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ON SOME TYPES OF VECTORIAL SADDLE-POINT PROBLEMS

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ABSTRACT. In the paper, we consider some types of vectorial saddle-point problems. We present some existence results of vectorial saddle-point problems. After that we consider a generalized vector equilibrium problem as an application.

1. Introduction

Let X and Y be two nonempty subsets of two real topological vector spaces, respectively. Let Z be a real topological vector space. A set $C \subset Z$ is said to be a cone if $\lambda x \in C$ for any $\lambda \geq 0$ and for any $x \in C$. In this paper, we assume every convex cone C is not the whole space and not only consisting of the origin, i.e., $C \neq Z$ and $C \neq \{\theta_Z\}$, where θ_Z denotes the origin point of Z. A cone C is said to be solid if its topological interior is nonempty and C is said to be pointed if $C \cap (-C) = \{\theta_Z\}$. For a set A of topological vector space, let $\operatorname{int} A$ and $\operatorname{co} A$ denote the topological interior and convex hull of A, respectively, where the convex hull of A is the smallest convex set containing A.

In the paper, we consider the following vectorial saddle-points problems: find a pair $(x_0, y_0) \in X \times Y$ such that

$$(\mathrm{P}_1)\qquad \begin{array}{ll} f(u,y_0) - f(x_0,y_0) \notin -\operatorname{int} C & \text{for all } u \in X, \text{ and} \\ f(x_0,y_0) - f(x_0,v) \notin -\operatorname{int} C & \text{for all } v \in Y, \end{array}$$

where $f : X \times Y \to Z$;

$$(\mathrm{P}_2)\qquad \begin{array}{ll} f(u,y_0) - f(x_0,y_0) \in C & \text{for all } u \in X, \text{ and} \\ f(x_0,y_0) - f(x_0,v) \notin -\operatorname{int} C & \text{for all } v \in Y; \end{array}$$

$$(\mathrm{P}_3)\qquad \begin{array}{ll} f(u,y_0) - f(x_0,y_0) \notin -\operatorname{int} C & \text{for all } u \in X, \text{ and} \\ f(x_0,y_0) - f(x_0,v) \in C & \text{for all } v \in Y; \end{array}$$

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$$(\mathrm{P}_4) \qquad \begin{array}{ll} f(u,y_0)-f(x_0,y_0)\in C & \text{for all } u\in X, \text{ and} \\ f(x_0,y_0)-f(x_0,v)\in C & \text{for all } v\in Y. \end{array}$$

In the literature, vectorial saddle-point problems are investigated by some researchers, [5, 6, 12, 13]. They established existence results for vectorial saddle-point problems, especially for (P_1) type problem. In [5], existence results have been established by assuming the following conditions:

(1) $x \mapsto f(x,y)$ is C-invex on X with respect to $\eta : X \times X \to X$;
(2) $T(x) = \{y \in Y : f(x,v) - f(x,y) \notin \operatorname{int} C \text{ for all } v \in Y\}$ is singleton.

The above theorem is based on Fan-KKM Lemma and utilize the following a kind of vector variational inequality problem: find a pair $x \in X$ and $y \in T(x)$ such that

$$F'(x,y)\eta(u,x) \notin -\operatorname{int} C \text{ for all } u \in X,$$

where $F'(x,y)$ stands for the Fréchet derivative of F with respect to the first variable at $(x,y) \in X \times Y$. In [12], some existence results have been obtained for problem (P_1) in the special case, where f takes the following forms:

$$f(x,y) = u(x) + v(y);$$
$$f(x,y) = u(x) + \beta(x)v(y).$$

The results obtained in [12] are based on a semi-continuity assumption of u and v and compactness property of the domains. In [12], some existence results of vectorial saddle-points for saddle-function f on topological vector spaces have been also established under the following assumptions:

(1) $x \mapsto f(x,y)$ is C-lower semicontinuous and naturally quasi C-convex on X for every $y \in Y$;
(2) $y \mapsto f(x,y)$ is C-upper semicontinuous and naturally quasi C-concave on Y for every $x \in X$.

This is utilizing Sion's minimax theorem in [9] for scalar function and an extended result of a theorem in [13].

Recently, Lin[7] investigated some types of quasi saddle-point problems for continuous functions on locally convex spaces.

For (P_4) type problem, Gong [3] have been studied the existence of strong saddle-point for vector-valued function on locally convex spaces. The results obtained in [3] are based on the Kakutani-Fan-Glicksberg fixed point theorem.

In Section 3, we present existence results and examples for problems (P_1), (P_2), (P_3) and (P_4). In Section 4, we consider a generalized vector equilibrium problem.

