (I-TT^*)^{1/2}a_2)$ be in $(\ker \mathbb{T})^\perp$, a_1 be in $(\ker T)^\perp$ and a_2 be in $\ker((I-TT^*)^{1/2})^\perp$. Then we have that

$$\begin{aligned}\|x\|^2_{\mathcal{M}(\mathbb{T})} &= \|\mathbb{T}(Ta_1, (I - TT^*)^{1/2}a_2)\|^2 \\ &= \|Ta_1\|^2_{\mathcal{M}(T)} + \|(I - TT^*)^{1/2}a_2\|^2_{\mathcal{H}(T)} \\ &= \|a_1\|^2_{\mathcal{H}_{G_1}} + \|a_2\|^2_{\mathcal{H}_{G_2}}.\end{aligned}$$

It follows from this identity that

$$\begin{aligned}\|x\|^4_{\mathcal{H}_{G_2}} &= |\langle x, x\rangle_{\mathcal{H}_{G_2}}|^2 \\ &= |\langle x, Ta_1 + (I - TT^*)^{1/2}a_2\rangle_{\mathcal{H}_{G_2}}|^2 \\ &= |\langle (T^*x, (I - TT^*)^{1/2}x), (a_1, a_2)\rangle_{\mathcal{H}_{G_1}\oplus\mathcal{H}_{G_2}}|^2 \\ &\le (\|T^*x\|^2_{\mathcal{H}_{G_1}} + \|(I - TT^*)^{1/2}x\|^2_{\mathcal{H}_{G_2}})(\|a_1\|^2_{\mathcal{H}_{G_1}} + \|a_2\|^2_{\mathcal{H}_{G_2}}) \\ &= \|x\|^2_{\mathcal{H}_{G_2}} \|x\|^2_{\mathcal{M}(\mathbb{T})}.\end{aligned}$$

Therefore we have that $\|x\|_{\mathcal{M}(\mathbb{T})} \ge \|x\|_{\mathcal{H}_{G_2}}$. This concludes the proof. □

Lemma 3.3. If T is compatible with S then $\mathbb{U} \ker \mathbb{T} = \ker \mathbb{S}$.

Proof. Let (u, v) be in $\ker \mathbb{T}$. Then we have that $(u, v) = (u, -u)$ where u belongs to $\mathcal{M}(|T^*|) \cap \mathcal{H}(|T^*|)$. Hence u can be represented as follows:

$$u = \sum_{x_2\in V_2} a_{x_2} TT^* k^{G_2}_{x_2} = \sum_{x_2\in V_2} b_{x_2} (I - TT^*) k^{G_2}_{x_2}.$$

Since T is compatible with S, we have that

$$\sum_{x_2 \in V_2} a_{x_2} SS^* k^{H_2}_{\Psi(x_2)} = \sum_{x_2 \in V_2} b_{x_2} (I - SS^*) k^{H_2}_{\Psi(x_2)}.$$

Further, by Lemma 3.1, we have that

$$\begin{aligned}&\mathbb{U}(\sum_{x_2 \in V_2} a_{x_2} TT^* k^{G_2}_{x_2}, -\sum_{x_2 \in V_2} b_{x_2} (I - TT^*) k^{G_2}_{x_2})\\ &= (\sum_{x_2 \in V_2} a_{x_2} SS^* k^{H_2}_{\Psi(x_2)}, -\sum_{x_2 \in V_2} b_{x_2} (I - SS^*) k^{H_2}_{\Psi(x_2)}).\end{aligned}$$

This concludes the inclusion $\mathbb{U} \ker \mathbb{T} \subseteq \ker \mathbb{S}$. Since $\mathbb{U}$ is an onto isometry, it is similar to see the converse inclusion. □

