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THE CHERN CHARACTER IN THE SIMPLICIAL DE RHAM COMPLEX

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Abstract. On the basis of Dupont's work, we exhibit a cocycle in the simplicial de Rham complex which represents the Chern character. We also prove the related conjecture due to Brylinski. This gives a way to construct a cocycle in a local truncated complex.

1. Introduction

It is well-known that there is one-to-one correspondence between the characteristic classes of G-bundles and the elements in the cohomology ring of the classifying space BG. So it is important to investigate $H^*(BG)$ in research on the characteristic classes. However, in general BG is not a manifold so we can not adapt the usual de Rham theory on it. To overcome this problem, a total complex of a double complex $\Omega^*(NG(*))$ which is associated to a simplicial manifold $\{NG(*)\}$ is often used. In brief, $\{NG(*)\}$ is a sequence of manifolds $\{NG(p) = G^p\}_{p=0,1,\cdots}$ together with face operators $\varepsilon_i : NG(p) \to NG(p-1)$ for $i = 0, \cdots, p$ satisfying relations $\varepsilon_i\varepsilon_j = \varepsilon_{j-1}\varepsilon_i$ for $i < j$ (The standard definition also involves degeneracy operators but we do not need them here). The cohomology ring of $\Omega^*(NG(*))$ is isomorphic to $H^*(BG)$ so we can use this complex as a candidate of the de Rham complex on BG.

In [5], Dupont introduced another double complex $A^{*,*}(NG)$ on NG and showed the cohomology ring of its total complex $A^*(NG)$ is also isomorphic to $H^*(BG)$. Then he used it to construct a homomorphism from $I^*(G)$, the G-invariant polynomial ring over Lie algebra $\mathcal{G}$, to $H^*(BG)$ for a classical Lie group G.

The images of this homomorphism in $\Omega^*(NG(*))$ are called the Bott-Shulman-Stasheff forms. The main purpose of this paper is to exhibit these cocycles precisely when they represent the Chern characters.

In addition, we also show that the conjecture due to Brylinski in [3] is true. This gives a way to construct a cocycle in a local truncated complex $[\sigma_{<p}\Omega^*_{\mathrm{loc}}(NG)]$

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whose cohomology class is mapped to the cohomology class of the Bott-Shulman-Stasheff form in a local double complex by a boundary map. His original motivation to introduce these complexes and the conjecture is to study the local cohomology group of the gauge group $\mathrm{Map}(X, G)$ and the Lie algebra cohomology of its Lie algebra. Actually, as a special case $X = S^1$, he constructed the standard Kac-Moody 2-cocycle for a loop Lie algebra by using the cocycle in the local truncated complex $[\sigma_{<2}\Omega^3_{\mathrm{loc}}(NG)]$.

The outline of this paper is as follows. In section 2, we briefly recall the universal Chern-Weil theory due to Dupont. In section 3, we obtain the Bott-Shulman-Stasheff form in $\Omega^*(NG(*))$ which represents the Chern character ch_p. In section 4, we introduce some result about the Chern-Simons forms. In section 5, we prove Brylinski's conjecture.

2. Review of the universal Chern-Weil Theory

In this section we recall the universal Chern-Weil theory following [6]. For any Lie group G, we have simplicial manifolds NG, $N\bar{G}$ and simplicial G-bundle $\gamma : N\bar{G} \to NG$ as follows:

$NG(q) = \overbrace{G \times \cdots \times G}^{q-times} \ni (h_1, \cdots, h_q)$:
face operators $\varepsilon_i : NG(q) \to NG(q-1)$

$$\varepsilon_i(h_1, \cdots, h_q) = \begin{cases} (h_2, \cdots, h_q) & i = 0 \\ (h_1, \cdots, h_i h_{i+1}, \cdots, h_q) & i = 1, \cdots, q-1 \\ (h_1, \cdots, h_{q-1}) & i = q. \end{cases}$$

$N\bar{G}(q) = \overbrace{G \times \cdots \times G}^{q+1-times} \ni (g_1, \cdots, g_{q+1})$:
face operators $\bar{\varepsilon}_i : N\bar{G}(q) \to N\bar{G}(q-1)$

$$\bar{\varepsilon}_i(g_0, \cdots, g_q) = (g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_q) \qquad i = 0, 1, \cdots, q.$$

We define $\gamma : N\bar{G} \to NG$ as $\gamma(g_0, \cdots, g_q) = (g_0 {g_1}^{-1}, \cdots, g_{q-1} {g_q}^{-1})$.

For any simplicial manifold $X = \{X_*\}$, we can associate a topological space $\| X \|$ called the fat realization. Since any G-bundle $\pi : E \to M$ can be realized as a pull-back of the fat realization of γ, $\| \gamma \|$ is the universal bundle $EG \to BG$ [8].