2. Preliminaries

Definition 2.1 (C-quasiconvexity, [2, 8, 11])**.** Let X be a convex subset of a real vector space and Z a real vector space. Suppose that C is a convex cone of Z and that $f : X \to Z$ is a vector-valued function. Then f is said to be *C-quasiconvex on X* if for each $z \in Z$,

$$A(z) = \{x \in X : f(x) \in z - C\}$$

is a convex set.

Definition 2.2 (Weak C-quasiconvexity)**.** Let X be a nonempty subset of a real vector space, Z a real vector space and D a nonempty subset of Z. Suppose that C is a convex cone of Z and that $f : X \to Z$ is a vector-valued function. Then we say that f is *weakly C-quasiconvex on X with respect to D* if for each $z \in D$,

$$A(z) = \{x \in X : f(x) \in z - C\}$$

is a convex set. Especially if $D = f(X)$ then we say that f is *weakly C-quasiconvex on X*, where $f(X) = \cup_{x \in X} f(x)$.

Remark 2.1. Obviously, f is C-quasiconvex on X means f is weakly C-quasiconvex on X. However the inverse is not true.

Example 2.1. Let $X = [0, 2]$, $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $f : X \to Z$ defined by

$$f(x) = \begin{cases} (-1, 1) & \text{if } x = 0, \\ (x - 1, x) & \text{if } 0 < x < 1, \\ (x - 1, x - 2) & \text{otherwise.} \end{cases}$$

Then f is weakly C-quasiconvex on X, but not C-quasiconvex.

Example 2.2. Let $X = \{(1, 0), (0, 1)\}$, $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $f : X \to Z$ defined by

$$f(x) = x.$$

Then f is weakly C-quasiconvex on X. This example shows that the weak C-quasiconvexity notion can be defined on non-convex set.

Definition 2.3 (C-proper quasiconvexity, [11])**.** Let X be a convex subset of a real vector space and Z a real vector space. Suppose that C is a convex cone of Z and that $f : X \to Z$ is a vector-valued function. Then, f is said to be *C-properly quasiconvex on X* if for every $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ we have either

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_1) - C,$$

or

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_2) - C.$$

Proposition 2.1. *Let X be a convex subset of a real vector space and Z a real vector space. Suppose that C is a convex cone of Z and that $f : X \to Z$ is a vector-valued function. Then f is C-properly quasiconvex on X if and only if the following set is convex for each $z \in Z$:*

$$A(z) = \{x \in X : f(x) \notin z + C\}.$$

Proof. Let $z \in Z$. Assume that $A(z)$ is convex. Suppose to the contrary that f is not C-properly quasiconvex on X. Then, there exist $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ such that

$$f(\lambda x_1 + (1 - \lambda)x_2) \notin (\{f(x_1), f(x_2)\} - C).$$

Then $x_1, x_2 \in A(f(\lambda x_1 + (1-\lambda)x_2))$ and clearly $(\lambda x_1 + (1-\lambda)x_2) \notin A(f(\lambda x_1 + (1-\lambda)x_2))$, i.e., $A(f(x))$ is not convex. This is a contradiction to the assertion.

Inversely, assume that f is C-properly quasiconvex on X. Let $z \in Z$ and $x_1, x_2 \in A(z)$. Since f is C-properly quasiconvex on X, we see that

$$f(x) \in (\{f(x_1), f(x_2)\} - C) \text{ for all } x \in [x_1, x_2]$$

and that $z \notin (\{f(x_1), f(x_2)\} - C)$, where $[x_1, x_2]$ denotes the line segment between x_1 and x_2. Since C is a convex cone, it follows

$$(z + C) \bigcap (\{f(x_1), f(x_2)\} - C) = \emptyset,$$

i.e., $A(z)$ is convex. □

Definition 2.4 (C-continuity, [8])**.** Let X be a topological space, Z a topological vector space and C a nonempty convex cone of Z. Suppose that $f : X \to Z$ is a vector-valued function. Then, f is said to be *C-continuous on X* if for any neighborhood $V_{f(x)} \subset Z$ of $f(x)$, there exists a neighborhood $U_x \subset X$ of x such that $f(u) \in V_{f(x)} + C$ for all $u \in U_x$.

Remark 2.2. In [12], the concept of continuity introduced in Definition 2.4 is called "C-lower semicontinuity." However, these days many papers adopt "C-continuity" appellation. So, in this paper we also call it "C-continuity."

If C is solid, we have the following proposition.