Proof of Theorem 3.2. It suffices to show (i) because Lemmas 3.1, 3.2 and 3.3 hold for T^* and S^*. We set $U = \mathbb{S}\mathbb{U}\mathbb{T}^{-1}$. Then U is a well-defined linear operator by Lemma 3.3, and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}(|T^*|) \oplus \mathcal{H}(|T^*|) & \xrightarrow{\mathbb{U}} & \mathcal{M}(|S^*|) \oplus \mathcal{H}(|S^*|) \\ {\scriptstyle \mathbb{T}}\downarrow & & \downarrow{\scriptstyle \mathbb{S}} \\ \mathcal{H}_{G_2} & \xrightarrow[U]{} & \mathcal{H}_{H_2}. \end{array}$$

Since $\{k^{G_2}_{x_2}\}_{x_2 \in V(G_2)}$ and $\{k^{H_2}_{\Psi(x_2)}\}_{x_2 \in V(G_2)}$ are linearly independent and $Uk^{G_2}_{x_2} = k^{H_2}_{\Psi(x_2)}$, we have that $U = C^*_\Psi$, and which is an invertible linear operator from $\mathcal{H}_{G_2}$ onto $\mathcal{H}_{H_2}$ satisfying $U\mathbb{T} = \mathbb{S}\mathbb{U}$. We shall show that U is an isometry. Let w be a function in $\mathcal{H}_{G_2}$, and let $w = u + v$ be its unique decomposition with respect to $\mathcal{M}(|T^*|)$ and $\mathcal{H}(|T^*|)$. Then (u, v) belongs to the orthogonal complement of $\ker \mathbb{T}$ in $\mathcal{M}(|T^*|) \oplus \mathcal{H}(|T^*|)$, and $\|w\|^2_{\mathcal{H}_{G_2}} = \|u\|^2_{\mathcal{M}(|T^*|)} + \|v\|^2_{\mathcal{H}(|T^*|)}$ by Lemma 3.2. Setting $\mathbb{U}(u, v) = (u', v')$, (u', v') belongs to the orthogonal complement of $\ker \mathbb{S}$ in $\mathcal{M}(|S^*|) \oplus \mathcal{H}(|S^*|)$ by Lemma 3.3 and $U(u+v) = \mathbb{S}\mathbb{U}\mathbb{T}^{-1}(u+v) = u' + v'$. Hence we have that

$$\begin{aligned}\|Uw\|_{\mathcal{H}_{H_2}} &= \|u' + v'\|^2_{\mathcal{H}_{H_2}}\\ &= \|u'\|^2_{\mathcal{M}(|S^*|)} + \|v'\|^2_{\mathcal{H}(|S^*|)}\\ &= \|u\|^2_{\mathcal{M}(|T^*|)} + \|v\|^2_{\mathcal{H}(|T^*|)}\\ &= \|w\|^2_{\mathcal{H}_{G_2}}.\end{aligned}$$

Next, we shall show that Ψ is an isomorphism. Since

$$\langle \delta_{x_2}, \delta_{y_2} \rangle_{\mathcal{H}_{G_2}} = \langle U^* \delta_{\Psi(x_2)}, U^* \delta_{\Psi(y_2)} \rangle_{\mathcal{H}_{G_2}} = \langle \delta_{\Psi(x_2)}, \delta_{\Psi(y_2)} \rangle_{\mathcal{H}_{H_2}},$$

we have that $(\Psi(x_2), \Psi(y_2))$ belongs to $E(H_2)$ if and only if (x_2, y_2) belongs to $E(G_2)$ by (i) in Lemma 2.1. Therefore G_2 and H_2 are isomorphic. □

Corollary 3.1. *Let G_1 and G_2 be graphs.*

(i) *If there exist a graph H, homomorphisms $\varphi : H \to G_1$ and $\psi : H \to G_2$ such that T and S are compatible, then G_1 and G_2 are isomorphic.*
(ii) *If there exist a graph H, homomorphisms $\varphi : G_1 \to H$ and $\psi : G_2 \to H$ such that T^* and S^* are compatible, then G_1 and G_2 are isomorphic.*

In general de Branges-Rovnyak theory, analysis of $\mathcal{H}(T)$ and $\mathcal{H}(T^*)$ is not easy. Next, we shall give a general property on $\mathcal{H}(T)$ and $\mathcal{H}(T^*)$ in our setting.

