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REMARKS ON THE SET OF POLES ON A POINTED COMPLETE SURFACE

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ABSTRACT. M. Tanaka ([2]) determined the radius of the ball which consists of all poles in a von Mangoldt surface of revolution. The purpose of the present paper is to give an alternative proof and a geometrical meaning of the radius. Furthermore, we estimate the radius of the maximal ball consisting of poles in a complete surface homeomorphic to the plane under a certain condition.

1. Introduction

Let (M, p) be a pointed complete Riemannian manifold with a base point at $p \in M$. We say that a pointed complete Riemannian manifold (M, p) with dimension 2 is a *surface of revolution with a vertex at* p if the Gaussian curvature $G(q)$ of M is constant on the metric t-circle $S_p(t) := \{q \in M \mid d(p, q) = t\}$ around p for $t > 0$, say $G(t)$. Namely, there exists a polar coordinates around p such that the Riemannian metric g on a surface of revolution M is expressed as

$$g : ds^2 = dr^2 + m(r)^2 d\theta^2, \tag{1.1}$$

where the smooth function $m : [0, \infty) \to [0, \infty)$ satisfies the differential equation $m''(t) + G(t)m(t) = 0; m(0) = 0, m'(0) = 1$ and is extendable to an odd function around 0. The length of $S_p(t)$ is $2\pi m(t)$.

Let $\gamma : I \to M$ be a geodesic with unit speed in a complete Riemannian manifold M. We say that $\gamma(t_0)$ and $\gamma(t_1)$ are called a *conjugate pair* along γ if there exists a non-trivial Jacobi field along γ that vanishes at $\gamma(t_0)$ and $\gamma(t_1)$. A point $q \in M$ is called a *pole* if there exist no points conjugate to q along every geodesic $\gamma : [0, \infty) \to M$ emanating from $q = \gamma(0)$. In a surface of revolution M the vertex is a pole if M is homeomorphic to the plane. H. von Mangoldt ([4]) proved that the set of all poles of every connected component of two-sheeted hyperboloid of revolution is a non-trivial closed ball centered at its vertex. We discuss his result under a general

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setting. Put

$$r(M) := \sup\{r \mid q \in M \text{ is a pole if } d(p,q) < r\}.$$

If M is a surface of revolution homeomorphic to the plane with a vertex at p, then $r(M)$ is equal to the distance between p and the farthest pole in M ([3, Lemma 1.1]). Tanaka ([2]) generalized von Mangoldt's result and showed a necessary and sufficient condition for $r(M) > 0$, and found the equation which determines the $r(M)$ for a von Mangoldt surface. Here a *von Mangoldt surface* is by definition a surface of revolution such that the Gaussian curvature is monotone non-increasing with respect to the distance to its vertex. The purpose of the present paper is to make his proof much simpler and the geometrical meaning of the equation clearer. Moreover, using these theorems, we estimate $r(M)$ for a Riemannian 2-manifold M whose Gaussian curvature is pinched by those of two von Mangoldt surfaces. More precisely, we prove the following.

Theorem 1.1. *Let (M,p) be a pointed complete surface homeomorphic to the plane. Assume that there exist two von Mangoldt surfaces (M_i,p_i) $(i=1,2)$ such that $G_1(d(p,q)) \le G(q) \le G_2(d(p,q))$ for all $q \in M$. Then, p is a pole in M, and*

$$r(M_1) \ge r(M) \ge r(M_2).$$

In Section 2 we review the theory of stable Jacobi field. In particular, we study when we can extend a disconjugate interval. In Section 3 we also review the theory of Jacobi field on the surface of revolution. In Section 4 we give an alternative proof of Tanaka's characterization of $r(M) > 0$ for a surface of revolution. Actually, we prove the following.

Theorem 1.2. *(*[2, Theorem 1.10]*.) Let M be a surface of revolution with a vertex at p. Then, $r(M) > 0$ if and only if M satisfies*

$$\int_1^\infty m(t)^{-2}dt < \infty \quad \textit{and} \quad \liminf_{t\to\infty} m(t) > 0.$$

Our proof is based on the disconjugate property for the solution of the differential equation of Jacobi type, and is seemed to be simpler than the original one. In Section 5 we prove the following.

Theorem 1.3. *Let M be a von Mangoldt surface such that $\int_1^\infty m(r)^{-2}dr < \infty$. Let*

$$y_\infty(t) = m(t)\int_t^\infty m(r)^{-2}dr \ (t > 0).$$

Then, $c(m) := 2y'_\infty(0)$ exists. Set

$$\bar{F}(x) := c(m) - \int_x^\infty m(r)^{-2}dr.$$

We then have the following.

(i) *If* $c(m) \leq 0$, *then* $r(M) = \infty$.
(ii) *If* $c(m) > 0$, *then* $r(M)$ *is the unique zero point of the function* $\bar{F}$.

Tanaka first proved Theorem 1.3 ([2, Theorem 2.1]), where he defined the constant $c(m)$ as:

$$c(m) := \int_0^\infty \frac{m(r) - rm'(r)}{m(r)^3} dr.$$

However, the geometrical meaning of this constant arising in the equation was not explained. We emphasize that the constant is expressed by means of the stable Jacobi field. Our method is based on the disconjugate property of Jacobi field along a ray emanating from the vertex. In Section 6 we prove Theorem 1.1.