Proposition 2.2. [12, Proposition 2.1] *Let X be a topological space, Z a topological vector space and C a solid convex cone. Suppose that $f : X \to Z$ is a vector-valued function. Then, the following three statements are equivalent:*

(1) *f is C-continuous on X;*
(2) *for every $z \in Z$, $f^{-1}(z + \operatorname{int} C)$ is open;*
(3) *for each $x \in X$ and for any $k \in \operatorname{int} C$, there exists a neighborhood $U_x \subset X$ of x such that $f(u) \in f(x) - k + \operatorname{int} C$ for all $u \in U_x$.*

Lemma 2.1. [10, Theorem 6.1.7] *Let X be a nonempty compact convex subset of a real Hausdorff topological vector space and let $A : X \to 2^X$ satisfying the following conditions:*

(1) *for each $y \in X$, $A^{-1}(y) = \{x \in X : y \in A(x)\}$ is open;*
(2) *for each $x \in X$, $A(x)$ is convex;*
(3) *for each $x \in X$, $x \notin A(x)$.*

Then there exists $x_0 \in X$ such that $A(x_0) = \emptyset$.

Next we obtain a system result of Lemma 2.1.

Theorem 2.1. *Let $I = \{1, \ldots, n\}$ be an index set. For each $i \in I$, let X_i be a nonempty convex compact subset of a real Hausdorff topological vector space and $A_i : \mathbb{X} \to 2^{X_i}$ a set-valued mapping, where $\mathbb{X}$ denotes the product space of $X_1, \ldots, X_n$, i.e., $\mathbb{X} = \prod_{j \in I} X_j$. For each $i \in I$, we also assume the following conditions:*

(1) *for each $y_i \in X_i$, $A^{-1}(y_i) = \{\boldsymbol{x} \in \mathbb{X} : y_i \in A_i(\boldsymbol{x})\}$ is open;*
(2) *for each $\boldsymbol{x} \in \mathbb{X}$, $A_i(\boldsymbol{x})$ is convex;*
(3) *for each $\boldsymbol{x} \in \mathbb{X}$, $x_i \notin A_i(\boldsymbol{x})$, where x_i denotes the i th element of $\boldsymbol{x}$.*

Then there exists $\bar{\boldsymbol{x}} \in \mathbb{X}$ such that $A_i(\bar{\boldsymbol{x}}) = \emptyset$ for every $i \in I$.

Proof. For each $i \in I$, let $B_i : \mathbb{X} \to 2^{\mathbb{X}}$ be a set-valued mapping defined by

$$B_i(\boldsymbol{x}) = \prod_{j \in I} X'_j,$$

where $X'_j = \begin{cases} A_i(\boldsymbol{x}) & \text{if } j = i, \\ X_j & \text{if } j \in I \setminus \{i\}. \end{cases}$
Let $\mathbb{A} : \mathbb{X} \to 2^{\mathbb{X}}$ be a set-valued mapping defined by

$$\mathbb{A}(\boldsymbol{x}) = \begin{cases} \bigcap_{i \in I(\boldsymbol{x})} B_i(\boldsymbol{x}), & \text{if } I(\boldsymbol{x}) \neq \emptyset, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $I(\boldsymbol{x}) = \{i \in I : A_i(\boldsymbol{x}) \neq \emptyset\}$. Then we have the following facts:

(1) for each $\boldsymbol{y} \in \mathbb{X}$, $\mathbb{A}^{-1}(\boldsymbol{y})$ is open;
(2) for each $\boldsymbol{x} \in \mathbb{X}$, $\mathbb{A}(\boldsymbol{x})$ is convex;
(3) for each $\boldsymbol{x} \in \mathbb{X}$, $\boldsymbol{x} \notin \mathbb{A}(\boldsymbol{x})$.

Therefore by Lemma 2.1, there exists $\bar{\boldsymbol{x}}$ such that $\mathbb{A}(\bar{\boldsymbol{x}}) = \emptyset$, i.e., for every $i \in I$ $A_i(\bar{\boldsymbol{x}}) = \emptyset$. □

More general results are investigated in [1]. However for reader's convenience, we include a simple result and a plain proof.

3. Existence results for vectorial saddle-points problems.

In this section we consider existence of solution for vectorial saddle-points.

3.1. Existence results and examples for type 1 problem

Theorem 3.1. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Let $F : X \times Y \to 2^X$ and $G : X \times Y \to 2^Y$ be two set-valued mappings defined by*

$$F(x, y) = \{u \in X : f(u, y) - f(x, y) \in -\operatorname{int} C\}$$

and

$$G(x, y) = \{v \in Y : f(x, y) - f(x, v) \in -\operatorname{int} C\}.$$

Suppose the following conditions hold:

(1) *$F(x, y)$ is convex for each $(x, y) \in X \times Y$;*
(2) *$G(x, y)$ is convex for each $(x, y) \in X \times Y$;*
(3) *$F^{-1}(u)$ is open for each $u \in X$;*
(4) *$G^{-1}(v)$ is open for each $v \in Y$.*