Lemma 3.4. *Let φ be a homomorphism from G_1 to G_2. Then*

$$\dim \mathcal{M}(T) = \dim \mathcal{M}(T^*) = |\varphi(V_1)|.$$

Proof. First, it is trivial that $\dim \mathcal{M}(T) = |\varphi(V_1)|$, because $Tk_{x_1}^{G_1} = k_{\varphi(x_1)}^{G_2}/N_\varphi$ and $\{k_{x_1}^{G_1}\}_{x\in V_1}$ is linearly independent. Moreover, since $\mathcal{M}(T^*) = \operatorname{ran} T^* = (\ker T)^\perp$ as linear spaces in $\mathcal{H}_{G_1}$, we have

$$\dim \mathcal{M}(T^*) = |V_1| - \dim \ker T = |V_1| - (|V_1| - |\varphi(V_1)|) = |\varphi(V_1)|.$$

Thus we have the conclusion. □

We set

$$\operatorname{ind} T = \dim \mathcal{H}(T^*) - \dim \mathcal{H}(T).$$

It is easy to see that this quantity is invariant under isomorphisms in the sense of Theorem 3.1.

Theorem 3.3. *Let φ be a homomorphism from G_1 to G_2. Then*

$$|\varphi(V_1)| - |V_2| \leq \operatorname{ind} T \leq |V_1| - |\varphi(V_1)|.$$

Proof. By the decomposition (3.2) and (I-9) in [3], we have that

$$\begin{aligned}
|V_1| &= \dim \mathcal{H}_{G_1} \\
&= \dim \mathcal{M}(T^*) + \dim \mathcal{H}(T^*) - \dim(\mathcal{M}(T^*) \cap \mathcal{H}(T^*)) \\
&= \dim \mathcal{M}(T^*) + \dim \mathcal{H}(T^*) - \dim T^*\mathcal{H}(T) \\
&\geq \dim \mathcal{M}(T^*) + \dim \mathcal{H}(T^*) - \dim \mathcal{H}(T).
\end{aligned}$$

Similarly, by (3.1), we have that

$$|V_2| \geq \dim \mathcal{M}(T) + \dim \mathcal{H}(T) - \dim \mathcal{H}(T^*).$$

These inequalities concludes the following inequality:

$$\dim \mathcal{M}(T) - |V_2| \leq \dim \mathcal{H}(T^*) - \dim \mathcal{H}(T) \leq |V_1| - \dim \mathcal{M}(T^*).$$

By Lemma 3.4, we have the conclusion. □

4. Injective homomorphisms

In this section, we deal with injective homomorphisms. First, we shall give a partial converse of Theorem 3.1:

Theorem 4.1. *Let $\varphi : G_1 \to G_2$ and $\psi : H_1 \to H_2$ be injective homomorphisms. If T and T^* are compatible S and S^*, respectively, then there exists isomorphisms $\Phi : G_1 \to H_1$, $\Psi : G_2 \to H_2$ and a unitary operator U on $\mathcal{H}_{H_2}$ such that $UC_\Psi^* C_\varphi^* = C_\psi^* C_\Phi^*$ on $\mathcal{H}_{G_1}$.*

Proof. By Theorem 3.2, there exist isomorphisms $\Phi : G_1 \to H_1$ and $\Psi : G_2 \to H_2$. Then we have that