Now we construct a double complex associated to a simplicial manifold.

Definition 2.1. For any simplicial manifold $\{X_*\}$ with face operators $\{\varepsilon_*\}$, we define a double complex as follows:

$$\Omega^{p,q}(X) := \Omega^q(X_p)$$

Derivatives are:

$$d' := \sum_{i=0}^{p+1}(-1)^i\varepsilon_i^*, \qquad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).$$

For NG and $N\bar{G}$ the following holds [2] [6] [7].

Theorem 2.1. *There exist ring isomorphisms*

$$H(\Omega^*(NG)) \cong H^*(BG), \qquad H(\Omega^*(N\bar{G})) \cong H^*(EG).$$

Here $\Omega^*(NG)$ and $\Omega^*(N\bar{G})$ mean the total complexes.

For example, the derivative $d' + d'' : \Omega^p(NG) \to \Omega^{p+1}(NG)$ is given as follows:

$$\begin{array}{cccccc}
\Omega^p(G) & & & & & \\
\uparrow{\scriptstyle -d} & & & & & \\
\Omega^{p-1}(G) & \xrightarrow{\varepsilon_0^*-\varepsilon_1^*+\varepsilon_2^*} & \Omega^{p-1}(NG(2)) & & & \\
 & & \uparrow{\scriptstyle d} & & & \\
 & & \Omega^{p-2}(NG(2)) & & & \\
 & & & \ddots & & \\
 & & & & \Omega^1(NG(p)) & \\
 & & & & \uparrow{\scriptstyle (-1)^p d} & \\
 & & & & \Omega^0(NG(p)) & \xrightarrow{\sum_{i=0}^{p+1}(-1)^i\varepsilon_i^*} \ \Omega^0(NG(p+1))
\end{array}$$

Remark 2.1. Let $\pi : P \to M$ be a principal G-bundle and $\{g_{\alpha\beta} : U_{\alpha\beta} \to G\}$ be the transition functions of it. Then we can pull-back the cocycle in $\Omega^*(NG)$ to the Čech-de Rham complex of M by $\{g_{\alpha\beta}\}$. When κ is the characteristic class which corresponds to the cocycle in $\Omega^*(NG)$, the image of $g_{\alpha\beta}^*$ in $H^*_{\check{C}ech-deRham}(M)$ is the characteristic class $\kappa(P)$ of $\pi : P \to M$. For more details, see for instance [7].

There is another double complex associated to a simplicial manifold.

Definition 2.2 ([5])**.** A simplicial n-form on a simplicial manifold $\{X_p\}$ is a sequence $\{\phi^{(p)}\}$ of n-forms $\phi^{(p)}$ on $\Delta^p \times X_p$ such that

$$(\varepsilon^i \times id)^* \phi^{(p)} = (id \times \varepsilon_i)^* \phi^{(p-1)}.$$

Here ε^i is the canonical i-th face operator of Δ^p.

Let $A^{k,l}(X)$ be the set of all simplicial $(k+l)$-forms on $\Delta^p \times X_p$ which are expressed locally of the form

$$\sum a_{i_1\cdots i_k j_1\cdots j_l}(dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l})$$

where $(t_0, t_1, \cdots, t_p)$ are the barycentric coordinates in Δ^p and x_j are the local coordinates in X_p. We call these forms (k,l)-form on $\Delta^p \times X_p$ and define derivatives as:

$$d' := \text{the exterior differential on } \Delta^p$$

$$d'' := (-1)^k \times \text{the exterior differential on } X_p.$$

Then $(A^{k,l}(X), d', d'')$ is a double complex.

Let $A^*(X)$ denote the total complex of $A^{*,*}(X)$. We define a map $I_\Delta : A^*(X) \to \Omega^*(X)$ as follows:

$$I_\Delta(\alpha) := \int_{\Delta^p} (\alpha|_{\Delta^p \times X_p}).$$

Then the following theorem holds [5].

Theorem 2.2. I_Δ *induces a natural ring isomorphism*

$$I_\Delta^* : H(A^*(X)) \cong H(\Omega^*(X)).$$

Let $\mathcal{G}$ denote the Lie algebra of G. A connection on a simplicial G-bundle $\pi : \{E_p\} \to \{M_p\}$ is a sequence of 1-forms $\{\theta\}$ on $\{E_p\}$ with coefficients $\mathcal{G}$ such that θ restricted to $\Delta^p \times E_p$ is a usual connection form on a principal G-bundle $\Delta^p \times E_p \to \Delta^p \times M_p$.