2. The disconjugate properties for Jacobi fields

Let M be a complete Riemannian 2-manifold. Let $\gamma : [0, \infty) \to M$ be a unit speed geodesic. Let $\{\mathbf{e}_1(t) = \gamma'(t), \mathbf{e}_2(t)\}$ be an orthonormal parallel frame field along γ. We say that a vector field $Y(t)$ along γ is a *Jacobi field* if it satisfies the Jacobi equation $\nabla_{\gamma'} \nabla_{\gamma'} Y + R(Y, \gamma')\gamma' = \mathbf{0}$, where $R : \mathcal{X}(M)^3 \to \mathcal{X}(M)$ denotes the Riemannian curvature tensor. Define a linear map $F_t : M_{\gamma(t)} \to M_{\gamma(t)}$ by $F_t(\mathbf{x}) = R\big(\mathbf{x}, \gamma'(t)\big)\gamma'(t)$. We then have $F_t(\mathbf{e}_1) = \mathbf{0}, g\big(R(\mathbf{e}_2, \gamma')\gamma', \mathbf{e}_1\big) = 0, g\big(R(\mathbf{e}_2, \gamma')\gamma', \mathbf{e}_2\big) = G\big(\gamma(t)\big)$. Let $\mathcal{J}_\gamma$ be the set of all Jacobi vector fields along γ, which forms a vector space over $\mathbb{R}$. If $Y(t) = x(t)\mathbf{e}_1(t) + y(t)\mathbf{e}_2(t) \in \mathcal{J}_\gamma$, we then have

$$x''(t) = 0 \Longleftrightarrow x(t) = c_1 t + c_2, \tag{J$_1$}$$

$$y''(t) + G\big(\gamma(t)\big)y(t) = 0. \tag{J$_\mathrm{G}$}$$

We have following contents on the disconjugate property for later use by digesting Chapter XI in [1]. The differential equation ($\mathrm{J_G}$) is said to be *disconjugate* on I if every non-trivial solution $y : I \to \mathbb{R}$ of ($\mathrm{J_G}$) along γ vanishes at most once, where $y(t)$ means that $Y(t) = y(t)\mathbf{e}_2(t) \in \mathcal{J}_\gamma$. The disconjugate property is stated as follows: For each solution y_s of ($\mathrm{J_G}$) on I with $y_s(s) = 0$ and $y_s'(s) \neq 0$, we have $y_s(t) \neq 0$ for all $t \in I \setminus \{s\}$. This property implies that the solution of ($\mathrm{J_G}$) is uniquely determined by its values at two distinct points in I.

We have a general solution y of ($\mathrm{J_G}$) from a non-trivial solution z by using the variation method of constants as following formula:

$$y(t) = z(t) \left(\int z(t)^{-2} dt C_1 + C_2 \right), \tag{2.1}$$

where C_1 and C_2 are constants. Assume that ($\mathrm{J_G}$) is disconjugate on I and $c \in I$. Let y_c be the solution of ($\mathrm{J_G}$) with $y_c(c) = 0$ and $y_c'(c) = 1$. Then the solution

y_s of $(\mathrm{J_G})$ with $y_s(c) = 1$ and $y_s(s) = 0$ is given by the following formula for each $s \in I \setminus \{c\}$:

$$y_s(t) = y_c(t) \int_t^s y_c(w)^{-2} dw \tag{2.2}$$

for all t such that c is not in between t and s. We have from (2.2), $y_u(t) - y_s(t) = y_c(t) \int_s^u y_c(w)^{-2} dw$ for all $t \in [c, u]$. Differentiating it at $t = c$, we have

$$y'_u(c) - y'_s(c) = \int_s^u y_c(w)^{-2} dw. \tag{2.3}$$

Lemma 2.1. *Let $c < s$ and y_s be defined as in (2.2). Then*

$$y'_s(c) \to -\infty \quad \text{as} \quad s \to c + 0.$$

If the orientation of parameter is reversed, we then have $y'_s(c) \to +\infty$ as $s \to c-0$.

Lemma 2.2. *Assume that $(\mathrm{J_G})$ is disconjugate on I. Let $c < s$ $(c, s \in I)$ and let $y_s : [c, s] \to \mathbb{R}$ be defined as in (2.2). If $y : [c, s] \to \mathbb{R}$ satisfies $(\mathrm{J_G})$ such that $y(c) = 1$ and $y(t) \neq 0$ for all $t \in [c, s]$, then $y(t) > y_s(t)$ for all $t \in (c, s]$.*

Lemma 2.3. *Assume that there exists a solution $y : I \to \mathbb{R}$ of $(\mathrm{J_G})$ with $y(t) \neq 0$ for all $t \in I$. Then $(\mathrm{J_G})$ is disconjugate on I.*

Let $G : \mathbb{R} \to \mathbb{R}$ be the function as defined in $(\mathrm{J_G})$.

Theorem 2.1. *Assume that $(\mathrm{J_G})$ is disconjugate on $(c - \varepsilon, \infty)$ for some positive ε. Let $y_s, y_{c-\varepsilon} : \mathbb{R} \to \mathbb{R}$ be the solutions of $(\mathrm{J_G})$ with $y_s(c) = 1$, $y_s(s) = 0$ and with $y_{c-\varepsilon}(c) = 1$, $y_{c-\varepsilon}(c-\varepsilon) = 0$, respectively. Then $y_s(t)$ converges to $y(t)$ as $s \to \infty$ for each $t \in \mathbb{R}$. Moreover, $y : \mathbb{R} \to \mathbb{R}$ is the solution of $(\mathrm{J_G})$ such that $y_{c-\varepsilon}(t) \geq y(t) > y_s(t)$ for all $t \in (c, s)$. (cf. Figure 1 as the case $c < u < s$.)*

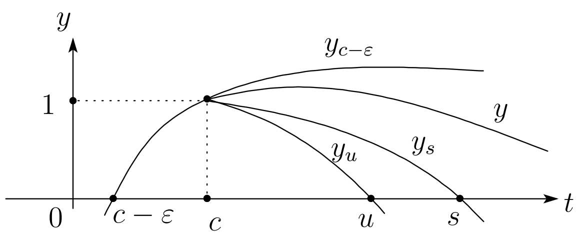

FIGURE 1. The solution of $(\mathrm{J_G})$ in the case of $c < u < s$.

