Nihonkai Math. J.
Vol.25(2014), 127–146

GENERALIZED SPLIT FEASIBILITY PROBLEM GOVERNED BY WIDELY MORE GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

MAYUMI HOJO AND WATARU TAKAHASHI

Abstract. Generalized split feasibility problem governed by a widely more generalized hybrid mapping is studied. In particular, strong convergence of this algorithm is proved. As tools, resolvents of maximal monotone operators are technically maneuvered to facilitate the argument of the proof to the main result. Applications to iteration methods for various nonlinear mappings and to equilibrium problem are included.

1. Introduction

Let H be a real Hilbert space and let C be a non-empty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x-y, Ux-Uy\rangle \geq \alpha\|Ux-Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be non-empty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [6] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [5] considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1},\ 1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2},\ 1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2,\ 1 \leq j \leq n$, the *split common null point problem* [5] is to find a point $z \in H_1$ such that

$$z \in \big(\cap_{i=1}^m A_i^{-1}0\big) \cap \big(\cap_{j=1}^n T_j^{-1}(B_j^{-1}0)\big),$$

2010 *Mathematics Subject Classification.* 47H05, 47H09, 47H20.

Key words and phrases. Maximal monotone operator, inverse strongly monotone mapping, widely more generalized hybrid mapping, fixed point, strong convergence theorem, equilibrium problem, split feasibility problem.

The second author was partially supported by Grant-in-Aid for Scientific Research No. 23540188 from Japan Society for the Promotion of Science.

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator, where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is non-empty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and generalized split feasibility problems including the split common null point problem; see, for instance, [5, 7, 16, 29]. In the study, they used established results for solving the problems. In particular, established convergence theorems have been used for finding solutions of the problems. On the other hand, we know many existence and convergence theorems for inverse strongly monotone mappings in Hilbert spaces; see, for instance, [9, 17, 19, 21, 25, 26].

In this paper, motivated by the ideas of these problems and results, we consider generalized split feasibility problem and then the problem governed by a widely more generalized hybrid mapping is studied. In particular, strong convergence of this algorithm is proved. As tools, resolvents of maximal monotone operators are technically maneuvered to facilitate the argument of the proof to the main result. Applications to iteration methods for various nonlinear mappings and to equilibrium problem are included.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [24] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle; \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a non-empty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x \rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [22]. Let $\alpha > 0$ be a given constant. A mapping $A\colon C \to H$ is said to be α-inverse strongly monotone if $\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. It is known that $\|Ax - Ay\| \leq (1/\alpha) \|x - y\|$ for all $x, y \in C$ if A is α-inverse-strongly monotone. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B on H is said to be monotone if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}\colon H \to D(B)$, which is called the resolvent of B for r. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of fixed points of J_r. It is also known that $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu| / \lambda) \|x - J_\lambda x\|$ holds for all $\lambda, \mu > 0$ and $x \in H$; see [10, 22] for more details. As a matter of fact, we know the following lemma [21].

Lemma 2.1. *Let H be a real Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s - t}{s} \langle J_s x - J_t x, J_s x - x \rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

We also know the following lemmas:

Lemma 2.2 ([2], [28]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, ...$ Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([15]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \dots$ *and* $\tau(n) \to \infty$*;*

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1},\ \forall n \geq n_0$.

Let H be a Hilbert space and let S be a firmly nonexpansive mapping of H into itself with $F(S) \neq \emptyset$. Then we have that

$$\langle x - Sx, Sx - y\rangle \geq 0 \tag{2.5}$$

for all $x \in H$ and $y \in F(S)$. In fact, we have that for all $x \in H$ and $y \in F(S)$

$$\begin{aligned}\langle x - Sx, Sx - y\rangle &= \langle x - y + y - Sx, Sx - y\rangle \\ &= \langle x - y, Sx - y\rangle + \langle y - Sx, Sx - y\rangle \\ &\geq \|Sx - y\|^2 - \|Sx - y\|^2 \\ &= 0.\end{aligned}$$

From [27], we also have the following lemmas.

Lemma 2.4. *Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then a mapping $A^*(I - T)A : H_1 \to H_1$ is $\frac{1}{2\|A\|^2}$-inverse strongly monotone.*

Lemma 2.5. *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*(I - T)A)z$;
(ii) $0 \in A^*(I - T)Az + Bz$;
(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Then, a mapping $T : C \to H$ is called generalized hybrid [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{2.6}$$

for all $x, y \in C$. We call such a mapping (α, β)-generalized hybrid. Notice that the mapping above covers several well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. Kawasaki and Takahashi [13] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping S from C into H is said to be widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\|Sx - Sy\|^2 + \beta\|x - Sy\|^2 + \gamma\|Sx - y\|^2 + \delta\|x - y\|^2 \\ &+\varepsilon\|x - Sx\|^2 + \zeta\|y - Sy\|^2 + \eta\|(x - Sx) - (y - Sy)\|^2 \leq 0\end{aligned} \tag{2.7}$$

for all $x, y \in C$. Such a mapping S is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [14] if $\alpha + \beta = -\gamma - \delta = 1$ and $\varepsilon = \zeta = \eta = 0$. A generalized hybrid mapping with a fixed point is quasinonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. We know the following theorem from Kawasaki and Takahashi [13].