Dupont constructed a canonical connection $\theta \in A^1(N\bar{G})$ on $\gamma : N\bar{G} \to NG$ in the following way:

$$\theta|_{\Delta^p \times N\bar{G}(p)} := t_0\theta_0 + \cdots + t_p\theta_p.$$

Here θ_i is defined by $\theta_i = \mathrm{pr}_i^*\bar{\theta}$ where $\mathrm{pr}_i : \Delta^p \times N\bar{G}(p) \to G$ is the projection into the i-th factor of $N\bar{G}(p)$ and $\bar{\theta}$ is the Maurer-Cartan form of G. We also obtain its curvature $\Omega \in A^2(N\bar{G})$ on γ as:

$$\Omega|_{\Delta^p \times N\bar{G}(p)} = d\theta|_{\Delta^p \times N\bar{G}(p)} + \frac{1}{2}[\theta|_{\Delta^p \times N\bar{G}(p)}, \theta|_{\Delta^p \times N\bar{G}(p)}].$$

Let $\mathrm{I}^*(G)$ denote the ring of G-invariant polynomials on $\mathcal{G}$. For $P \in I^*(G)$, we restrict $P(\Omega) \in A^*(N\bar{G})$ to each $\Delta^p \times N\bar{G}(p) \to \Delta^p \times NG(p)$ and apply the usual Chern-Weil theory then we have a simplicial $2k$-form $P(\Omega)$ on NG.

Now we have a canonical homomorphism

$$w : \mathrm{I}^*(G) \to H(\Omega^*(NG))$$

which maps $P \in I^*(G)$ to $w(P) = [I_\Delta(P(\Omega))]$.

3. The Chern character in the double complex

In this section we exhibit a cocycle in $\Omega^{*,*}(NG)$ which represents the Chern character. Throughout this section, $G = GL(n;\mathbb{C})$ and ch_p means the p-th Chern character.

Note that the diagram below is commutative, since I_Δ acts only on the differential forms on Δ^*, and so does γ^* on differential forms on each $NG(*)$.

$$\begin{array}{ccc} A^{*,*}(N\bar{G}) & \xrightarrow{I_\Delta} & \Omega^{*,*}(N\bar{G}) \\ {\scriptstyle\gamma^*}\uparrow & & \uparrow{\scriptstyle\gamma^*} \\ A^{*,*}(NG) & \xrightarrow{I_\Delta} & \Omega^{*,*}(NG) \end{array}$$

We first give the cocycle in $\Omega^{p+q}(N\bar{G}(p-q))(0 \leq q \leq p-1)$ which corresponds to the p-th Chern character by restricting $(1/p!)\,\mathrm{tr}\,((-\Omega/2\pi i)^p) \in A^{2p}(N\bar{G})$ to $A^{p-q,p+q}(\Delta^{p-q} \times N\bar{G}(p-q))$ and integrating it along Δ^{p-q}. Then we give the cocycle in $\Omega^{p+q}(NG(p-q))$ which hits to it by γ^*.

Since $[\theta_i, \theta_j] = \theta_i \wedge \theta_j + \theta_j \wedge \theta_i$ for any i, j,

$$\Omega|_{\Delta^{p-q} \times N\bar{G}(p-q)} = -\sum_{i=1}^{p-q} dt_i \wedge (\theta_0 - \theta_i) - \sum_{0 \leq i < j \leq p-q} t_i t_j (\theta_i - \theta_j)^2.$$

Now

$$dt_i \wedge (\theta_0 - \theta_i) = dt_i \wedge \{(\theta_0 - \theta_1) + (\theta_1 - \theta_2) + \cdots + (\theta_{i-1} - \theta_i)\}$$

and for any $\mathcal{G}$-valued differential forms α, β, γ and any integer $0 \leq \forall x \leq p-q-1$, the equation $\alpha \wedge (dt_i \wedge (\theta_x - \theta_{x+1})) \wedge \beta \wedge (dt_j \wedge (\theta_x - \theta_{x+1})) \wedge \gamma = -\alpha \wedge (dt_j \wedge (\theta_x - \theta_{x+1})) \wedge \beta \wedge (dt_i \wedge (\theta_x - \theta_{x+1})) \wedge \gamma$ holds, so the terms of the forms above cancel with each other in $\left(-\Omega|_{\Delta^{p-q} \times N\bar{G}(p-q)}\right)^p$. Then we see:

$$\left(-\Omega|_{\Delta^{p-q} \times N\bar{G}(p-q)}\right)^p = \left(\sum_{i=1}^{p-q} dt_i \wedge (\theta_{i-1} - \theta_i) + \sum_{0 \leq i < j \leq p-q} t_i t_j (\theta_i - \theta_j)^2\right)^p.$$

Now we obtain the following theorem.