$$\begin{aligned}
\langle k^{H_2}_{\Psi\circ\varphi(x_1)}, k^{H_2}_{\Psi\circ\varphi(y_1)} \rangle_{\mathcal{H}_{H_2}} &= \langle C_\Psi^* T k^{G_1}_{x_1}, C_\Psi^* T k^{G_1}_{y_1} \rangle_{\mathcal{H}_{H_2}} \\
&= \langle T k^{G_1}_{x_1}, T k^{G_1}_{y_1} \rangle_{\mathcal{H}_{G_2}} \\
&= \langle T^* T k^{G_1}_{x_1}, T^* T k^{G_1}_{y_1} \rangle_{\mathcal{M}(T^*)} \\
&= \langle S^* S k^{H_1}_{\Phi(x_1)}, S^* S k^{H_1}_{\Phi(y_1)} \rangle_{\mathcal{M}(S^*)} \\
&= \langle S k^{H_1}_{\Phi(x_1)}, S k^{H_1}_{\Phi(y_1)} \rangle_{\mathcal{H}_{H_2}} \\
&= \langle k^{H_2}_{\psi\circ\Phi(x_1)}, k^{H_2}_{\psi\circ\Phi(y_1)} \rangle_{\mathcal{H}_{H_2}}.
\end{aligned}$$

Hence there exists a unitary operator U on $\mathcal{H}_{H_2}$ such that $U : k^{H_2}_{\Psi\circ\varphi(x_1)} \mapsto k^{H_2}_{\psi\circ\Phi(x_1)}$, and which is equivalent to that $UC_\Psi^* C_\varphi^* = C_\psi^* C_\Phi^*$ on $\mathcal{H}_{G_1}$. This concludes the proof. □

Furthermore, $\operatorname{ind} T$ can be obtained explicitly for injective homomorphisms.

Theorem 4.2. *Let φ be an injective homomorphism from G_1 to G_2. Then*

$$\operatorname{ind} T = |V_1| - |V_2|.$$

Proof. If φ is injective, then so is T. Hence we have that $\dim \mathcal{M}(T) = \dim \mathcal{H}_{G_1} = |V_1|$. Moreover, from $\mathcal{M}(T) \cap \mathcal{H}(T) = T\mathcal{H}(T^*)$ by (I-9) in [3], it follows that

$$\dim(\mathcal{M}(T) \cap \mathcal{H}(T)) = \dim \mathcal{H}(T^*).$$

By the identity which follows from (3.1), we have that

$$\begin{aligned}
|V_2| &= \dim \mathcal{H}_{G_2} \\
&= \dim \mathcal{M}(T) + \dim \mathcal{H}(T) - \dim(\mathcal{M}(T) \cap \mathcal{H}(T)) \\
&= |V_1| + \dim \mathcal{H}(T) - \dim \mathcal{H}(T^*).
\end{aligned}$$

This concludes the proof. □

Remark 4.1. $\operatorname{ind} T$ is an integer valued function invariant under isomorphisms. It would be worth mentioning the following observation which is a consequence of Theorem 4.2. Let $\varphi_1 : G_1 \to G_2$ and $\varphi_2 : G_2 \to G_3$ be injective homomorphisms. Then since $T_2T_1 = C^*_{\varphi_2 \circ \varphi_1}$, we have that

$$\operatorname{ind} T_2T_1 = |V_1| - |V_3| = |V_1| - |V_2| + |V_2| - |V_3| = \operatorname{ind} T_1 + \operatorname{ind} T_2,$$

that is, $\operatorname{ind} T$ is additive for injective homomorphisms. We should note that Fredholm index in a finite dimensional case is just a difference between dimensions of underlying spaces.

Next, we shall see that the growth of numbers of edges by an injective homomorphism is encoded in the Hilbert space structure of $\mathcal{H}(T)$.

We will write $\varphi^{-1}(x)$, instead of $\varphi^{-1}(\{x\})$, for every x in V_2 if no confusion occurs, and set $A_{\varphi^{-1}(x)\varphi^{-1}(y)} = 0$ if $\varphi^{-1}(x)$ is empty.

Lemma 4.1. *Let φ be an injective homomorphism from G_1 to G_2. Then*

(i) $\|(I - TT^*)\delta_x\|^2_{\mathcal{H}(T)} = \begin{cases} \deg_{G_2}(x) - \deg_{G_1}(\varphi^{-1}(x)) & (\varphi^{-1}(x) \neq \emptyset), \\ 1 + \deg_{G_2}(x) & (\varphi^{-1}(x) = \emptyset), \end{cases}$

(ii) $\langle (I - TT^*)\delta_x, (I - TT^*)\delta_y \rangle_{\mathcal{H}(T)} = -A^{G_2}_{xy} + A^{G_1}_{\varphi^{-1}(x)\varphi^{-1}(y)}$ *if* $x \neq y$.