Theorem 2.1 ([13]). *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

Then S has a fixed point if and only if there exists $z \in C$ such that $\{S^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of S is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2).

The following lemmas for widely more generalized hybrid mappings are essential for proving our main theorem.

Lemma 2.6 ([13]). *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(S) \neq \emptyset$ and it satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\zeta + \eta \geq 0$ *and* $\alpha + \beta > 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\varepsilon + \eta \geq 0$ *and* $\alpha + \gamma > 0$.

Then S is quasi-nonexpansive.

Lemma 2.7 ([12]). *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $S : C \to H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta + \eta > 0$.

If $x_n \rightharpoonup z$ and $x_n - Sx_n \to 0$, then $z \in F(S)$.

3. Main result

In this section, we solve a generalized split feasibility problem governed by a widely more generalized hybrid mapping in Hilbert spaces.

Theorem 3.1. *Let H_1 and H_2 be Hilbert spaces and let C be a non-empty, closed and convex subset of H_1. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that $D(B) \subset C$ and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$.

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n) S J_{\lambda_n} (I - \lambda_n A^*(I - T)A) x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in B^{-1}0 \cap F(S) \cap A^{-1}F(T)$, where $z_0 = P_{B^{-1}0 \cap F(S) \cap A^{-1}F(T)} u$.

Proof. Let $z \in B^{-1}0 \cap F(S) \cap A^{-1}F(T)$. We have that $Sz = z$, $J_{\lambda_n} z = z$ and $(I - T)Az = 0$. Since S is quasi-nonexpansive (Lemma 2.6), J_{λ_n} is nonexpansive and $(I - T)$ is $\frac{1}{2}$-inverse strongly monotone, we obtain that for any $n \in \mathbb{N}$,

$$\begin{aligned} \|S J_{\lambda_n}(x_n - \lambda_n A^*(I - T)Ax_n) - z\|^2 &\leq \|J_{\lambda_n}(x_n - \lambda_n A^*(I - T)Ax_n) - z\|^2 \\ &\leq \|x_n - \lambda_n A^*(I - T)Ax_n - z\|^2 \qquad (3.1) \\ &= \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, A^*(I - T)Ax_n\rangle + (\lambda_n)^2 \|A^*(I - T)Ax_n\|^2 \\ &= \|x_n - z\|^2 - 2\lambda_n \langle Ax_n - Az, (I - T)Ax_n\rangle + (\lambda_n)^2 \|A^*(I - T)Ax_n\|^2 \\ &\leq \|x_n - z\|^2 - \lambda_n \|(I - T)Ax_n\|^2 + (\lambda_n)^2 \|A\|^2 \|(I - T)Ax_n\|^2 \\ &= \|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - 1) \|(I - T)Ax_n\|^2 . \\ &\leq \|x_n - z\|^2 . \end{aligned}$$

Let $y_n = \alpha_n u_n + (1 - \alpha_n) S J_{\lambda_n}(x_n - \lambda_n A^*(I - T)Ax_n)$. We have that

$$\begin{aligned} \|y_n - z\| &= \|\alpha_n(u_n - z) + (1 - \alpha_n)(S J_{\lambda_n}(x_n - \lambda_n A^*(I - T)Ax_n) - z)\| \\ &\leq \alpha_n \|u_n - z\| + (1 - \alpha_n) \|x_n - z\| . \end{aligned}$$

Using this, we get that

$$\|x_{n+1} - z\|$$

$$
\begin{aligned}
&= \|\beta_n(x_n - z) + (1-\beta_n)(y_n - z)\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n)\|y_n - z\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n)(\alpha_n \|u_n - z\| + (1-\alpha_n)\|x_n - z\|) \\
&= (1 - \alpha_n(1-\beta_n))\|x_n - z\| + \alpha_n(1-\beta_n)\|u_n - z\|.
\end{aligned}
$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n - z\| \leq M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \leq K$. Suppose that $\|x_k - z\| \leq K$ for some $k \in \mathbb{N}$. Then we have that

$$
\begin{aligned}
\|x_{k+1} - z\| &\leq (1 - \alpha_k(1-\beta_k))\|x_k - z\| + \alpha_k(1-\beta_k)\|u_k - z\| \\
&\leq (1 - \alpha_k(1-\beta_k))K + \alpha_k(1-\beta_k)K = K.
\end{aligned}
$$