Combining Theorem 2.1 and (2.2), we have the following.

Corollary 2.1. *Assume that $(\mathrm{J_G})$ is disconjugate on $(c - \varepsilon, \infty)$ for some positive ε. Let y_s for each $s > c$ be defined as in (2.2). Then $y_s(t)$ for each $t \in [c, \infty)$ converges* *to* $y_\infty(t)$ *as* $s \to \infty$*, which is the solution of* $(\mathrm{J_G})$*. Moreover,* $y_\infty(t)$ *is given by the following formula:*

$$y_\infty(t) = y_c(t) \int_t^\infty y_c(w)^{-2} dw \quad (t > c).$$

Remark 2.1. For the statements in Theorem 1.3, $m(t)$ is equal to $y_0(t)$ as above, that is, $m(t)\mathbf{e}_2(t) \in \mathcal{J}_\mu$, where μ is some unit speed meridian, and so y_∞ is the solution of $(\mathrm{J_G})$ along a ray emanating from the vertex.

Conversely, $\int_{c+1}^\infty y_c(w)^{-2}dw < \infty$ shows that there exists a positive ε such that $(\mathrm{J_G})$ is disconjugate on $(c-\varepsilon, \infty)$. The following corollary will play an important role in our proof of Theorem 1.2.

Corollary 2.2. *Assume that* $(\mathrm{J_G})$ *is disconjugate on* $[c, \infty)$ *and* $\int_{c+1}^\infty y_c(w)^{-2}dw < \infty$*. Then,* $[c, \infty)$ *is extendable to a disconjugate interval* $[c-\varepsilon, \infty)$ *of* $(\mathrm{J_G})$ *for some positive* ε*.*

3. The properties of Jacobi fields on a surface of revolution

Let M be a complete surface of revolution with a vertex at p, homeomorphic to the plane, whose metric is expressed as (1.1). It is known that the Gaussian curvature of M at each point $q \in S_p(t)$ is given by $G(t) = -m''(t)/m(t)$. Let $\gamma : [0, \infty) \to M$ be a unit speed geodesic and put $\gamma(t) := \big(r(t), \theta(t)\big)$ for all $t \in [0, \infty)$. Let ν be a constant. Since $\theta'(t) = \nu\big/m(r(t))^2$ from the differential equation for a geodesic and (1.1), we have

$$r'(t) = \pm\sqrt{m(r(t))^2 - \nu^2}\big/m(r(t)). \tag{3.1}$$

A 1-parameter family of geodesics $\gamma_\varepsilon : [0, \infty) \times (-\varepsilon_0, \varepsilon_0) \to M$, $\gamma_\varepsilon(t) = \big(r(t), \theta(t) + \varepsilon\big)$ is a geodesic variation. Thus, $(\partial/\partial\varepsilon)_{\varepsilon=0}\gamma_\varepsilon(t) = (\partial/\partial\theta)_{\gamma(t)} \in \mathcal{J}_\gamma$. Put $(\partial/\partial\theta)_{\gamma(t)} := a(t)\mathbf{e}_1(t) + b(t)\mathbf{e}_2(t)$. We have the following from $(\mathrm{J_1})$

$$\begin{cases} a(t) = g_{\gamma(t)}\left(\partial/\partial\theta, \mathbf{e}_1\right) = m(r(t)) \cos\xi(t) = \nu \\ b(t) = g_{\gamma(t)}\left(\partial/\partial\theta, \mathbf{e}_2\right) = m(r(t)) \sin\xi(t) = \pm\sqrt{m(r(t))^2 - \nu^2}\ , \end{cases}$$

where $\xi(t)$ denotes the angle between $\gamma'(t)$ and $\partial/\partial\theta(\gamma(t))$. The first formula is called Clairaut's relation.

Let $\tau_q : [0, \infty) \to M$ for each $q \in M$ be the geodesic emanating from $q = \tau_q(0)$ through p and let $\mu_q : [0, \infty) \to M$ denote the meridian emanating from $p = \mu_q(0)$ through q. With the above notation, we state the following four lemmas in [2].

Lemma 3.1. *(Compare Lemma 1.1 in* [2]*.) Let* $\gamma : [0, \infty) \to M$ *be a geodesic. If* $r'(t) = 0$ *at two distinct parameter values, then* γ *is not a ray.*

Proof. Let the first zero point of $r' : (0, \infty) \to \mathbb{R}$ be t_0 and the second t_1. From (3.1) and that $y(t) = \sqrt{m(r(t))^2 - \nu^2}$ is the solution of $(\mathrm{J_G})$ $\gamma(t_0)$ and $\gamma(t_1)$ is a conjugate pair along γ. $\square$

Lemma 3.2. *(See Lemma 1.2 in* [2]*.) Let $\gamma : [0, \infty) \to M$ be a geodesic. If $r_0 := \lim_{t\to\infty} d(p, \gamma(t)) < \infty$, then $m'(r_0) = 0$, that is, the parallel circle $S_p(r_0)$ is a geodesic.*

Lemma 3.3. *(See Lemma 1.3 in* [2]*.) If $\liminf_{t\to\infty} m(t) = 0$, then $\mu_q|[d(p, q), \infty)$ for every $q \in M \setminus \{p\}$ is a unique ray emanating from q.*

We give an alternative proof for the following lemma. Put $\rho := d(p, q)$.