Then (P_1) has at least one solution, i.e., there exists a pair $(x_0, y_0) \in X \times Y$ such that

$$\begin{aligned} &f(u, y_0) - f(x_0, y_0) \notin -\operatorname{int} C \quad \text{for all } u \in X, \text{ and} \\ &f(x_0, y_0) - f(x_0, v) \notin -\operatorname{int} C \quad \text{for all } v \in Y. \end{aligned}$$

Proof. By the definitions of F and G, $x \notin F(x, y)$ and $y \notin G(x, y)$ for each $(x, y) \in X \times Y$. Therefore by Theorem 2.1, there exists $(x_0, y_0) \in X \times Y$ such that

$$F(x_0, y_0) = \emptyset \text{ and } G(x_0, y_0) = \emptyset.$$

Hence

$$\begin{aligned} &f(u, y_0) - f(x_0, y_0) \notin -\operatorname{int} C \quad \text{for all } u \in X, \text{ and} \\ &f(x_0, y_0) - f(x_0, v) \notin -\operatorname{int} C \quad \text{for all } v \in Y, \end{aligned}$$

i.e., $(x_0, y_0) \in X \times Y$ is a solution of (P_1). □

Lemma 3.1. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid convex cone. Let $f : X \times Y \to Z$ be a vector-valued function and $F : X \times Y \to 2^X$ be a set-valued mapping defined by*

$$F(x, y) = \{u \in X : f(u, y) - f(x, y) \in -\operatorname{int} C\}.$$

If $f(\cdot, y)$ is weakly C-quasiconvex on X for each $y \in Y$, then $F(x, y)$ is convex for each $(x, y) \in X \times Y$.

Proof. Let $u_1, u_2 \in F(x,y)$, i.e.,

$$f(u_1,y) - f(x,y) \in -\operatorname{int} C \text{ and } f(u_2,y) - f(x,y) \in -\operatorname{int} C.$$

Therefore, $f(u_1,y), f(u_2,y) \in f(x,y) - \operatorname{int} C$. Since $f(\cdot,y)$ is weakly C-quasiconvex on X, $u' \in [u_1,u_2]$ implies $f(u',y) \in f(x,y) - \operatorname{int} C$. Hence $F(x,y)$ is convex for each $(x,y) \in X \times Y$. □

Corollary 3.1. *In Lemma 3.1, let $G : X \times Y \to 2^X$ be a set-valued mapping defined by*

$$G(x,y) = \{v \in Y : f(x,y) - f(x,v) \in -\operatorname{int} C\}.$$

If $f(x,\cdot)$ is weakly $(-C)$-quasiconvex on Y for each $x \in X$, then $G(x,y)$ is convex for each $(x,y) \in X \times Y$.

Lemma 3.2. *Let X and Y be two topological spaces. Let Z be a real topological vector space and $C \subset Z$ a solid convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Let $F : X \times Y \to 2^X$ be a set-valued mapping defined by*

$$F(x,y) = \{u \in X : f(u,y) - f(x,y) \in -\operatorname{int} C\}$$

and $g : X \times X \times Y \to Z$ a vector-valued function defined by

$$g(u,x,y) = f(u,y) - f(x,y).$$

If $g(u,\cdot,\cdot)$ is $(-C)$-continuous on $X \times Y$ for each $u \in X$, then $F^{-1}(u)$ is open for each $u \in X$.

Proof. Let $(x,y) \in X \times Y$ and $u \in F(x,y)$. Then $g(u,x,y) \in -\operatorname{int} C$. Since g is $(-C)$-continuous on $X \times Y$ and $-\operatorname{int} C$ is a neighborhood of $g(u,x,y)$, there exists a neighborhood U of (x,y) such that $g(u,x',y') \in -\operatorname{int} C$ for all $(x',y') \in U$, i.e., $U \subset F(x,y)$. Hence $F(x,y)$ is open. □

Corollary 3.2. *In Lemma 3.2, let $G : X \times Y \to 2^Y$ be a set-valued mapping defined by*

$$G(x,y) = \{v \in Y : f(x,y) - f(x,v) \in -\operatorname{int} C\}$$

and $g : Y \times X \times Y \to Z$ a vector-valued function defined by

$$g(v,x,y) = f(x,y) - f(x,v).$$

If $g(v,\cdot,\cdot)$ is $(-C)$-continuous on $X \times Y$ for each $v \in Y$, then $G^{-1}(v)$ is open for each $u \in X$.