By induction, we obtain that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$, $\{y_n\}$ and $\{J_{\lambda_n}(x_n - \lambda_n A^*(I-T)Ax_n)\}$ are bounded. Putting $z_n = J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n$, from the definition of $\{x_n\}$ we have that

$$
x_{n+1} - x_n = \beta_n x_n + (1-\beta_n)\{\alpha_n u_n + (1-\alpha_n)Sz_n\} - x_n
$$

and hence

$$
\begin{aligned}
x_{n+1} - x_n - (1-\beta_n)\alpha_n u_n &= \beta_n x_n + (1-\beta_n)(1-\alpha_n)Sz_n - x_n \\
&= (1-\beta_n)\{(1-\alpha_n)Sz_n - x_n\} \\
&= (1-\beta_n)\{Sz_n - x_n - \alpha_n Sz_n\}.
\end{aligned}
$$

Thus we have that

$$
\begin{aligned}
&\langle x_{n+1} - x_n - (1-\beta_n)\alpha_n u_n, x_n - z_0\rangle \\
&\quad = (1-\beta_n)\langle Sz_n - x_n, x_n - z_0\rangle - (1-\beta_n)\langle \alpha_n Sz_n, x_n - z_0\rangle \qquad (3.2)\\
&\quad = -(1-\beta_n)\langle x_n - Sz_n, x_n - z_0\rangle - (1-\beta_n)\alpha_n\langle Sz_n, x_n - z_0\rangle.
\end{aligned}
$$

From (2.3) and (3.1), we have that

$$
\begin{aligned}
2\langle x_n - Sz_n, x_n - z_0\rangle &= \|x_n - z_0\|^2 + \|Sz_n - x_n\|^2 - \|Sz_n - z_0\|^2 \\
&\geq \|x_n - z_0\|^2 + \|Sz_n - x_n\|^2 - \|x_n - z_0\|^2 \qquad (3.3)\\
&= \|Sz_n - x_n\|^2.
\end{aligned}
$$

From (3.2) and (3.3), we have that

$$
\begin{aligned}
-2\langle x_n - x_{n+1}, x_n - z_0\rangle &= 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\quad - 2(1-\beta_n)\langle x_n - Sz_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle Sz_n, x_n - z_0\rangle \qquad (3.4)\\
&\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\quad - (1-\beta_n)\|Sz_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle Sz_n, x_n - z_0\rangle.
\end{aligned}
$$

Furthermore, using (2.3) and (3.4), we have that

$$\begin{aligned}\|x_{n+1} - z_0\|^2 - \|x_n - x_{n+1}\|^2 - \|x_n - z_0\|^2 &\le 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|Sz_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle Sz_n, x_n - z_0\rangle.\end{aligned}$$

Setting $\Gamma_n = \|x_n - z_0\|^2$, we have that

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n - \|x_n - x_{n+1}\|^2 &\le 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|Sz_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle Sz_n, x_n - z_0\rangle.\end{aligned} \tag{3.5}$$

Since

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-\beta_n)\alpha_n(u_n - Sz_n) + (1-\beta_n)(Sz_n - x_n)\| \\ &\le (1-\beta_n)\big(\|Sz_n - x_n\| + \alpha_n\|u_n - Sz_n\|\big),\end{aligned} \tag{3.6}$$

we have that

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &\le (1-\beta_n)^2\big(\|Sz_n - x_n\| + \alpha_n\|u_n - Sz_n\|\big)^2 \\ &= (1-\beta_n)^2\|Sz_n - x_n\|^2 \\ &\quad + (1-\beta_n)^2\big(2\alpha_n\|Sz_n - x_n\|\|u_n - Sz_n\| + \alpha_n^2\|u_n - Sz_n\|^2\big).\end{aligned} \tag{3.7}$$

Thus we have from (3.5) and (3.7) that

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n &\le \|x_n - x_{n+1}\|^2 + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ &\quad - (1-\beta_n)\|Sz_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle Sz_n, x_n - z_0\rangle \\ &\le (1-\beta_n)^2\|Sz_n - x_n\|^2 \\ &\quad + (1-\beta_n)^2\big(2\alpha_n\|Sz_n - x_n\|\|u_n - Sz_n\| + \alpha_n^2\|u_n - Sz_n\|^2\big) \\ &\quad + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle - (1-\beta_n)\|Sz_n - x_n\|^2 \\ &\quad - 2(1-\beta_n)\alpha_n\langle Sz_n, x_n - z_0\rangle\end{aligned}$$

and hence

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n + \beta_n(1-\beta_n)\|Sz_n - x_n\|^2 &\le (1-\beta_n)^2\big(2\alpha_n\|Sz_n - x_n\|\|u_n - Sz_n\| + \alpha_n^2\|u_n - Sz_n\|^2\big) \\ &\quad + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle Sz_n, x_n - z_0\rangle.\end{aligned} \tag{3.8}$$