Lemma 3.4. *(Compare Lemma 1.4 in* [2]*.) If $\int_1^\infty m(r)^{-2}dr = \infty$, then τ_q is not a ray for any $q \in M \setminus \{p\}$.*

Proof. Let $y_\rho(t) = m(t - \rho)$ for all $t \geq 0$. Then y_ρ is the solution of $(\mathrm{J_G})$ along τ_q with $y_\rho(\rho) = m(0) = 0$, $y'_\rho(\rho) = m'(0) = 1$. From (2.2), the solution y_s of $(\mathrm{J_G})$ with $y_s(\rho) = 1$ and $y_s(s) = 0$ is written as follows:

$$y_s(t) = m(t - \rho) \int_t^s m(w - \rho)^{-2}dw = m(t - \rho) \int_{t-\rho}^{s-\rho} m(r)^{-2}dr$$

for all $t > \rho$. If τ_q is a ray, then there exists no conjugate pair along τ_q. By Corollary 2.1, we have $\int_{t-\rho}^\infty m(r)^{-2}dr < \infty$, a contradiction. $\square$

4. The proof of Theorem 1.2

In this section we give an alternative proof for Theorem 1.2. Let M be a complete surface of revolution with a vertex at p, homeomorphic to the plane. Combining Lemmas 3.3 and 3.4, we have the following Corollary 4.1. We give a necessary condition that there exists a pole $q \in M \setminus \{p\}$.

Corollary 4.1. *If $\liminf_{t\to\infty} m(t) = 0$ or $\int_1^\infty m(r)^{-2}dr = \infty$, then the vertex p is the unique pole on M.*

We next prove the converse of Corollary 4.1. In order to do so we need some preparations. For a point $q \in M \setminus \{p\}$ and for each $\nu \in [-m(\rho), m(\rho)]$ we define two geodesics $\beta_\nu, \gamma_\nu : [0, \infty) \to M$ emanating from q, whose velocity vectors at $t = 0$ are given by

$$\beta'_\nu(0) = \sqrt{1 - (\nu/m(\rho))^2}\, (\partial/\partial r)_{\beta_\nu(0)} + \nu/m(\rho)^2\, (\partial/\partial\theta)_{\beta_\nu(0)}\,, \tag{4.1}$$

$$\gamma'_\nu(0) = -\sqrt{1 - (\nu/m(\rho))^2}\, (\partial/\partial r)_{\gamma_\nu(0)} + \nu/m(\rho)^2\, (\partial/\partial\theta)_{\gamma_\nu(0)}\,, \tag{4.2}$$

respectively. Thus we have smooth 1-parameter family of geodesics whose variation vector fields are Jacobi fields $X_\nu(t) := \partial/\partial\nu(\beta_\nu(t))$ and $Y_\nu(t) := \partial/\partial\nu(\gamma_\nu(t))$ along β_ν and γ_ν, respectively. The next proposition contains Lemma 3.4 as its special case.

Proposition 4.1. *If $\int_1^\infty m(r)^{-2}dr = \infty$, then for all point $q \in M \setminus \{p\}$ the geodesic $\gamma_\nu|[0,\infty)$ is not a ray emanating from $q = \gamma_\nu(0)$ for any $\nu \in (-m(\rho), m(\rho))$.*

Proof. When $\nu \neq 0$, if $\lim_{t\to\infty} r(t) = r_0 < \infty$, then γ_ν is not a ray by Lemma 3.2. Let $\lim_{t\to\infty} r(t) = \infty$. In the case there exist more than one zero points of r', Lemma 3.1 implies that γ_ν is not a ray. In the case where r' has a zero only at t_0, we observe that

$$y_{t_0}(t) = \sqrt{m(r(t))^2 - \nu^2}/m'(r(t_0))$$

is the solution of $(\mathrm{J_G})$ along γ_ν with $y_{t_0}(t_0) = 0$ and $y'_{t_0}(t_0) = 1$. If y_s is the solution of $(\mathrm{J_G})$ with $y_s(s) = 0$ and $y_s(t_0) = 1$, we then have from (2.2) that

$$\begin{aligned} y_s(t) &= m'(r(t_0))\sqrt{m(r(t))^2 - \nu^2}\int_t^s \frac{1}{m(r(w))^2 - \nu^2}dw \\ &= m'(r(t_0))\sqrt{m(r(t))^2 - \nu^2}\int_{r(t)}^{r(s)} \frac{m(r)}{(m(r)^2 - \nu^2)^{3/2}}dr \\ &\geq m'(r(t_0))\sqrt{m(r(t))^2 - \nu^2}\int_{r(t)}^{r(s)} m(r)^{-2}dr \end{aligned}$$

for all $t \in (t_0, s)$. By assumption, $y_s(t)$ does not converge as $s \to \infty$. Therefore, $(\mathrm{J_G})$ is not disconjugate on $(t_0 - \varepsilon, \infty)$ for any positive ε. Thus, γ_ν is not a ray. When $\nu = 0$, τ_q is not a ray by Lemma 3.4. □

Recall that $\beta : [0,\infty) \to M, \beta(t) = \big(r(t), \theta(t)\big)$ and γ are geodesics whose velocity vectors at $t = 0$ are given in (4.1) and (4.2), respectively.