Corollary 3.3. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid convex cone, and $f : X \times Y \to Z$ be a vector-valued function. Suppose the following conditions hold:*

(1) *for each* $y \in Y$, $f(\cdot, y)$ *is weakly* C*-quasiconvex on* X*;*
(2) *for each* $x \in X$, $f(x, \cdot)$ *is weakly* $(-C)$*-quasiconvex on* Y*;*
(3) *for each* $u \in X$, $f(u, y) - f(x, y)$ *is* $(-C)$*-continuous on* $X \times Y$*;*
(4) *for each* $u \in Y$, $f(x, y) - f(x, v)$ *is* $(-C)$*-continuous on* $X \times Y$*.*

Then (P_1) *has at least one solution, i.e., there exists a pair* $(x_0, y_0) \in X \times Y$ *such that*

$$\begin{array}{ll} f(u, y_0) - f(x_0, y_0) \notin -\operatorname{int} C & \textit{for all } u \in X, \textit{ and} \\ f(x_0, y_0) - f(x_0, v) \notin -\operatorname{int} C & \textit{for all } v \in Y. \end{array}$$

Proof. Let $F : X \times Y \to 2^X$ and $G : X \times Y \to 2^Y$ be two set-valued mappings defined by

$$F(x, y) = \{u \in X : f(u, y) - f(x, y) \in -\operatorname{int} C\}$$

and

$$G(x, y) = \{v \in Y : f(x, y) - f(x, v) \in -\operatorname{int} C\}.$$

Then, we see that

(a) by condition (i) and Lemma 3.1, $F(x, y)$ is convex for each $(x, y) \in X \times Y$;
(b) by condition (ii) and Corollary 3.1, $G(x, y)$ is convex for each $(x, y) \in X \times Y$;
(c) by condition (iii) and Lemma 3.2, $F^{-1}(u)$ is open for each $u \in X$;
(d) by condition (iv) and Corollary 3.2, $G^{-1}(v)$ is open for each $v \in Y$;
(e) by the definitions of F and G, $x \notin F(x, y)$ and $y \notin G(x, y)$ for each $(x, y) \in X \times Y$.

Therefore by Theorem 2.1, there exists $(x_0, y_0) \in X \times Y$ such that

$$F(x_0, y_0) = \emptyset \text{ and } G(x_0, y_0) = \emptyset.$$

Hence

$$\begin{array}{ll} f(u, y_0) - f(x_0, y_0) \notin -\operatorname{int} C & \text{for all } u \in X, \text{ and} \\ f(x_0, y_0) - f(x_0, v) \notin -\operatorname{int} C & \text{for all } v \in Y, \end{array}$$

i.e., $(x_0, y_0) \in X \times Y$ is a solution of (P_1). □

Example 3.1. Let $X = [0, 1]$, $Y = [0, 1]$, $Z = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Suppose that f is a vector-valued function from $X \times Y$ to Z defined by

$$f(x, y) = \begin{cases} (x, x + xy + 1), & \text{if } 0 \le x \le \frac{1}{2}, \\ (x, x - 1), & \text{otherwise.} \end{cases}$$

Then all assumptions of Corollary 3.3 are satisfied. Hence f has at least one vectorial saddle-point in the scene of (P_1). Indeed, $(0, 0)$ is a solution of (P_1).

Example 3.2. Let $X, Y = [-2, 2]$, $Z = \mathbb{R}^2$ and $C = \{(z_1, z_2) : z_1 \ge |z_2|\}$. Suppose that $f : X \times Y \to Z$ is a vector-valued function defined by

$$f(x, y) = g(x) + xy + h(y),$$

where $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ are the following functions:

$$g(x) = \begin{cases} (x, -|x|), & \text{if } |x| \le 1, \\ (x, |x| - 1), & \text{otherwise} \end{cases} \text{ and } h(y) = \begin{cases} (y, -|y|), & \text{if } |x| < 1, \\ (y, |y| - 1), & \text{otherwise.} \end{cases}$$

Then X and Y are two compact and convex sets of $\mathbb{R}$, C is a solid closed pointed convex cone of Z, where $\mathbb{R}$ denotes the set of all real numbers. Moreover, we see that for each $u \in X$ $f(u, y) - f(x, y)$ is $(-C)$-continuous on $X \times Y$ and for each $v \in Y$ $f(x, y) - f(x, v)$ is $(-C)$-continuous on $X \times Y$ and that $f(\cdot, y)$ is weakly C-quasiconvex on X for each $y \in Y$ and $f(x, \cdot)$ is weakly C-quasiconcave on Y for each $x \in X$. Hence, f has at least one vectorial saddle-point in the sense of (P_1). Indeed, $(0, 0) \in X \times Y$ is a solution.

Lemma 3.3. *Let X and Y be two topological spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid convex cone. Let $f : X \times Y \to Z$ be a vector-valued function and $g : X \times X \times Y \to Z$ be a vector-valued function defined by*

$$g(u, x, y) = f(u, y) - f(x, y).$$

If f is C-continuous and $(-C)$-continuous on $X \times Y$, then $g(u, \cdot, \cdot)$ is $(-C)$-continuous on $X \times Y$ for each $u \in X$.