Theorem 3.1. *We set:*

$$\bar{S}_{p-q} = \sum_{\sigma \in \mathfrak{S}_{p-q-1}} (\mathrm{sgn}(\sigma))(\theta_{\sigma(1)} - \theta_{\sigma(1)+1}) \cdots (\theta_{\sigma(p-q-1)} - \theta_{\sigma(p-q-1)+1})$$

Then the cocycle in $\Omega^{p+q}(N\bar{G}(p-q))$ $(0 \le q \le p-1)$ *which corresponds to the* p*-th Chern character* ch_p *is*

$$\frac{1}{p!}\left(\frac{1}{2\pi i}\right)^p (-1)^{(p-q)(p-q-1)/2} \times$$

$$\mathrm{tr} \sum \left((p(\theta_0 - \theta_1)) \wedge \bar{H}_q(\bar{S}_{p-q}) \times \int_{\Delta^{p-q}} \prod_{i<j} (t_i t_j)^{a_{ij}(\bar{H}_q(\bar{S}_{p-q}))} dt_1 \wedge \cdots \wedge dt_{p-q} \right).$$

Here $\bar{H}_q(\bar{S}_{p-q})$ *means the terms that* $(\theta_i - \theta_j)^2$ $(1 \le i < j \le p-q+1)$ *are put* q *-times between* $(\theta_{k-1} - \theta_k)$ *and* $(\theta_l - \theta_{l+1})$ *in* $\bar{S}_{p-q}$ *permitting overlaps;* $a_{ij}(\bar{H}_q(\bar{S}_{p-q}))$ *means the number of* $(\theta_i - \theta_j)^2$ *in it.* $\sum$ *means the sum of all such terms.*

Proof. The cocycle in $\Omega^{p+q}(N\bar{G}(p-q))$ which corresponds to ch_p is given by

$$\int_{\Delta^{p-q}} \frac{1}{p!} \mathrm{tr} \left(\left(\frac{-\Omega|_{\Delta^{p-q} \times N\bar{G}(p-q)}}{2\pi i} \right)^p \right)$$

$$= \frac{1}{p!}\left(\frac{1}{2\pi i}\right)^p \int_{\Delta^{p-q}} \mathrm{tr} \left(\left(\sum_{i=1}^{p-q} dt_i \wedge (\theta_{i-1} - \theta_i) + \sum_{0 \le i < j \le p-q} t_i t_j (\theta_i - \theta_j)^2 \right)^p \right).$$

By calculating this equation, we can check that the statement of Theorem 3.1 is true. □

For the purpose of getting the differential forms in $\Omega^{*,*}(NG)$ which hit the cocycles in Theorem 3.1 by γ^*, we set

$$\varphi_s := h_1 \cdots h_{s-1} dh_s h_s^{-1} \cdots h_1^{-1}.$$

Here h_i is the i-th factor of $NG(*)$.

A straightforward calculation shows that

$$\gamma^* \mathrm{tr}(\varphi_{i_1} \varphi_{i_2} \cdots \varphi_{i_{p-1}} \varphi_{i_p}) = \mathrm{tr}(\theta_{i_1 - 1} - \theta_{i_1})(\theta_{i_2 - 1} - \theta_{i_2}) \cdots (\theta_{i_p - 1} - \theta_{i_p}).$$

From the above, we conclude:

Theorem 3.2. *We set:*

$$R_{ij} = (\varphi_i + \varphi_{i+1} + \cdots + \varphi_{j-1})^2 \qquad (1 \le i < j \le p-q+1)$$

$$S_{p-q} = \sum_{\sigma \in \mathfrak{S}_{p-q-1}} \mathrm{sgn}(\sigma) \varphi_{\sigma(1)+1} \cdots \varphi_{\sigma(p-q-1)+1}.$$

Then the cocycle in $\Omega^{p+q}(NG(p-q))$ $(0 \leq q \leq p-1)$ *which represents the p-th Chern character* ch_p *is*

$$\frac{1}{(p-1)!}\left(\frac{1}{2\pi i}\right)^p(-1)^{(p-q)(p-q-1)/2}\times$$

$$\mathrm{tr}\sum\left(\varphi_1\wedge H_q(S_{p-q})\times\int_{\Delta^{p-q}}\prod_{i<j}(t_{i-1}t_{j-1})^{a_{ij}(H_q(S))}dt_1\wedge\cdots\wedge dt_{p-q}\right).$$

Here $H_q(S_{p-q})$ *means the term that* R_{ij} $(1 \leq i < j \leq p-q+1)$ *are put* q *-times between* ψ_k *and* ψ_l *in* S_{p-q} *permitting overlaps;* $a_{ij}(H_q(S_{p-q}))$ *means the number of* R_{ij} *in it.* $\sum$ *means the sum of all such terms.*