Proof. First, since $T^*\delta_x = \delta_x \circ \varphi$, we note that $T^*\delta_x = 0$ if $\varphi^{-1}(x)$ is empty. For any x in V_2, we have that

$$\begin{aligned} \|(I - TT^*)\delta_x\|^2_{\mathcal{H}(T)} &= \langle (I - TT^*)\delta_x, (I - TT^*)\delta_x \rangle_{\mathcal{H}(T)} \\ &= \langle (I - TT^*)\delta_x, \delta_x \rangle_{\mathcal{H}_{G_2}} \\ &= \|\delta_x\|^2_{\mathcal{H}_{G_2}} - \|T^*\delta_x\|^2_{\mathcal{H}_{G_1}} \\ &= \deg_{G_2}(x) - \deg_{G_1}(\varphi^{-1}(x)). \end{aligned}$$

Hence we have (i). Next, we shall show (ii). For any x, y in V_2 such that $x \neq y$, we have that

$$\begin{aligned} \langle (I - TT^*)\delta_x, (I - TT^*)\delta_y \rangle_{\mathcal{H}(T)} &= \langle (I - TT^*)\delta_x, \delta_y \rangle_{\mathcal{H}_{G_2}} \\ &= \langle \delta_x, \delta_y \rangle_{\mathcal{H}_{G_2}} - \langle T^*\delta_x, T^*\delta_y \rangle_{\mathcal{H}_{G_1}} \\ &= -A^{G_2}_{xy} + A^{G_1}_{\varphi^{-1}(x)\varphi^{-1}(y)}. \end{aligned}$$

Thus we have (ii). This concludes the proof. □

Remark 4.2. Suppose that $G_2 = \varphi(G_1)$ and $V_2 = \{x_1, \ldots, x_n\}$. Then, by (i) and (ii) in Lemma 4.1, we have that

$$\|(I - TT^*) \sum_{j=1}^{n} c_j \delta_{x_j}\|^2_{\mathcal{H}(T)} = \langle (L_{G_2} - U^* L_{G_1} U)^t (c_1, \ldots, c_n), {}^t(c_1, \ldots, c_n) \rangle_{\mathbb{R}^n},$$

where L_G denotes the Laplacian matrix of G and U denotes the unitary matrix induced by φ as a permutation.

In general de Branges-Rovnyak theory, calculation of $\dim \mathcal{H}(T)$ is important, but not easy. However, in our case, it is possible under some conditions. We shall see how to calculate it with several examples.

Example 4.1. If $G_2 = \varphi(G_1)$ and $|E(\varphi(G_1))| - |E(G_1)| = 1$ then $\dim \mathcal{H}(T) = 1$. Indeed, we assume that x_1 and x_2 are in $V_2 = \varphi(V_1)$ and $A_{x_1x_2} > A_{\varphi^{-1}(x_1)\varphi^{-1}(x_2)}$. By Lemma 4.1, for any function $u = \sum_{x\in V_2} c_x\delta_x$ in $\mathcal{H}_{G_2}$, we have that

$$\begin{aligned}
&\langle (I_{\mathcal{H}_{G_2}} - TT^*)u, u\rangle_{\mathcal{H}_{G_2}}\\
&= \|(I_{\mathcal{H}_{G_2}} - TT^*)u\|^2_{\mathcal{H}(T)}\\
&= \sum_{x\in V_2} c_x^2(\deg(x) - \deg(\varphi^{-1}(x))) + \sum_{x,y\in V_2} c_xc_y(-A_{xy} + A_{\varphi^{-1}(x)\varphi^{-1}(y)})\\
&= (c_{x_1} - c_{x_2})^2.
\end{aligned}$$

Hence we have that $\dim\ker(I_{\mathcal{H}_{G_2}} - TT^*) = |V_2| - 1$,

$$\dim \mathcal{H}(T) = \dim \operatorname{ran}(I_{\mathcal{H}_{G_2}} - TT^*) = \dim(\ker(I_{\mathcal{H}_{G_2}} - TT^*))^\perp = 1$$

and $\mathcal{H}(T)$ is generated by $(I_{\mathcal{H}_{G_2}} - TT^*)^{1/2}(\delta_{x_1} - \delta_{x_2})$.