Lemma 4.1. *(Compare Lemma 1.5 in* [2]*.) If a geodesic $\beta : [0,\infty) \to M$ does not pass through p, and if $r'(t) \neq 0$ for all $t \in (0,\infty)$, then β contains no conjugate pair.*

Proof. Clearly, $y(t) = \sqrt{m(r(t))^2 - \nu^2}$ is the solution of $(\mathrm{J_G})$ along β. If $r'(t) \neq 0$ for all $t \in (0,\infty)$, then $y(t) \neq 0$ on $(0,\infty)$ from (3.1). By Lemma 2.3, $(\mathrm{J_G})$ is disconjugate on $(0,\infty)$. □

From now on, let $\liminf_{t\to\infty} m(t) := m_0 > 0$ and β be a geodesic with $r(\beta(0)) = r_1, \beta'(0) = (0, 1/m(r_1))$. Fix a k with $0 < k < 1$. Then there exists a number $a_1 > 0$ (cf. Figure 2) such that if $0 \leq r_1 \leq a_1$, then $m(r_1) < km_0$ and $m(r_1) < m(r)$ for all $r > r_1$. We have the following.

Lemma 4.2. *If $0 \leq r_1 \leq a_1 < r_2$ and $r_2 := r(t_2)$, then $\int_{t_2}^\infty 1/\big(m(r(t))^2 - m(r_1)^2\big)dt < \infty$ if and only if $\int_{r_2}^\infty m(r)^{-2}dr < \infty$.*

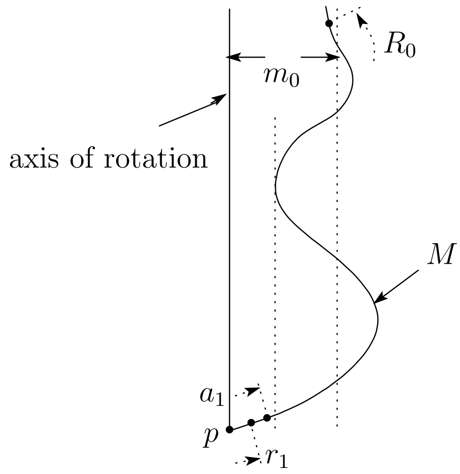

FIGURE 2. The number r_1, a_1 and R_0.

Proof. (cf. Figure 2.) Since $r'(t) = \sqrt{m(r(t))^2 - m(r_1)^2}/m(r(t))$ from (3.1),

$$\int_u^v \frac{1}{m(r(t))^2 - m(r_1)^2} dt = \int_{r(u)}^{r(v)} \frac{1}{m(r)^2 - m(r_1)^2} \cdot \frac{m(r)}{\sqrt{m(r)^2 - m(r_1)^2}} dr.$$

It follows

$$\int_{t_2}^v \frac{1}{m(r(t))^2 - m(r_1)^2} dt = \int_{r_2}^{r(v)} \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr \geq \int_{r_2}^{r(v)} m(r)^{-2} dr.$$

Therefore, if the right hand side diverges, then the left hand side diverges. There exists an $R_0 > 0$ such that if $R_0 < r$, then $m(r_1) < km(r)$. If $R_0 < r(u) < r(v)$, then

$$\int_{r(u)}^{r(v)} \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr \leq \frac{1}{(1-k^2)^{3/2}} \int_{r(u)}^{r(v)} m(r)^{-2} dr.$$

Therefore, if the right hand side converges, then the left hand side converges. □

Recall that $y(t) = \sqrt{m(r(t))^2 - m(r_1)^2}/m'(r_1)$ is the solution of $(\mathrm{J_G})$ along β with $y(0) = 0$ and $y'(0) = 1$. It follows from (2.2) that for each $s > 0$ the solution y_s of $(\mathrm{J_G})$ with $y_s(0) = 1$ and $y_s(s) = 0$ can be written as follows:

$$y_s(t) = m'(r_1)\sqrt{m(r(t))^2 - m(r_1)^2} \int_t^s \frac{1}{m(r(w))^2 - m(r_1)^2} dw \ (s > t > 0).$$

By putting $c = 0$, we have the following from (2.3).

Lemma 4.3. *Let $u > s > 0$. Then it follows*

$$y_u'(0) - y_s'(0) = \int_s^u y(w)^{-2} dw = \int_s^u \frac{m'(r_1)^2}{m(r(w))^2 - m(r_1)^2} dw. \tag{4.3}$$

In particular, if $\int_s^\infty y(w)^{-2} dw < \infty$, then $y_\infty'(0) = \int_s^\infty y(w)^{-2} dw + y_s'(0)$.

Here $y_s(t)$ and $y_\infty(t)$ are defined as in (2.2), Corollary 2.1, respectively. The values $y_s(t), y_\infty(t)$ and $y_s'(0), y_\infty'(0)$ depend also on r_1. In order to show that these values, especially, $y_s'(0), y_\infty'(0)$ are continuous on r_1 in some neighborhood of p, we use the following notations: $y_{r_1,\infty} := y_\infty, y_{r_1,s} := y_s$, and so on. Let $0 \le r_1 < a_1$ and $\int_s^\infty m(r)^{-2}dr < \infty$. Then

$$h(r_1) := \int_s^\infty \frac{1}{m(r(w))^2 - m(r_1)^2} dw < \infty$$

by Lemma 4.2. The function $y_{r_1,\infty}$ is the solution of ($\mathrm{J_G}$) along β as stated in Remark 2.1.

Lemma 4.4. *Assume that* $\int_s^\infty m(r)^{-2}dr < \infty$. *Then there exists a neighborhood* U *of the vertex* p *such that* $h(r(q))$ *is continuous in* $U \ni q$.