We will divide the proof into two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using

$\lim_{n\to\infty}\alpha_n = 0$ and $0 < c \le \beta_n \le d < 1$, we have from (3.8) that

$$\lim_{n\to\infty} \|Sz_n - x_n\| = 0. \tag{3.9}$$

From (3.6) we have that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

We also have that

$$\begin{aligned} \|y_n - Sz_n\| &= \|\alpha_n u_n + (1-\alpha_n)Sz_n - Sz_n\| \\ &= \alpha_n \|u_n - Sz_n\| \to 0. \end{aligned} \tag{3.11}$$

Furthermore, from $\|y_n - x_n\| \le \|y_n - Sz_n\| + \|Sz_n - x_n\|$, we have that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.12}$$

We show $\lim_{n\to\infty}\|Sz_n - z_n\| = 0$. Since $\|\cdot\|^2$ is a convex function, we have that

$$\|x_{n+1} - z_0\|^2 \le \beta_n \|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2. \tag{3.13}$$

From (2.1) we also have that

$$\begin{aligned} \|y_n - z_0\|^2 &= \|\alpha_n(u_n - z_0) + (1-\alpha_n)(Sz_n - z_0)\|^2 \\ &\le (1-\alpha_n)^2\|Sz_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\ &\le (1-\alpha_n)^2\|z_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\ &\le \|z_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\ &\le \|x_n - z_0\|^2 + \lambda_n(\lambda_n\|A\|^2 - 1)\|(I-T)Ax_n\|^2 \\ &\qquad + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle. \end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), we have that

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &\le \beta_n \|x_n - z_0\|^2 + (1-\beta_n)\|x_n - z_0\|^2 \\ &\quad + (1-\beta_n)\lambda_n(\lambda_n\|A\|^2 - 1)\|(I-T)Ax_n\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\ &= \|x_n - z_0\|^2 + (1-\beta_n)\lambda_n(\lambda_n\|A\|^2 - 1)\|(I-T)Ax_n\|^2 \\ &\quad + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle). \end{aligned} \tag{3.15}$$

Thus we have that

$$\begin{aligned} &(1-\beta_n)\lambda_n(1 - \lambda_n\|A\|^2)\|(I-T)Ax_n\|^2 \\ &\qquad \le \|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2 + (1-\beta_n)2\alpha_n\langle u_n - z_0, y_n - z_0\rangle. \end{aligned} \tag{3.16}$$

Then we have that

$$\lim_{n\to\infty} \|(I-T)Ax_n\| = 0. \tag{3.17}$$

Since J_{λ_n} is firmly nonexpansive, we have that

$$2\|z_n - z_0\|^2 = 2\|J_{\lambda_n}(x_n - \lambda_n A^*(I-T)Ax_n) - J_{\lambda_n} z_0\|^2$$

$$
\begin{aligned}
&\le 2\left\langle x_n - \lambda_n A^*(I-T)Ax_n - z_0, z_n - z_0\right\rangle \\
&= \|x_n - \lambda_n A^*(I-T)Ax_n - z_0\|^2 + \|z_n - z_0\|^2 \\
&\quad - \|x_n - \lambda_n A^*(I-T)Ax_n - z_0 - (z_n - z_0)\|^2 \\
&\le \|x_n - z_0\|^2 + \|z_n - z_0\|^2 \\
&\quad - \|x_n - z_n - \lambda_n A^*(I-T)Ax_n\| \\
&= \|x_n - z_0\|^2 + \|z_n - z_0\|^2 - \|x_n - z_n\|^2 \\
&\quad + 2\lambda_n \left\langle x_n - z_n, A^*(I-T)Ax_n\right\rangle - (\lambda_n)^2 \|A^*(I-T)Ax_n\|^2.
\end{aligned}
$$

Thus we get that

$$
\begin{aligned}
\|z_n - z_0\|^2 &\le \|x_n - z_0\|^2 - \|x_n - z_n\|^2 \\
&\quad + 2\lambda_n \left\langle x_n - z_n, A^*(I-T)Ax_n\right\rangle - (\lambda_n)^2 \|A^*(I-T)Ax_n\|^2.
\end{aligned}
$$

