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UNIQUENESS OF THE EXTENSION OF ISOMETRIES ON THE UNIT SPHERES IN NORMED LINEAR SPACES

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Abstract. In this paper we show that the extension of a surjective isometry on the unit sphere in a normed linear space is unique.

1. Introduction

In [3], Mazur and Ulam studied a property of the isometries T from a normed real-linear space X onto a normed real-linear space Y. They proved the so-called Mazur-Ulam theorem stating that $T - T(0)$ must be a real-linear map. We refer to [1, 5] for the proof of the theorem. Mankiewicz [2] gave a generalization of the theorem which if U is a non-empty open connected set of X, V is a open set of Y and $f : U \to V$ is a surjective isometry, then there exists an affine isometry T from X onto Y such that the restriction $T|_U$ to U is equal to f.

Moreover, the Mazur-Ulam theorem has been generalized in many directions. Tingley [4] have proposed the so-called Tingley problem. The problem is as follows: let S_X, S_Y be the unit sphere in X, Y, respectively, and $f : S_X \to S_Y$ a surjective isometry. Is f necessarily the restriction to S_X of a linear, or affine, transformation? In this paper, we study the uniqueness of the extension of f for the problem when f can be extended. The following is a main theorem in this paper.

Theorem 1.1. *Let X, Y be normed real-linear spaces with the unit sphere S_X, S_Y, respectively, and $f : S_X \to S_Y$ a surjective isometry. If there exists a surjective isometry $T : X \to Y$ such that the restriction $T|_{S_X}$ to S_X is equal to f, then such a map is unique.*

The above theorem can be proved by applying Theorem 1.3.4 in [1]. In this paper, we give an alternative simple proof of Theorem 1.1 by applying Lemma 2.1 in the next section. We also note that Lemma 2.1 gives a short and simple proof of Theorem 1.3.4 in [1].

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2. Proof of Theorem 1.1

We begin with the following lemma.

Lemma 2.1. *Let y be an element in X. Then, $\|y - x\| = 1$ for any $x \in S_X$ if and only if $y = 0$.*

Proof. If $y = 0$, then $\|y-x\| = 1$ for every $x \in S_X$. We verify the converse. Suppose that there exists an element $z \neq 0$ such that $\|z - x\| = 1$ holds for any $x \in S_X$. Putting $z_1 = \frac{z}{\|z\|} \in S_X$, we have the equation

$$1 = \|z - z_1\| = \left\| z - \frac{z}{\|z\|} \right\| = \left| 1 - \frac{1}{\|z\|} \right| \|z\| = |\|z\| - 1|.$$

Therefore we get $\|z\| = 2$ by the above equality. However,

$$1 = \|z - (-z_1)\| = \|z + z_1\| = \left\| z + \frac{z}{2} \right\| = \frac{3}{2}\|z\| = 3,$$

since $-z_1 \in S_X$. This is a contradiction. □

The main theorem is proved by applying Lemma 2.1.

Proof of Theorem 1.1. Let $T' : X \to Y$ be a surjective isometry such that the restriction $T'|_{S_X}$ to S_X is equal to f. We prove $T'(0) = 0$. For any $y \in S_Y$, there exists an element $x \in S_X$ such that $T'(x) = f(x) = y$ since f is a surjection. Therefore we have

$$\|T'(0) - y\| = \|T'(0) - T'(x)\| = \|0 - x\| = 1.$$

Applying Lemma 2.1 for $T'(0)$, we obtain $T'(0) = 0$. We also get $T(0) = 0$ in the same way. By the Mazur-Ulam theorem, T and T' are real-linear maps. As T and T' are homogeneous, we deduce the equation

$$T(z) = \|z\|T\left(\frac{z}{\|z\|}\right) = \|z\|f\left(\frac{z}{\|z\|}\right) = \|z\|T'\left(\frac{z}{\|z\|}\right) = T'(z)$$

for any $z \neq 0$. This imply that $T = T'$, and the proof is completed. □

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