Proof. Set $U = \{q \in M \mid r(q) < a_1\}$. There exists for any $\varepsilon > 0$ an $R_2 > 0$ such that if $0 < r_1 < a_1$, then

$$\int_{R_2}^\infty \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr \le \frac{1}{(1-k^2)^{3/2}} \int_{R_2}^\infty m(r)^{-2} dr < \varepsilon/3.$$

We have

$$\begin{aligned} h(r_1) &= \int_{r(\beta(s))}^\infty \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr \\ &= \int_{r(\beta(s))}^{R_2} \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr + \int_{R_2}^\infty \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr. \end{aligned}$$

Let $\bar\beta : [0,\infty) \to M$ be the geodesic with $r(\bar\beta(0)) = \bar r_1, \bar\beta'(0) = (0, 1/m(\bar r_1)), r_1 \risingdotseq \bar r_1$. Then

$$\begin{aligned} h(r_1) - h(\bar r_1) = &\int_{r(\beta(s))}^{R_2} \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr - \int_{r(\bar\beta(s))}^{R_2} \frac{m(r)}{(m(r)^2 - m(\bar r_1)^2)^{3/2}} dr \\ &+ \int_{R_2}^\infty \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr - \int_{R_2}^\infty \frac{m(r)}{(m(r)^2 - m(\bar r_1)^2)^{3/2}} dr \end{aligned}$$

and

$$\begin{aligned} &\big| h(r_1) - h(\bar r_1) \big| \\ &\quad < \left| \int_{r(\beta(s))}^{R_2} \frac{m(r)}{(m(r)^2 - m(r_1)^2)^{3/2}} dr - \int_{r(\bar\beta(s))}^{R_2} \frac{m(r)}{(m(r)^2 - m(\bar r_1)^2)^{3/2}} dr \right| + 2\varepsilon/3. \end{aligned}$$

There exists a $\delta > 0$ such that if $|\, r_1 - \bar r_1 \,| < \delta$, then $\big| h(r_1) - h(\bar r_1) \big| < \varepsilon/3 + 2\varepsilon/3 = \varepsilon$. Thus, $h \circ r$ is continuous in U. □

As $u \to \infty$ in (4.3), we have $y'_{r_1,\infty}(0) = m'(r_1)^2 h(r_1) + y'_{r_1,s}(0)$. In this consequence, $y'_{r_1,\infty}(0)$ is continuous at $r_1 \in [0, a_1]$, where $y_{r_1,s}$ is the solution of ($\mathrm{J_G}$) along β with $y_{r_1,s}(s) = 0$ and $y_{r_1,s}(0) = 1$ for each $s > 0$. From Corollary 2.1 we have

$$y_{r_1,\infty}(t) = m'(r_1)\sqrt{m(r(t))^2 - m(r_1)^2} \int_t^\infty \frac{1}{m(r(w))^2 - m(r_1)^2} dw \ (t > 0).$$

We remark that the right hand side of the above equation is an expression of a Jacobi field on the interval $(0, \infty)$ and the expression is not available in any interval containing 0. We think that it is the restriction of a Jacobi vector field $y_{r_1,\infty}$ defined along a whole geodesic $\beta : (-\infty, \infty) \to M$. We can extend an interval with no conjugate pair as follows.

Lemma 4.5. *Assume that* $\int_1^\infty m(r)^{-2} dr < \infty$. *If a geodesic* $\beta : (-\infty, \infty) \to M$ *through* $q = \beta(0) \in U$ *is tangent to the parallel circle around* p *at* q, *that is,* $\beta'(0) = (0, 1/m(r_1))$, *then there exists a* $\delta_{r_1} > 0$ *such that there is no conjugate pair on* $(-\delta_{r_1}, \infty)$ *along the geodesic* β *where* $r_1 = r(\beta(0))$. *Furthermore,* δ_{r_1} *is continuous on* r_1.

Proof. We observe from Lemmas 4.3 and 4.4 that $y'_{r_1,\infty}(0)$ exists and that $h(r_1)$ is continuous on $r_1 \in [0, a_1)$. Since $y_{r_1,\infty}(0) = 1$ and $y'_{r_1,\infty}(0)$ exists, we can extend the disconjugate interval of $y_{r_1,\infty}$ as follows. If there are zeros of $y_{r_1,\infty}$, we then put $\delta_{r_1} := -t(r_1)$, where $t(r_1)$ is the maximum zero of zeros of $y_{r_1,\infty}$. Clearly, $t(r_1) < 0$. If there are no zeros, we put $\delta_{r_1} = \infty$. In this consequence, the interval which has no conjugate pairs extends from $[0, \infty)$ to $(-\delta_{r_1}, \infty)$ as showed in Corollary 2.2 and this is the maximal disconjugate interval. Since the solution of ($\mathrm{J_G}$) depends continuously on the initial condition, the function δ_{r_1} is continuous on r_1 (cf. Figure 3). $\square$

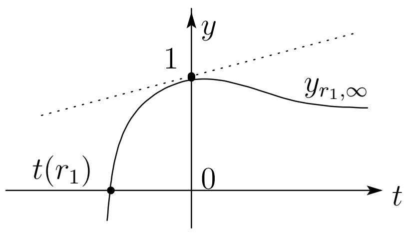

FIGURE 3. The maximum zero of zeros of $y_{r_1,\infty}$.

We enter our final stage to the proof of Theorem 1.2.