Proof. Let $u, x \in X$, $y \in Y$, and $z \in f(u, y) - f(x, y) + \operatorname{int} C$. Then we have $f(u, y) - f(x, y) - z \in -\operatorname{int} C$. Since f is $(-C)$-continuous and C-continuous on $X \times Y$, there exist two neighborhoods U and V of (u, y) and (x, y) such that

$$f(u, \tilde{y}) \in f(u, y) + \frac{1}{2}(z - g(u, x, y)) - \operatorname{int} C \text{ for all } (u, \tilde{y}) \in U,$$

where u is fixed and

$$-f(\tilde{x}, \tilde{y}) \in -f(x, y) + \frac{1}{2}(z - g(u, x, y)) - \operatorname{int} C \text{ for all } (\tilde{x}, \tilde{y}) \in V.$$

Therefore

$$f(u, \tilde{y}) - f(\tilde{x}, \tilde{y}) - z \in -\operatorname{int} C \text{ for all } (\tilde{x}, \tilde{y}) \in U \cap V,$$

i.e.,

$$g(u, \tilde{x}, \tilde{y}) \in z - \operatorname{int} C \text{ for all } (\tilde{x}, \tilde{y}) \in U \cap V.$$

Hence $g(u, \cdot, \cdot)$ is $(-C)$-continuous on $X \times Y$. $\square$

Corollary 3.4. *In Lemma 3.3, let $g : Y \times X \times Y \to Z$ be a vector-valued function defined by*

$$g(v, x, y) = f(x, y) - f(x, v).$$

If f is C-continuous and $(-C)$-continuous on $X \times Y$, then $g(v, \cdot, \cdot)$ is $(-C)$-continuous on $X \times Y$ for each $u \in X$,

Corollary 3.5. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid convex cone and $f : X \times Y \to Z$ be a vector-valued function. Suppose the following conditions hold:*

(1) *for each $y \in Y$, $f(\cdot, y)$ is weakly C-quasiconvex on X;*
(2) *for each $x \in X$, $f(x, \cdot)$ is weakly $(-C)$-quasiconvex on Y;*
(3) *f is C-continuous and $(-C)$-continuous on $X \times Y$.*

Then (P_1) has at least one solution, i.e., there exists a pair $(x_0, y_0) \in X \times Y$ such that

$$\begin{aligned} f(u, y_0) - f(x_0, y_0) &\notin -\operatorname{int} C \quad \textit{for all } u \in X, \textit{ and} \\ f(x_0, y_0) - f(x_0, v) &\notin -\operatorname{int} C \quad \textit{for all } v \in Y. \end{aligned}$$

Proof. By condition (iii) and Lemma 3.3, $f(u, y) - f(x, y)$ is $(-C)$-continuous on $X \times Y$ for each $u \in X$. By condition (iii) and Corollary 3.4, $f(x, y) - f(x, v)$ is $(-C)$-continuous on $X \times Y$ for each $v \in Y$. Therefore by Corollary 3.3, there exists $(x_0, y_0) \in X \times Y$ such that

$$\begin{aligned} f(u, y_0) - f(x_0, y_0) &\notin -\operatorname{int} C \quad \text{for all } u \in X, \text{ and} \\ f(x_0, y_0) - f(x_0, v) &\notin -\operatorname{int} C \quad \text{for all } v \in Y, \end{aligned}$$

i.e., $(x_0, y_0) \in X \times Y$ is a solution of (P_1). □

Remark 3.1. Whenever a vector-valued function $g : X \to Z$ is continuous at $x \in X$, g is C-continuous and $(-C)$-continuous at a point $x \in X$. However the inverse is not true, even if C is a pointed closed convex cone of Z.

Example 3.3. Let $X, Y = [-1, 1]$ and $Z = l^\infty$. Suppose that C is a cone in Z defined by

$$C := \left\{ z \in Z : z_1 \geq \frac{1}{i}|z_i|,\ i = 2, 3, \dots \right\},$$

where z_i denotes the i th element of $z \in Z$ and that $f : X \times Y \to Z$ is a vector-valued function defined by

$$f(x, y) = \begin{cases} n(n+1)\Big(\big(1 - (\frac{1}{n} - |x|)\big)\boldsymbol{e}_n^\sigma + (|x| - \frac{1}{n+1})\boldsymbol{e}_{n+1}^\sigma\Big) + \frac{1}{3}|y| \cdot |x| \cdot \boldsymbol{e}_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } |x| \in (\frac{1}{n+1}, \frac{1}{n}], \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise, i.e., } x = 0, \end{cases}$$

where $\boldsymbol{e}_i, i = 1, 2, \dots$ denote fundamental vectors of Z and $\boldsymbol{e}_i^\sigma = \sum_{k=i}^\infty \boldsymbol{e}_k$.