Using (3.14), we obtain

$$
\begin{aligned}
\|x_{n+1} - z_0\|^2 &\le \beta_n \|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2 \\
&\le \beta_n \|x_n - z_0\|^2 + (1-\beta_n)(\|z_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle) \\
&\le \beta_n \|x_n - z_0\|^2 + (1-\beta_n)\|x_n - z_0\|^2 \\
&\quad - (1-\beta_n)\|x_n - z_n\|^2 + 2(1-\beta_n)\lambda_n \left\langle x_n - z_n, A^*(I-T)Ax_n\right\rangle \\
&\quad - (1-\beta_n)\lambda_n^2 \|A^*(I-T)Ax_n\|^2 + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle \\
&= \|x_n - z_0\|^2 - (1-\beta_n)\|x_n - z_n\|^2 \\
&\quad + 2(1-\beta_n)\lambda_n \left\langle x_n - z_n, A^*(I-T)Ax_n\right\rangle - (1-\beta_n)\lambda_n^2 \|A^*(I-T)Ax_n\|^2 \\
&\quad + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle,
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
(1-\beta_n)\|x_n - z_n\|^2 &\le \|x_n - z_0\|^2 \\
&\quad - \|x_{n+1} - z_0\|^2 + 2(1-\beta_n)\lambda_n \left\langle x_n - z_n, A^*(I-T)Ax_n\right\rangle \\
&\quad - (1-\beta_n)\lambda_n^2 \|A^*(I-T)Ax_n\|^2 + 2(1-\beta_n)\alpha_n\langle u_n - z_0, y_n - z_0\rangle.
\end{aligned}
$$

Then we have that

$$
\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.18}
$$

Since $\|Sz_n - z_n\| \le \|Sz_n - x_n\| + \|x_n - z_n\|$, we have that

$$
\lim_{n\to\infty} \|Sz_n - z_n\| = 0. \tag{3.19}
$$

Take $\lambda_0 \in \mathbb{R}$ with $0 < a \le \lambda_0 \le b < \frac{1}{\|A\|^2}$. Put $s_n = (I - \lambda_n)A^*(I-T)Ax_n$. Using $z_n = J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n$, we have from Lemma 2.1 that

$$
\|J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n - z_n\|
$$

$$
\begin{aligned}
&= \|J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)x_n - J_{\lambda_n}(I-\lambda_n A^*(I-T)A)x_n\| \\
&= \|J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)x_n - J_{\lambda_0}(I-\lambda_n A^*(I-T)A)x_n \\
&\qquad + J_{\lambda_0}(I-\lambda_n A^*(I-T)A)x_n - J_{\lambda_n}(I-\lambda_n A^*(I-T)A)x_n\| \qquad (3.20)\\
&\le \|(I-\lambda_0 A^*(I-T)A)x_n - (I-\lambda_n A^*(I-T)A)x_n\| + \|J_{\lambda_0}s_n - J_{\lambda_n}s_n\| \\
&\le |\lambda_0-\lambda_n|\|A^*(I-T)Ax_n\| + \frac{|\lambda_0-\lambda_n|}{\lambda_0}\|J_{\lambda_0}s_n - s_n\|.
\end{aligned}
$$

We also have from (3.20) that

$$
\begin{aligned}
&\|x_n - J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)x_n\| \qquad (3.21)\\
&\qquad \le \|x_n - z_n\| + \|z_n - J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)x_n\|.
\end{aligned}
$$

We will use (3.20) and (3.21) later.

Let us show that $\limsup_{n\to\infty} \langle z_0-u, x_n-z_0\rangle \ge 0$. Put

$$\ell = \limsup_{n\to\infty} \langle z_0-u, x_n-z_0\rangle .$$

Without loss of generality, we may assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\ell = \lim_{i\to\infty} \langle z_0-u, x_{n_i}-z_0\rangle$ and $\{x_{n_i}\}$ converges weakly to some point $w \in C$. From $\|x_n - z_n\| \to 0$, we also have that $\{z_{n_i}\}$ converges weakly to $w \in C$. On the other hand, from $\{\lambda_{n_i}\} \subset [a,b]$ there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\lambda_{n_{i_j}} \to \lambda_0$ for some $\lambda_0 \in [a,b]$. Without loss of generality, we assume that $z_{n_i} \rightharpoonup w$ and $\lambda_{n_i} \to \lambda_0$. From (3.19) we know $\lim_{n\to\infty}\|Sz_n - z_n\| = 0$. Thus we have from Lemma 2.7 that $w = Sw$. Since $\lambda_{n_i} \to \lambda_0$, we have from (3.20) that

$$\|J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)x_{n_i} - z_{n_i}\| \to 0.$$

Furthermore, we have from (3.21) that

$$\|x_{n_i} - J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)x_{n_i}\| \to 0.$$

Since $J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)$ is nonexpansive, we have that

$$w = J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)w.$$

This means that $0 \in B^{-1}0 \cap A^{-1}F(T)$. Thus we have

$$w \in F(S) \cap B^{-1}0 \cap A^{-1}F(T).$$

Then we have

$$\ell = \lim_{i\to\infty}\langle z_0-u, x_{n_i}-z_0\rangle = \langle z_0-u, w-z_0\rangle \ge 0. \qquad (3.22)$$