Example 4.2. If $G_2 = \varphi(G_1)$ and $|E(\varphi(G_1))| - |E(G_1)| = 2$ then $\dim \mathcal{H}(T) = 2$. We assume that x_i is in $V_2 = \varphi(V_1)$ $(i = 1, 2, 3, 4)$ such that $A_{x_ix_{i+1}} > A_{\varphi^{-1}(x_i)\varphi^{-1}(x_{i+1})}$ for $i = 1, 3$.

(Case 1) If $\{x_1, x_2\}$ is not connected with $\{x_3, y_4\}$, for any function $u = \sum_{x\in V_2} c_x\delta_x$ in $\mathcal{H}_{G_2}$, we have that

$$\langle (I_{\mathcal{H}_{G_2}} - TT^*)u, u\rangle_{\mathcal{H}_{G_2}} = (c_{x_1} - c_{x_2})^2 + (c_{x_3} - c_{x_4})^2.$$

Hence we have that $\dim\ker(I_{\mathcal{H}_{G_2}} - TT^*) = |V_2| - 4 + 2 = |V_2| - 2$. This concludes that $\dim \mathcal{H}(T) = 2$.

(Case 2) If $\{x_1, x_2\}$ is connected with $\{x_3, x_4\}$, then we may assume that $x_2 = x_4$. For any function $u = \sum_{x\in V_2} c_x\delta_x$ in $\mathcal{H}_{G_2}$, we have that

$$\langle (I_{\mathcal{H}_{G_2}} - TT^*)u, u\rangle_{\mathcal{H}_{G_2}} = (c_{x_1} - c_{x_2})^2 + (c_{x_2} - c_{x_3})^2.$$

Hence we have that $\dim\ker(I_{\mathcal{H}_{G_2}} - TT^*) = |V_2| - 3 + 1 = |V_2| - 2$. This concludes that $\dim \mathcal{H}(T) = 2$.

Example 4.3. Suppose that $G_2 = \varphi(G_1)$ and $|E(\varphi(G_1))| - |E(G_1)| = 3$. Then $\dim \mathcal{H}(T) = 3$ does not always hold. Indeed, we assume that x_1, x_2 and x_3 are in $V_2 = \varphi(V_1)$ and $\{x_1, x_2\}$, $\{x_2, x_3\}$ and $\{x_3, x_1\}$ are in $|E(\varphi(G_1))|$, however neither

$\{\varphi^{-1}(x_1), \varphi^{-1}(x_2)\}$, $\{\varphi^{-1}(x_2), \varphi^{-1}(x_3)\}$ nor $\{\varphi^{-1}(x_3), \varphi^{-1}(x_1)\}$ is in $|E(G_1)|$. Then for any function $u = \sum_{x \in V_2} c_x \delta_x$ in $\mathcal{H}_{G_2}$, we have that

$$\langle (I_{\mathcal{H}_{G_2}} - TT^*)u, u \rangle_{\mathcal{H}_{G_2}} = (c_{x_1} - c_{x_2})^2 + (c_{x_2} - c_{x_3})^2 + (c_{x_3} - c_{x_1})^2.$$

Hence we have that $\dim \ker(I_{\mathcal{H}_{G_2}} - TT^*) = |V_2| - 3 + 1 = |V_2| - 2$. This concludes that $\dim \mathcal{H}(T) = 2$.