Then X and Y are two compact and convex sets of $\mathbb{R}$, C is a solid closed pointed convex cone of Z. Moreover, we see that f is C-continuous and $(-C)$-continuous on $X \times Y$ but not continuous and that $f(\cdot, y)$ is C-quasiconvex on X for each $y \in Y$ and $f(x, \cdot)$ is C-quasiconcave on Y for each $x \in X$. Hence, f has at least one vectorial saddle-point in the sense of (P_1). Indeed, $(0, 0) \in X \times Y$ is a solution.

3.2. Existence results and examples for type 2 problem

Next we consider existence of solution for (P_2).

Theorem 3.2. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Let $F : X \times Y \to 2^X$ and $G : X \times Y \to 2^Y$ be two set-valued mappings defined by*

$$F(x, y) = \{u \in X : f(u, y) - f(x, y) \notin C\}$$

and

$$G(x, y) = \{v \in Y : f(x, y) - f(x, v) \in -\operatorname{int} C\}.$$

Suppose the following conditions hold:

(1) *$F(x, y)$ is convex for each $(x, y) \in X \times Y$;*
(2) *$G(x, y)$ is convex for each $(x, y) \in X \times Y$;*
(3) *$F^{-1}(u)$ is open for each $u \in X$;*
(4) *$G^{-1}(v)$ is open for each $v \in Y$.*

Then (P_2) has at least one solution, i.e., there exists a pair $(x_0, y_0) \in X \times Y$ such that

$$\begin{array}{ll} f(u, y_0) - f(x_0, y_0) \in C & \text{for all } u \in X, \text{ and} \\ f(x_0, y_0) - f(x_0, v) \notin -\operatorname{int} C & \text{for all } v \in Y. \end{array}$$

Proof. By the definitions of F and G, $x \notin F(x, y)$ and $y \notin G(x, y)$ for each $(x, y) \in X \times Y$. Therefore by Theorem 2.1, there exists $(x_0, y_0) \in X \times Y$ such that

$$\begin{array}{ll} f(u, y_0) - f(x_0, y_0) \in C & \text{for all } u \in X, \text{ and} \\ f(x_0, y_0) - f(x_0, v) \notin -\operatorname{int} C & \text{for all } v \in Y, \end{array}$$

i.e., $(x_0, y_0) \in X \times Y$ is a solution of (P_2). □

Note that Theorem 3.2 does not require the closedness of C, but the following Corollaries 3.6 and 3.7 need the closedness of C.

Corollary 3.6. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid closed convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Assume the following conditions:*

(1) *for each $y \in Y$, $f(\cdot, y)$ is properly C-quasiconvex on X;*
(2) *for each $x \in X$, $f(x, \cdot)$ is weakly $(-C)$-quasiconvex on Y;*
(3) *for each $u \in X$, $f(u, y) - f(x, y)$ is $(-C)$-continuous on $X \times Y$;*
(4) *for each $u \in Y$, $f(x, y) - f(x, v)$ is $(-C)$-continuous on $X \times Y$.*

Then (P_2) *has at least one solution, i.e., there exists a pair* $(x_0, y_0) \in X \times Y$ *such that*

$$\begin{array}{ll} f(u,y_0) - f(x_0,y_0) \in C & \textit{for all } u \in X, \textit{ and} \\ f(x_0,y_0) - f(x_0,v) \notin -\operatorname{int} C & \textit{for all } v \in Y. \end{array}$$

Proof. Let $F : X \times Y \to 2^X$ and $G : X \times Y \to 2^Y$ be two set-valued mappings defined by

$$F(x,y) = \{u \in X : f(u,y) - f(x,y) \notin C\}$$

and

$$G(x,y) = \{v \in Y : f(x,y) - f(x,v) \in -\operatorname{int} C\}.$$

By condition (i) and Proposition 2.1, $F(x,y)$ is convex for each $(x,y) \in X \times Y$. By condition (ii) and Lemma 3.1, $G(x,y)$ is convex for each $(x,y) \in X \times Y$. Since C is a convex cone and $f(u,y) - f(x,y)$ is $(-C)$-continuous on $X \times Y$ for each $u \in X$, $F^{-1}(u)$ is open for each $u \in X$. By condition (iv) and Corollary 3.2, $G^{-1}(v)$ is open for each $v \in Y$. Therefore by Theorem 3.2, there exists $(x_0, y_0) \in X \times Y$ such that

$$\begin{array}{ll} f(u,y_0) - f(x_0,y_0) \in C & \text{for all } u \in X, \text{ and} \\ f(x_0,y_0) - f(x_0,v) \notin -\operatorname{int} C & \text{for all } v \in Y, \end{array}$$

i.e., $(x_0, y_0) \in X \times Y$ is a solution of (P_2). □

Corollary 3.7. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid closed convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Assume the following conditions:*