Since $y_n - z_0 = \alpha_n(u_n - z_0) + (1-\alpha_n)(Sz_n - z_0)$, we have

$$\|y_n - z_0\|^2 \le (1-\alpha_n)^2\|Sz_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, y_n - z_0\rangle .$$

Thus we have from (3.1) that

$$\|y_n - z_0\|^2 \leq (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle u_n - z_0, y_n - z_0 \rangle .$$

Consequently we have that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \beta_n \|x_n - z_0\|^2 + (1-\beta_n) \|y_n - z_0\|^2 \\
&\leq \beta_n \|x_n - z_0\|^2 + (1-\beta_n) \left((1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle u_n - z_0, y_n - z_0 \rangle \right) \\
&= \left(\beta_n + (1-\beta_n)(1-\alpha_n)^2 \right) \|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n \langle u_n - z_0, y_n - z_0 \rangle \\
&\leq \left(\beta_n + (1-\beta_n)(1-\alpha_n) \right) \|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n \langle u_n - z_0, y_n - z_0 \rangle \\
&= (1-(1-\beta_n)\alpha_n) \|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n \langle u_n - z_0, y_n - z_0 \rangle \\
&= (1-(1-\beta_n)\alpha_n) \|x_n - z_0\|^2 + 2(1-\beta_n)\alpha_n \langle u_n - u, y_n - z_0 \rangle \\
&\quad + 2(1-\beta_n)\alpha_n \langle u - z_0, y_n - z_0 \rangle .
\end{aligned}$$

By this inequality and Lemma 2.2, we obtain that $x_n \to z_0$, where

$$z_0 = P_{F(S) \cap B^{-1}0 \cap A^{-1}F(T)} u.$$

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.3 that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. Thus we have from (3.8) that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\beta_{\tau(n)}(1-\beta_{\tau(n)}) &\|Sz_{\tau(n)} - x_{\tau(n)}\|^2 \\
&\leq (1-\beta_{\tau(n)})^2 2\alpha_{\tau(n)} \|Sz_{\tau(n)} - x_{\tau(n)}\| \|u_{\tau(n)} - Sz_{\tau(n)}\| \\
&\quad + (1-\beta_{\tau(n)})^2 \alpha_{\tau(n)}^2 \|u_{\tau(n)} - Sz_{\tau(n)}\|^2 \\
&\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)} \langle u_{\tau(n)}, x_{\tau(n)} - z_0 \rangle \\
&\quad - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)} \langle Sz_{\tau(n)}, x_{\tau(n)} - z_0 \rangle .
\end{aligned} \tag{3.23}$$

Using $\lim_{n\to\infty} \alpha_n = 0$ and $0 < c \leq \beta_n \leq d < 1$, we have that

$$\lim_{n\to\infty} \|Sz_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.24}$$

As in the proof of Case 1 we have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \tag{3.25}$$

and

$$\lim_{n\to\infty} \|y_{\tau(n)} - Sz_{\tau(n)}\| = 0. \tag{3.26}$$

Since $\|y_{\tau(n)} - x_{\tau(n)}\| \leq \|y_{\tau(n)} - Sz_{\tau(n)}\| + \|Sz_{\tau(n)} - x_{\tau(n)}\|$, we have that

$$\lim_{n\to\infty} \|y_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.27}$$

For $z_0 = P_{B^{-1}0\cap F(S)\cap A^{-1}F(T)}u$, we show that $\limsup_{n\to\infty}\left\langle z_0 - u, y_{\tau(n)} - z_0\right\rangle \geq 0$. Put

$$\ell = \limsup_{n\to\infty}\left\langle z_0 - u, y_{\tau(n)} - z_0\right\rangle.$$

Without loss of generality, there exists a subsequence $\{y_{\tau(n_i)}\}$ of $\{y_{\tau(n)}\}$ such that $\ell = \lim_{i\to\infty}\left\langle u - z_0, y_{\tau(n_i)} - z_0\right\rangle$ and $\{y_{\tau(n_i)}\}$ converges weakly to some point $w \in C$. From $\|y_{\tau(n_i)} - x_{\tau(n_i)}\| \to 0$, we also have that $\{x_{\tau(n_i)}\}$ converges weakly to $w \in C$. As in the proof of Case 1 we have that $w \in B^{-1}0 \cap F(S) \cap A^{-1}F(T)$. Then we have

$$\ell = \lim_{i\to\infty}\langle z_0 - u, y_{\tau(n_i)} - z_0\rangle = \langle z_0 - u, w - z_0\rangle \geq 0.$$