Example 4.4. Let O_n denote the graph having no edge with n vertices, and K_n denote the complete graph with n vertices. We consider an injective homomorphisms $\varphi : O_n \to K_n$ such that $\varphi(O_n) = K_n$. For any function $u = \sum c_x \delta_x$ in $\mathcal{H}_{K_n}$, we have that

$$\langle (I_{\mathcal{H}_{K_n}} - TT^*)u, u \rangle_{\mathcal{H}_{K_n}} = (n-1) \sum_j c_{x_j}^2 - 2 \sum_{i>j} c_{x_i} c_{x_j} = \sum_{i \neq j} (c_{x_i} - c_{x_j})^2.$$

Hence $\ker(I - TT^*)$ is generated by $u = \sum_{x \in V} \delta_x = 1$. Therefore we have that $\dim \mathcal{H}(T) = n - 1$.

Let $\varphi : G_1 \to G_2$ be an injective homomorphism such that $G_2 = \varphi(G_1)$. We set

$$\Delta_\varphi E = \{\{x_i, x_j\} \in E(\varphi(G_1)) : \{\varphi^{-1}(x_i), \varphi^{-1}(x_j)\} \notin E(G_1)\}.$$

Then, in Examples 4.1, 4.2, 4.3 and 4.4, it is essentially shown that

$$\|(I_{\mathcal{H}_{G_2}} - TT^*) \sum_{x \in V_2} c_x \delta_x\|_{\mathcal{H}(T)}^2 = \sum_{\{x_i, x_j\} \in \Delta_\varphi E} (c_{x_i} - c_{x_j})^2,$$

and which implies that $\dim \mathcal{H}(T) \leq |\Delta_\varphi E|$.

Theorem 4.3. *Let $\varphi : G \to H$ be an injective homomorphism such that $H = \varphi(G)$. We set $n = |\Delta_\varphi E|$. Then $\mathcal{H}(T)$ can be decomposed into n one-dimensional subspaces in the sense of quasi-orthogonal decomposition.*

Proof. Let $\varphi : G \to H$ be decomposed as follows:

$$G = G_n \xrightarrow{\varphi_{n-1,n}} G_{n-1} \xrightarrow{\varphi_{n-2,n-1}} \cdots \xrightarrow{\varphi_{1,2}} G_1 \xrightarrow{\varphi_{0,1}} G_0 = H, \quad \varphi = \varphi_{0,1} \circ \cdots \circ \varphi_{n-1,n},$$

$|E(\varphi_{j,j+1}(G_{j+1}))| - |E(G_j)| = 1$ and $\varphi_{j,j+1}(G_{j+1}) = G_j$ for $j = 0, 1, \ldots, n-1$. Setting $\varphi_j = \varphi_{0,1} \circ \cdots \circ \varphi_{j-1,j}$, $\varphi_j : G_j \to G_0 = H$ is an injective homomorphism. Furthermore we set $T_{j-1,j} = C^*_{\varphi_{j-1,j}} : \mathcal{H}_{G_j} \to \mathcal{H}_{G_{j-1}}$ and $T_j = C^*_{\varphi_j} : \mathcal{H}_{G_j} \to \mathcal{H}_{G_0}$. Then trivially, we have that $T_{j+1} = T_j T_{j,j+1}$. and we note that $\dim \mathcal{H}(T_{j,j+1}) = 1$ by Example 4.1. Using (I-10) in [3] inductively or by Theorem A140 in Vasyunin-Nikol'skiĭ [5], $\mathcal{H}(T)$ can be decomposed as follows:

$$\mathcal{H}(T) = \mathcal{H}(T_n) = \sum_{j=0}^{n-1} \mathcal{H}(T_j, T_{j+1}).$$

Since $\mathcal{H}(T_j, T_{j+1}) = \mathcal{M}(T_j(I - T_{j,j+1}T_{j,j+1}^*)^{1/2})$ and T_j is injective, we have that

$$\dim \mathcal{H}(T_j, T_{j+1}) = \dim \mathcal{H}(T_{j,j+1}) = 1.$$

This concludes the proof. □

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