Proof. We can easily check that the cocycle in Theorem 3.2 is mapped to the cochain in Theorem 3.1 by $\gamma^*:\Omega^{p+q}(NG(p-q))\to\Omega^{p+q}(N\bar{G}(p-q))$. The statement folllows from this. □

Remark 3.1. The coefficients in Theorem 3.2 are calculated using the following famous formula.

$$\int_{\Delta^r}t_0^{b_0}t_1^{b_1}\cdots t_r^{b_r}dt_1\wedge\cdots\wedge dt_r=\frac{b_0!\,b_1!\cdots b_r!}{(b_0+b_1+\cdots+b_r+r)!}.$$

Corollary 3.1. *The cochain* ω_p *in* $\Omega^{2p-1}(NG(1))$ *which corresponds to the p-th Chern character is given as follows:*

$$\omega_1=\frac{1}{p!}\left(\frac{1}{2\pi i}\right)^p\frac{1}{{}_{2p-1}C_{p-1}}\mathrm{tr}(h^{-1}dh)^{2p-1}.$$

Corollary 3.2. *The cochain* ω_p *in* $\Omega^p(NG(p))$ *which corresponds to the p-th Chern character is given as follows:*

$$\omega_p=(-1)^{p(p-1)/2}\frac{1}{p!(p-1)!}\left(\frac{1}{2\pi i}\right)^p\mathrm{tr}\left(\varphi_1\wedge\sum_{\sigma\in\mathfrak{S}_{p-1}}\mathrm{sgn}(\sigma)\varphi_{\sigma(1)+1}\cdots\varphi_{\sigma(p-1)+1}\right).$$

Example 3.1. The cocycle which represents the second Chern character ch_2 in $\Omega^4(NG)$ is the sum of the following $C_{1,3}$ and $C_{2,2}$:

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle d''} & & & & \\
C_{1,3}\in\Omega^3(G) & \xrightarrow{d'} & \Omega^3(NG(2)) & & \\
 & & \uparrow{\scriptstyle d''} & & \\
 & & C_{2,2}\in\Omega^2(NG(2)) & \xrightarrow{d'} & 0
\end{array}$$

$$C_{1,3}=\left(\frac{1}{2\pi i}\right)^2\frac{1}{6}\mathrm{tr}(h^{-1}dh)^3,\qquad C_{2,2}=\left(\frac{1}{2\pi i}\right)^2\frac{-1}{2}\mathrm{tr}(dh_1dh_2h_2^{-1}h_1^{-1}).$$

Corollary 3.3. *The cocycle which represents the second Chern class* c_2 *in* $\Omega^4(NG)$ *is the sum of the following* $c_{1,3}$ *and* $c_{2,2}$*:*

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle d''} & & & & \\
c_{1,3}\in\Omega^3(G) & \xrightarrow{d'} & \Omega^3(NG(2)) & & \\
 & & \uparrow{\scriptstyle d''} & & \\
 & & c_{2,2}\in\Omega^2(NG(2)) & \xrightarrow{d'} & 0
\end{array}$$

$$c_{1,3}=\left(\frac{1}{2\pi i}\right)^2\frac{-1}{6}\mathrm{tr}(h^{-1}dh)^3$$

$$c_{2,2}=\left(\frac{1}{2\pi i}\right)^2\frac{1}{2}\mathrm{tr}(dh_1dh_2h_2^{-1}h_1^{-1})-\left(\frac{1}{2\pi i}\right)^2\frac{1}{2}\mathrm{tr}(h_1^{-1}dh_1)\mathrm{tr}(h_2^{-1}dh_2).$$

Example 3.2. The cocycle which represents the 3rd Chern character ch_3 in $\Omega^6(NG)$ is the sum of the following $C_{1,5}, C_{2,4}$ and $C_{3,3}$:

$$\begin{array}{ccccccc}
0 & & & & & & \\
\uparrow{\scriptstyle d''} & & & & & & \\
C_{1,5}\in\Omega^5(G) & \xrightarrow{d'} & \Omega^5(NG(2)) & & & & \\
 & & \uparrow{\scriptstyle d''} & & & & \\
 & & C_{2,4}\in\Omega^4(NG(2)) & \xrightarrow{d'} & \Omega^4(NG(3)) & & \\
 & & & & \uparrow{\scriptstyle d''} & & \\
 & & & & C_{3,3}\in\Omega^3(NG(3)) & \xrightarrow{d'} & 0
\end{array}$$

$$C_{1,5}=\frac{1}{3!}\left(\frac{1}{2\pi i}\right)^3\frac{1}{10}\mathrm{tr}(h^{-1}dh)^5$$