Lemma 4.6. *Assume that* $\liminf_{t\to\infty} m(t) > 0$ *and* $\int_1^\infty m(r)^{-2} dr < \infty$. *Then there exists a positive* b *such that any point* q *with* $d(p, q) \leq b$ *is a pole.*

Proof. By assumption that $\int_1^\infty m(r)^{-2} dr < \infty$, we have a $\delta_0 > 0$, where δ_0 is given by putting $r_1 = 0$ for δ_{r_1} in Lemma 4.5. There exists an $a_2 > 0$ such that if

$0 \le r_1 < a_2 < a_1$, then $|\delta_{r_1} - \delta_0| \le \delta_0/2$, that is, $\delta_{r_1} \ge \delta_0/2$. Put $b := \min\{a_2, \delta_0/2\}$. For any point q in the b-neighborhood of p, there is no conjugate pair along any geodesic emanating from q. For a geodesic $\beta : [0,\infty) \to M$ with $r(\beta(0)) = r_1 < b$ whose velocity vector at $t = 0$ is defined as (4.1), we have $y(t) = \sqrt{m(r(t))^2 - c^2} \ne 0$ on $(0,\infty)$ for any fixed $c \in [0, m(r_1)]$. Therefore, ($\mathrm{J_G}$) is disconjugate on $(0,\infty)$ along β by Lemma 2.3. For a geodesic $\gamma : [0,\infty) \to M$ whose velocity vector at $t = 0$ is defined as (4.2), the following is true. Let q_0 be a point such that $r'(q_0) = 0$, that is, $d(p, q_0) = d\big(p, \gamma([0,\infty))\big)$ with $r(q_0) < r_1$. Let q_1 be a point such that $d(p, q_1) = d(p, q)$, $q_1 \ne q$ and $q_1 \in \gamma([0,\infty))$. Since

$$d(q_0, q) = d(q, q_1)/2 \le \big(d(q,p) + d(p,q_1)\big)/2 \le b \le \delta_0/2 \le \delta_{r_1},$$

there also exist no points conjugate to q along γ by Lemma 4.5 (cf. Figure 4). Therefore, every point q in the b-neighborhood of p is a pole. $\square$

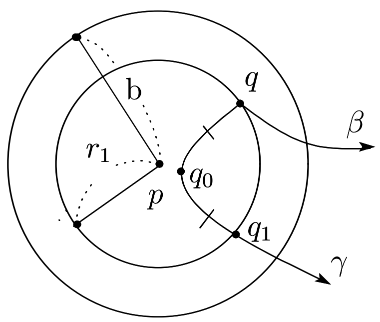

FIGURE 4. The geodesics β, γ emanating from q.

By Corollary 4.1 and Lemma 4.6, we have Theorem 1.2.

5. The proof of Theorem 1.3

In this section we prove Theorem 1.3. Let M be a von Mangoldt surface. We determine the number $r(M)$. In addition, we make the meaning of the constant $c(m)$ clear. The proof is based on the following lemma.

Lemma 5.1. ([3, Lemma 1.2].) *Let M be a von Mangoldt surface with a vertex at p. Let $q \in M \setminus \{p\}$. If the geodesic $\tau_q : [0,\infty) \to M$ emanating from $q = \tau_q(0)$ through p has no points conjugate to q along itself, then q is a pole.*

As a result of this lemma we have

$$r(M) = \max\{r(q) \mid \text{there are no points conjugate to } q = \tau_q(0) \text{ along } \tau_q\}$$

for a von Mangoldt surface M. We find the equation whose solution is $r(M)$. Since $m(0) = 0, m'(0) = 1$ and from (2.3) we have

$$y_u'(0) - y_s'(0) = \int_s^u m(r)^{-2} dr = \int_1^u m(r)^{-2} dr - \int_1^s m(r)^{-2} dr.$$

Thus,

$$y_u'(0) - \int_1^u m(r)^{-2} dr = y_s'(0) - \int_1^s m(r)^{-2} dr.$$

This shows that these values do not depend on parameter s. Then we can set

$$C = y_s'(0) - \int_1^s m(r)^{-2} dr = y_1'(0)$$

where C is a constant. From Corollary 2.1 and the assumption, both

$$y_\infty(t) = m(t) \int_t^\infty m(r)^{-2} dr \ (t > 0) \ \text{ and } \ y_\infty'(0) = \int_1^\infty m(r)^{-2} dr + C$$

exist. Let an $x > 0$ be a number such that the maximal disconjugate interval of $(\mathrm{J_G})$ along τ_q is $(-x, \infty)$. Then

$$\begin{aligned} y_\infty'(0) &= \int_1^\infty m(r)^{-2} dr + y_x'(0) - \int_1^x m(r)^{-2} dr \\ &= \int_x^\infty m(r)^{-2} dr + y_x'(0). \end{aligned}$$

Since the Gaussian curvature $G\big(\tau_q(t)\big)$ along τ_q is symmetric with respect to the

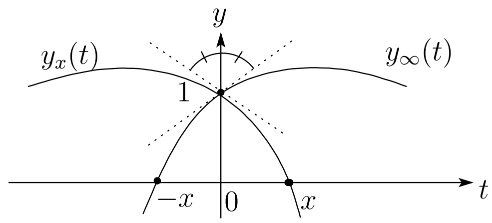

FIGURE 5. The relation of y_x to y_∞.

vertex p, the x satisfies $y_\infty'(0) = -y_x'(0)$ (cf. Figure 5). Since $y_s'(0)$ is monotone increasing on s, we have $y_\infty'(0) > y_x'(0)$. In the case where $c(m) \leq 0$, we have $-y_x'(0) \leq 0$, a contradiction. Namely, $(-\infty, \infty)$ is the disconjugate interval of $(\mathrm{J_G})$ We then have $r(M) = \infty$. If $c(m) > 0$, it follows that

$$y_\infty'(0) = \int_x^\infty m(r)^{-2} dr + y_x'(0) = -y_x'(0).$$

Therefore,

$$0 = 2y_x'(0) + \int_x^\infty m(r)^{-2}dr = 2\left(y_\infty'(0) - \int_x^\infty m(r)^{-2}dr\right) + \int_x^\infty m(r)^{-2}dr$$
$$= c(m) - \int_x^\infty m(r)^{-2}dr.$$

Thus, we have $\bar{F}(x) = 0$ and the results. □

The geometrical meaning of the constant is $c(m) = 2y_\infty'(0)$ as above.