As in the proof of Case 1, we also have that

$$\left\|y_{\tau(n)} - z_0\right\|^2 \leq (1-\alpha_{\tau(n)})^2\left\|x_{\tau(n)} - z_0\right\|^2 + 2\alpha_{\tau(n)}\left\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\right\rangle$$

and then

$$\begin{aligned}\|x_{\tau(n)+1} - z_0\|^2 &\leq \beta_{\tau(n)}\left\|x_{\tau(n)} - z_0\right\|^2 + (1-\beta_{\tau(n)})\left\|y_{\tau(n)} - z_0\right\|^2\\ &\leq \big(1 - (1-\beta_{\tau(n)})\alpha_{\tau(n)}\big)\left\|x_{\tau(n)} - z_0\right\|^2\\ &\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\left\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\right\rangle.\end{aligned}$$

From $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have that

$$(1-\beta_{\tau(n)})\alpha_{\tau(n)}\left\|x_{\tau(n)} - z_0\right\|^2 \leq 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle.$$

Since $(1-\beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}\left\|x_{\tau(n)} - z_0\right\|^2 &\leq 2\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle\\ &= 2\langle u_{\tau(n)} - u, y_{\tau(n)} - z_0\rangle + 2\langle u - z_0, y_{\tau(n)} - z_0\rangle.\end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty}\left\|x_{\tau(n)} - z_0\right\|^2 \leq 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$. From (3.25), we also have that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.3 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

4. Applications

Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x \rangle \leq f(y), \ \forall y \in H\}$$

for all $x \in H$. By Rockafellar [18], it is shown that ∂f is maximal monotone. Let C be a non-empty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Then $i_C : H \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function on H and hence ∂i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$ as follows:

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x, \quad \forall x \in H, \ \lambda > 0.$$

On the other hand, for any $u \in C$, we also define the normal cone $N_C(u)$ of C at u as follows:

$$N_C(u) = \{z \in H : \langle z, y - u \rangle \leq 0, \ \forall y \in C\}.$$

Then we have that for any $x \in C$

$$\begin{aligned} \partial i_C(x) &= \{z \in H : i_C(x) + \langle z, y - x \rangle \leq i_C(y), \ \forall y \in H\} \\ &= \{z \in H : \langle z, y - x \rangle \leq 0, \ \forall y \in C\} \\ &= N_C(x). \end{aligned}$$

Thus we have that

$$\begin{aligned} u = J_\lambda x &\Leftrightarrow (I + \lambda \partial i_C)^{-1} x = u \Leftrightarrow x \in u + \lambda \partial i_C(u) \\ &\Leftrightarrow x \in u + \lambda N_C(u) \Leftrightarrow x - u \in \lambda N_C(u) \\ &\Leftrightarrow \langle x - u, y - u \rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow P_C(x) = u. \end{aligned}$$

Putting $B = \partial i_C$ in Theorem 3.1, we have $J_\lambda = P_C$. Thus we obtain the following theorem from Theorem 3.1.

Theorem 4.1. *Let H_1 and H_2 be Hilbert spaces and let C be a non-empty, closed and convex subset of H_1. Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$.

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)SP_C(I - \lambda_n A^*(I-T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in F(S) \cap A^{-1}F(T)$, where $z_0 = P_{F(S)\cap A^{-1}F(T)}u$.

Proof. Set $B = \partial i_C$ in Theorem 3.1. Then we have that $J_\lambda = P_C$ for all $\lambda > 0$. Thus we have the desired result from Theorem 3.1. □

Replacing a widely more generalized hybrid mapping in Theorem 3.1 by a generalized hybrid mapping, we have the following theorem.

Theorem 4.2. *Let H_1 and H_2 be Hilbert spaces and let C be a non-empty closed convex subset of H_1. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that $D(B) \subset C$ and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let S be a generalized hybrid mapping from C into C. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)SJ_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in B^{-1}0 \cap F(S) \cap A^{-1}F(T)$, where $z_0 = P_{B^{-1}0\cap F(S)\cap A^{-1}F(T)}u$.

Proof. Since $S : C \to C$ is generalized hybrid, there exist $s, t \in \mathbb{R}$ such that

$$s\|Sx - Sy\|^2 + (1-s)\|x - Sy\|^2 \leq t\|Sx - y\|^2 + (1-t)\|x - y\|^2$$

for all $x, y \in C$. This implies that

$$s\|Sx - Sy\|^2 + (1-s)\|x - Sy\|^2 - t\|Sx - y\|^2 - (1-t)\|x - y\|^2 \leq 0.$$

Since (1) $\alpha+\beta+\gamma+\delta = s+(1-s)-(1-t)-t = 0$, $\alpha+\beta = s-(1-s) = 1$ and $\varepsilon+\eta = 0$ in Theorem 3.1 are satisfied, we have the desired result from Theorem 3.1. □

We also get the following theorem from Theorem 4.2.