$$\begin{aligned}C_{2,4}=\frac{-1}{3!}\left(\frac{1}{2\pi i}\right)^3\Bigg(&\frac{1}{2}\mathrm{tr}(dh_1{h_1}^{-1}dh_1{h_1}^{-1}dh_1dh_2{h_2}^{-1}{h_1}^{-1})\\
&+\frac{1}{4}\mathrm{tr}(dh_1dh_2{h_2}^{-1}{h_1}^{-1}dh_1dh_2{h_2}^{-1}{h_1}^{-1})\\
&\qquad+\frac{1}{2}\mathrm{tr}(dh_1dh_2{h_2}^{-1}dh_2{h_2}^{-1}dh_2{h_2}^{-1}{h_1}^{-1})\Bigg)\end{aligned}$$

$$C_{3,3} = \frac{-1}{3!}\left(\frac{1}{2\pi i}\right)^3 \left(\frac{1}{2}\mathrm{tr}(dh_1 dh_2 dh_3 {h_3}^{-1} {h_2}^{-1} {h_1}^{-1}) \right.$$
$$\left. -\frac{1}{2}\mathrm{tr}(dh_1 h_2 dh_3 {h_3}^{-1} {h_2}^{-1} dh_2 {h_2}^{-1} {h_1}^{-1})\right).$$

4. The Chern-Simons form

We briefly recall the notion of the Chern-Simons form in [4].

Let $\pi : E \to M$ be any principal G-bundle and θ, Ω denote its connection form and the curvature. For any $P \in \mathrm{I}^k(G)$, we define the $(2k-1)$-form $TP(\theta)$ on E as:

$$TP(\theta) := k\int_0^1 P(\theta \wedge \phi_t^{k-1})dt.$$

Here $\phi_t := t\Omega + \frac{1}{2}t(t-1)[\theta,\theta]$. Then the equation $d(TP(\theta)) = P(\Omega^k)$ holds and $TP(\theta)$ is called the Chern-Simons form of $P(\Omega^k)$. When the bundle is flat, its curvature vanishes and hence $d(TP(\theta)) = P(\Omega^k) = 0$.

Now we put the simplicial connection into TP and using the same argument in section 3, then we obtain the Chern-Simons form in $\Omega^{2p-1}(N\bar{G})$.

Proposition 4.1. *The Chern-Simons form in $\Omega^3(N\overline{U(n)})$ which corresponds to the second Chern class c_2 is the sum of the following $Tc_{0,3}$, $Tc_{1,2}$:*

$$\begin{array}{ccccc}
0 & & & & \\
\uparrow{\scriptstyle d''} & & & & \\
Tc_{0,3} \in \Omega^3(U(n)) & \xrightarrow{d'} & \Omega^3(N\overline{U(n)}(1)) & & \\
 & & \uparrow{\scriptstyle d''} & & \\
 & & Tc_{1,2} \in \Omega^2(N\overline{U(n)}(1)) & \xrightarrow{d'} & \Omega^2(N\overline{U(n)}(2))
\end{array}$$

$$Tc_{0,3} = \left(\frac{1}{2\pi i}\right)^2 \frac{1}{6}\mathrm{tr}(g^{-1}dg)^3$$

$$Tc_{1,2} = \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{2}\mathrm{tr}(g_0^{-1}dg_0 g_1^{-1}dg_1) - \frac{1}{2}\mathrm{tr}(g_0^{-1}dg_0)\mathrm{tr}(g_1^{-1}dg_1)\right).$$

Remark 4.1. The term $\left(\frac{1}{2\pi i}\right)^2 \frac{1}{2}\mathrm{tr}(g_0^{-1}dg_0)\mathrm{tr}(g_1^{-1}dg_1)$ vanishes when we restrict it to $SU(n)$.

5. Formulas for a cocycle in a truncated complex

In this section, we prove the conjecture due to Brylinski in [3].

At first, we introduce the filtered local simplicial de Rham complex.

Definition 5.1 ([3])**.** The filtered local simplicial de Rham complex $F^p\Omega^{*,*}_{\text{loc}}(NG)$ over a simplicial manifold NG is defined as follows:

$$F^p\Omega^{r,s}_{\text{loc}}(NG) = \begin{cases} \varinjlim_{1 \in V \subset G^r} \Omega^s(V) & \text{if } s \geq p \\ 0 & \text{otherwise.} \end{cases}$$

Let $F^p\Omega^*(NG)$ be a filtered complex

$$F^p\Omega^{r,s}(NG) = \begin{cases} \Omega^s(NG(r)) & \text{if } s \geq p \\ 0 & \text{otherwise} \end{cases}$$

and $[\sigma_{<p}\Omega^*(NG)]$ a truncated complex

$$[\sigma_{<p}\Omega^{r,s}(NG)] = \begin{cases} 0 & \text{if } s \geq p \\ \Omega^s(NG(r)) & \text{otherwise.} \end{cases}$$