Remark 5.1. Furthermore, put

$$c(m, r_1) := 2y_{r_1,\infty}'(0) \quad \text{and} \quad \bar{F}(r_1, x) := c(m, r_1) - \int_x^\infty \frac{m'(r_1)^2}{m(r(t))^2 - m(r_1)^2}dt.$$

If $c(m, r_1) > 0$, then there exists an $x = x(r_1)$ such that $\bar{F}(r_1, x(r_1)) = 0$ and $\delta_{r_1} = x(r_1)$. Then, $(-x(r_1), \infty)$ is the maximal disconjugate interval along a geodesic β such that $r\big(\beta(0)\big) = r_1$ and $r'\big(\beta(0)\big) = 0$.

6. The proof of Theorem 1.1

In this section we prove Theorem 1.1. Let (M, p) be a pointed complete surface homeomorphic to the plane with a base point at p, whose Gaussian curvature is G. Let M_i be a von Mangoldt surface with a vertex at p_i, whose Gaussian curvature is G_i $(i = 1, 2)$. Let $\tau_q : [0, \infty) \to M$ be a geodesic with $\tau_q(0) = q, \tau_q(d(p, q)) = p$ and let $\tau_{q_i} : [0, \infty) \to M_i$ be geodesics with $\tau_{q_i}(0) = q_i, \tau_{q_i}(d(p_i, q_i)) = p_i$ $(i = 1, 2)$.

(1) Proof of $r(M) \geq r(M_2)$ (cf. Figure 6). We show that if $q \in M$ is not a

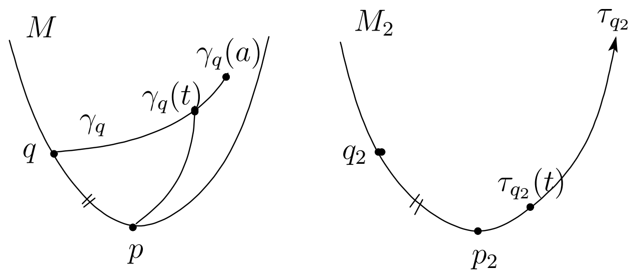

FIGURE 6. In the case $G(q) \leq G_2(d(p, q))$.

pole, then every point $q_2 \in M_2$ with $d(q_2, p_2) = d(q, p)$ is not a pole. Suppose q is not a pole in M. We then observe that the cut locus of q is not empty. Since M is homeomorphic to the plane, an endpoint of the cut locus of q is a point conjugate to q along every minimizing geodesic joining q to the end cut point. Let $\gamma_q : [0, a] \to M$ be the minimizing geodesic which satisfies that $\gamma_q(0) = q$ and $\gamma_q(a)$ is conjugate to $\gamma_q(0)$ along γ_q. Let $\tau_{q_2} : [0, \infty) \to M_2$ be the geodesic with $\tau_{q_2}(0) = q_2$ and $\tau_{q_2}(d(p, q)) = p_2$. Then it follows from the triangle inequality that

$$d(p, \gamma_q(t)) \geq |\, d(p, q) - t \,| = |\, d(p_2, q_2) - t \,| = d(p_2, \tau_{q_2}(t))$$

for all $t \in [0, a]$. Thus, $G(\gamma_q(t)) \leq G_2(d(p, \gamma_q(t))) \leq G_2(d(p_2, \tau_{q_2}(t)))$. Since $(\mathrm{J_G})$ is not disconjugate on $[0, a]$, $(\mathrm{J_G})$ is not disconjugate on $[0, a]$ from Comparison Theorem for the solutions of the differential equations of Jacobi type. Namely, there exists a point conjugate to $q_2 = \tau_{q_2}(0)$ along τ_{q_2}. Therefore, q_2 is not a pole. We then have $r(M) \geq r(M_2)$. The above argument proves that p is a pole in M because so is p_2.

(2) Proof of $r(M_1) \geq r(M)$ (cf. Figure 7). Let $q \in M$ satisfy $d(p, q) < r(M)$

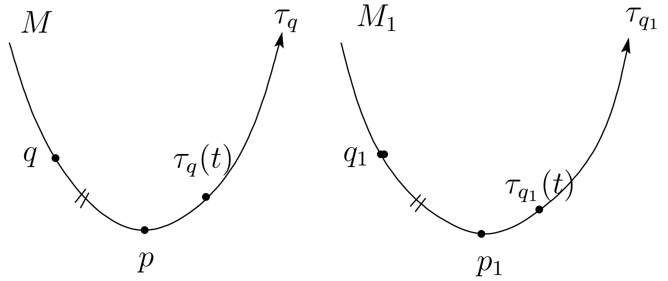

FIGURE 7. In the case $G_1(d(p, q)) \leq G(q)$.

and $q_1 \in M_1$ with $d(p, q) = d(p_1, q_1)$. Then q is a pole. We must prove that q_1 is a pole. Since M_1 is a von Mangoldt surface, it is sufficient to show that there exist no points conjugate to q_1 along τ_{q_1} by Lemma 5.1. Since q is a pole, there exist no points conjugate to q along τ_q. By the definitions of τ_q and τ_{q_1}, it follows $d(\tau_q(t), p) = d(\tau_{q_1}(t), p_1)$ for all $t \in [0, \infty)$. By assumption, $G_1(\tau_{q_1}(t)) \leq G(\tau_q(t))$. If $(\mathrm{J_G})$ is disconjugate on $[0, \infty)$, then $(\mathrm{J_{G_1}})$ is disconjugate on $[0, \infty)$ from Comparison Theorem. Therefore, q_1 is a pole of M_1 by Lemma 5.1, that is, $r(M_1) \geq r(M)$. We have just completed the proof of Theorem 1.1.

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