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces and let C be a non-empty closed convex subset of H_1. Let $S : C \to C$ be a nonexpansive mapping and let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)P_C(I - \lambda_n A^*(I-T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point z_0 of $F(S) \cap A^{-1}F(T)$, where $z_0 = P_{F(S)\cap A^{-1}F(T)}u$.

Let C be a non-empty, closed and convex subset of a real Hilbert space H, and let $f : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following equilibrium problem: Find $z \in C$ such that

$$f(z, y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of such $z \in C$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{z \in C : f(z, y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.

We know the following lemmas; see, for instance, [4] and [8].

Lemma 4.1 ([4]). *Let C be a non-empty closed convex subset of H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

Lemma 4.2 ([8]). *For $r > 0$ and $x \in H$, define the resolvent $T_r : H \to C$ of f for $r > 0$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}.$$

Then, the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) *$F(T_r) = EP(f)$;*

(iv) *$EP(f)$ is closed and convex.*

Takahashi, Takahashi and Toyoda [21] showed the following. See [1] for a more general result.

Lemma 4.3 ([21]). *Let C be a non-nempty, closed and convex subset of a Hibert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define A_f as follows:*

$$A_f(x) = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle,\ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}(0)$ and A_f is maximal monotone with the domain of A_f in C. Furthermore,

$$T_r(x) = (I + rA_f)^{-1}(x), \quad \forall r > 0.$$

We obtain the following theorem from Theorem 3.1.

Theorem 4.4. *Let H_1 and H_2 be Hilbert spaces and let C be a non-empty closed convex subset of H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 4.3. Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into C which satisfies the conditions* (1) *or* (2)*:*

(1) *$\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$;*

(2) *$\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$.*

Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n) S T_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfy

$$0 < a \le \lambda_n \le b < \frac{1}{\|A\|^2}, \quad 0 < c \le \beta_n \le d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad and \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in EP(f) \cap F(S) \cap A^{-1}F(T)$, where $z_0 = P_{EP(f)\cap F(S)\cap A^{-1}F(T)}u$.

Proof. Define A_f for the bifunction f and set $B = A_f$ in Theorem 3.1. Thus we have the desired result from Theorem 3.1. □

Acknowledgements. The authors would like to thank the referee for his careful reading.

References

[1] K. Aoyama, Y. Kimura, and W. Takahashi, *Maximal monotone operators and maximal monotone functions for equilibrium problems*, J. Convex Anal. **15** (2008), 395–409.

[2] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[3] ———, *On a strongly nonexpansive sequence in Hilbert spaces*, J. Nonlinear Convex Anal. **8** (2007), 471–489.

[4] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[5] C. Byrne, Y. Censor, A. Gibali, and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), 759–775.

[6] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms **8** (1994), 221–239.

[7] Y. Censor and A. Segal, *The split common fixed-point problem for directed operators*, J. Convex Anal. **16** (2009), 587–600.

[8] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[9] H. Cui and F. Wang, *Strong convergence of the gradient-projection algorithm in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 245–251.

[10] K. Eshita and W. Takahashi, *Approximating zero points of accretive operators in general Banach spaces*, JP J. Fixed Point Theory Appl. **2** (2007), 105–116.

[11] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[12] M. Hojo, T. Suzuki, and W. Takahashi, *Fixed point theorems and convergence theorems for generalized hybrid non-self mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 363–376.

[13] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 71–87.

[14] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[15] P. E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal. **16** (2008), 899–912.

[16] A. Moudafi, *The split common fixed point problem for demicontractive mappings*, Inverse Problems **26** (2010), 055007, 6 pp.

[17] N. Nadezhkina and W. Takahashi, *Strong convergence theorem by hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings*, SIAM J. Optim. **16** (2006), 1230–1241.

[18] R. T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. **33** (1970), 209–216.

[19] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (2007), 506–515.

[20] ———, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. **69** (2008), 1025–1033.

[21] S. Takahashi, W. Takahashi, and M. Toyoda, *Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces*, J. Optim. Theory Appl. **147** (2010), 27–41.

[22] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[23] ———, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).

[24] ———, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[25] W. Takahashi, *Strong convergence theorems for maximal and inverse-strongly monotone mappings in Hilbert spaces and applications*, J. Optim. Theory Appl. **157** (2013), 781–802.

[26] W. Takahashi, N.-C. Wong, and J.-C. Yao, *Weak and strong mean convergence theorems for extended hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 553–575.

[27] W. Takahashi, H.-K. Xu, and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal., to appear.

[28] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109–113.

[29] ———, *A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem*, Inverse Problems **22** (2006), 2021–2034.

(Mayumi Hojo) Center for Promotion of educational Innovation, Shibaura Institute of Technology, 307 Fukasaku Minuma-ku, Saitama 337-8570, Japan
E-mail address: mayumi-h@shibaura-it.ac.jp

(Wataru Takahashi) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp; wataru@a00.itscom.net

Received July 15, 2014