Then there is an exact sequence:

$$0 \to F^p\Omega^*(NG) \to \Omega^*(NG) \to [\sigma_{<p}\Omega^*(NG)] \to 0$$

which induces a boundary map

$$\beta : H^l(NG, [\sigma_{<p}\Omega^*_{\text{loc}}]) \to H^{l+1}(NG, [F^p\Omega^*_{\text{loc}}]).$$

Let $\omega_1 + \cdots + \omega_p$, $\omega_{p-q} \in \Omega^{p+q}(NG(p-q))$ be the cocycle in $\Omega^{2p}(NG)$ which represents the p-th Chern character. By using this cocycle, Brylinski constructed a cochain η in $[\sigma_{<p}\Omega^*_{\text{loc}}(NG)]$ in the following way.

We take a contractible open set $U \subset G$ containing 1. Using the same argument in [6, Lemma 9.7], we can construct mappings $\{\sigma_l : \Delta^l \times U^l \to U\}_{0 \leq l}$ inductively with the following properties:

(1) $\sigma_0(pt) = 1$;

(2)

$$\sigma_l(\varepsilon^j(t_0, \cdots, t_{l-1}); h_1, \cdots, h_l) = \begin{cases} \sigma_{l-1}(t_0, \cdots, t_{l-1}; \varepsilon_j(h_1, \cdots, h_l)) & \text{if } j \geq 1 \\ h_1 \cdot \sigma_{l-1}(t_0, \cdots, t_{l-1}; h_2, \cdots, h_l) & \text{if } j = 0. \end{cases}$$

Then we define mappings $\{f_{m,q} : \Delta^q \times U^{m+q-1} \to G^m\}$ by

$$f_{m,q}(t_0, \cdots, t_q; h_1, \cdots, h_{m+q-1}) := (h_1, \cdots, h_{m-1}, \sigma_q(t_0, \cdots, t_q; h_m, \cdots, h_{m+q-1})).$$

We can check $f_{m,q} \circ \varepsilon_j = f_{m,q+1} \circ \varepsilon^{j-m+1} : \Delta^q \times U^{m+q} \to G^m$ if $m \leq j \leq m+q$ and $f_{m,q} \circ \varepsilon_j = \varepsilon_j \circ f_{m+1,q}$ if $m-1 \geq j \geq 0$ and $\varepsilon_m \circ f_{m+1,q} = f_{m,q+1} \circ \varepsilon^0$ holds.

We define a $(2p-m-q)$-form $\beta_{m,q}$ on U^{m+q-1} by $\beta_{m,q} = (-1)^m \int_{\Delta^q} f_{m,q}^* \omega_m$. Then the cochain η is defined as the sum of following η_l on U^{2p-1-l} for $0 \le l \le p-1$:

$$\eta_l := \sum_{m+q=2p-l,\ m\ge 1} \beta_{m,q}.$$

Now we are ready to state the theorem whose statement is conjectured by Brylinski [3].

Theorem 5.1. *$\eta := \eta_0 + \cdots + \eta_{p-1}$ is a cocycle in $[\sigma_{<p}\Omega^*_{\mathrm{loc}}(NG)]$ whose cohomology class is mapped to $[\omega_1 + \cdots + \omega_p]$ in $H^{2p}(NG, [F^p\Omega^*_{\mathrm{loc}}])$ by a boundary map $\beta : H^{2p-1}(NG, [\sigma_{<p}\Omega^*_{\mathrm{loc}}]) \to H^{2p}(NG, [F^p\Omega^*_{\mathrm{loc}}])$.*

Proof. To prove this, it suffices to show the equation below holds true for any l which satisfies $0 \le l \le 2p-1$ since $\omega_{2p-l} = 0$ if $0 \le l \le p-1$:

$$\sum_{i=0}^{2p-l} (-1)^i \varepsilon_i^* \eta_l = (-1)^{2p-l+1} d\eta_{l-1} + \omega_{2p-l}.$$

The left side of this equation is equal to

$$\sum_{m+q=2p-l,\ m\ge 1} (-1)^m \left(\int_{\Delta^q} \sum_{i=0}^{m-1} (-1)^i (f_{m,q}\circ\varepsilon_i)^*\omega_m + \int_{\Delta^q} \sum_{i=m}^{m+q} (-1)^i (f_{m,q}\circ\varepsilon_i)^*\omega_m \right).$$

We can check that

$$\sum_{i=0}^{m-1} (-1)^i (f_{m,q}\circ\varepsilon_i)^*\omega_m = f_{m+1,q}^*(\sum_{i=0}^{m-1}(-1)^i\varepsilon_i^*\omega_m)$$

hence by using the cocycle relation $\sum_{i=0}^{m+1}(-1)^i\varepsilon_i^*\omega_m = (-1)^m d\omega_{m+1}$, we can see the following holds:

$$\int_{\Delta^q} \sum_{i=0}^{m-1} (-1)^i (f_{m,q}\circ\varepsilon_i)^*\omega_m = \int_{\Delta^q} (-1)^m d f_{m+1,q}^*\omega_{m+1}$$
$$- \left((-1)^m \int_{\Delta^q} (\varepsilon_m \circ f_{m+1,q})^*\omega_m + (-1)^{m+1} \int_{\Delta^q} (\varepsilon_{m+1}\circ f_{m+1,q})^*\omega_m \right).$$

Note that $\int_{\Delta^q}(\varepsilon_{m+1}\circ f_{m+1,q})^*\omega_m = 0$ for $q \ge 1$ and $\int_{\Delta^q}(\varepsilon_{m+1}\circ f_{m+1,q})^*\omega_m = \omega_{2p-l}$ if $q = 0$.

We can also check that

$$\int_{\Delta^q} \sum_{i=m}^{m+q} (-1)^i (f_{m,q}\circ\varepsilon_i)^*\omega_m = \int_{\Delta^q} \sum_{i=m}^{m+q} (-1)^i (f_{m,q+1}\circ\varepsilon^{i-m+1})^*\omega_m.$$

We set $j = i - m + 1$, then we see that $\int_{\Delta^q} \sum_{i=m}^{m+q} (-1)^i (f_{m,q+1} \circ \varepsilon^{i-m+1})^* \omega_m$ is equal to

$$\sum_{j=0}^{q+1} \left((-1)^{j+m-1} \int_{\Delta^q} (f_{m,q+1} \circ \varepsilon^j)^* \omega_m \right) - (-1)^{m-1} \int_{\Delta^q} (\varepsilon_m \circ f_{m+1,q})^* \omega_m$$

since $\varepsilon_m \circ f_{m+1,q} = f_{m,q+1} \circ \varepsilon^0$.

From above, we can see that $\sum_{i=0}^{2p-l} (-1)^i \varepsilon_i^* \eta_l$ is equal to

$$\omega_{2p-l} + \sum_{m+q=2p-l,\ m \geq 1} \left(\int_{\Delta^q} df^*_{m+1,q} \omega_{m+1} + \sum_{j=0}^{q+1} (-1)^{j-1} \int_{\Delta^q} (f_{m,q+1} \circ \varepsilon^j)^* \omega_m \right).$$

On the other hand, for any (m', q') which satisfies $m' + q' = 2p - (l-1)$ the following equation holds:

$$(-1)^{q'} d \int_{\Delta^{q'}} f^*_{m',q'} \omega_{m'} = \int_{\Delta^{q'}} df^*_{m',q'} \omega_{m'} - \sum_{j=0}^{q'} \int_{\Delta^{q'-1}} (-1)^j \varepsilon^{j*} f^*_{m',q'} \omega_{m'}.$$

Therefore $(-1)^{2p-l+1} d\eta_{l-1}$ is equal to

$$\sum_{m'+q'=2p-l+1,\ m' \geq 1} \left(\int_{\Delta^{q'}} df^*_{m',q'} \omega_{m'} - \sum_{j=0}^{q'} \int_{\Delta^{q'-1}} (-1)^j \varepsilon^{j*} f^*_{m',q'} \omega_{m'} \right).$$

This completes the proof.

□

Remark 5.1. Let me explain Brylinski's motivation in [3] to introduce these complexes and the conjecture briefly. Let LU be the free loop group of a contractible open set $U \subset G$ containing 1 and $\mathrm{ev} : LU \times S^1 \to U$ be the evaluation map, i.e. for $\gamma \in LU$ and $\theta \in S^1$, $\mathrm{ev}(\gamma, \theta)$ is defined as $\gamma(\theta)$. Then $\int_{S^1} \mathrm{ev}^*$ maps $\eta_1 \in \Omega^1(U^{2p-2})$ to a cochain in $\Omega^0(LU^{2p-2})$. This cochain defines a cohomology class in local cohomology group $H^{2p-2}_{\mathrm{loc}}(LU, \mathbb{C})$. Brylinski constructed a natural map from $H^{2p-2}_{\mathrm{loc}}(LU, \mathbb{C})$ to the the Lie algebra cohomology $H^{2p-2}(L\mathcal{G}, \mathbb{C})$. Then as a special case $p = 2$, he used the cocycle in the local truncated complex $[\sigma_{<2} \Omega^3_{\mathrm{loc}}(NG)]$ to construct the standard Kac-Moody 2-cocycle. He treated not only the free loop group but also the gauge group $\mathrm{Map}(X, G)$ for a compact oriented manifold X.

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