(1) *for each $y \in Y$, $f(\cdot, y)$ is C-properly quasiconvex on X;*
(2) *for each $x \in X$, $f(x, \cdot)$ is weakly $(-C)$-quasiconvex on Y;*
(3) *f is C-continuous and $(-C)$-continuous on $X \times Y$.*

Then (P_2) *has at least one solution, i.e., there exists a pair* $(x_0, y_0) \in X \times Y$ *such that*

$$\begin{array}{ll} f(u,y_0) - f(x_0,y_0) \in C & \textit{for all } u \in X, \textit{ and} \\ f(x_0,y_0) - f(x_0,v) \notin -\operatorname{int} C & \textit{for all } v \in Y. \end{array}$$

Proof. By Lemma 3.3 and condition (iii), $f(u,y) - f(x,y)$ is $(-C)$-continuous on $X \times Y$ for each $u \in X$. By Corollary 3.4 and condition (iii), $f(x,y) - f(x,v)$ is $(-C)$-continuous on $X \times Y$ for each $v \in Y$. Therefore by Corollary 3.6, (P_2) has at least one solution. □

Example 3.4. Let $X, Y = [-1,1]^\infty$ and $Z = l^\infty$. Suppose that C is a cone in Z defined by

$$C := \{z \in Z : z_1 \geq 0,\ z_1 \geq |z_i|,\ i = 2,4,6,\ldots\}$$

and that $f : X \times Y \to Z$ is a vector-valued function defined by

$$f(x,y) = -\|x\| \cdot \|y\| \cdot \boldsymbol{e}_1 + \sum_{i=1}^{\infty}(x_{2i}\boldsymbol{e}_{2i} + \lfloor y_{2i+1} \rfloor \boldsymbol{e}_{2i+1}),$$

where for each $a, k \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the largest integer satisfying $k \leq a$. Then X and Y are compact and convex sets of R^∞; C is a solid closed convex cone of Z. Moreover, we see that f is C-continuous and $(-C)$-continuous on $X \times Y$ and that $f(\cdot, y)$ is C-properly quasiconvex on X for each $y \in Y$ and $f(x, \cdot)$ is C-quasiconcave on Y for each $x \in X$. Hence, f has at least one vectorial saddle-point in the sense of (P_2). Indeed, $(\theta_X, 0) \in X \times Y$ is a solution.

3.3. Existence results and examples for type 3 problem

Similarly, we obtain the following existence result for (P_3).

Theorem 3.3. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Let $F : X \times Y \to 2^X$ and $G : X \times Y \to 2^Y$ be two set-valued mappings defined by*

$$F(x,y) = \{u \in X : f(u,y) - f(x,y) \in -\operatorname{int} C\}$$

and

$$G(x,y) = \{v \in Y : f(x,y) - f(x,v) \in C\}.$$

Suppose the following conditions hold:

(1) *$F(x,y)$ is convex for each $(x,y) \in X \times Y$;*
(2) *$G(x,y)$ is convex for each $(x,y) \in X \times Y$;*
(3) *$F^{-1}(u)$ is open for each $u \in X$;*
(4) *$G^{-1}(v)$ is open for each $v \in Y$.*

Then (P_2) has at least one solution, i.e., there exists a pair $(x_0, y_0) \in X \times Y$ such that

$$\begin{aligned} &f(u,y_0) - f(x_0,y_0) \notin -\operatorname{int} C \quad && \text{for all } u \in X, \text{ and} \\ &f(x_0,y_0) - f(x_0,v) \in C && \text{for all } v \in Y. \end{aligned}$$

Corollary 3.8. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid closed convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Assume the following conditions:*

(1) *for each $y \in Y$, $f(\cdot, y)$ is weakly C-quasiconvex on X;*
(2) *for each $x \in X$, $f(x, \cdot)$ is $(-C)$-properly quasiconvex on Y;*
(3) *for each $u \in X$ $f(u,y) - f(x,y)$ is $(-C)$-continuous on $X \times Y$;*

(4) *for each* $v \in Y$ $f(x,y) - f(x,v)$ *is* $(-C)$*-continuous on* $X \times Y$.

Then (P_3) *has at least one solution, i.e., there exists a pair* $(x_0, y_0) \in X \times Y$ *such that*

$$\begin{aligned} &f(u,y_0) - f(x_0,y_0) \notin -\operatorname{int} C \quad &&\text{for all } u \in X, \text{ and} \\ &f(x_0,y_0) - f(x_0,v) \in C &&\text{for all } v \in Y. \end{aligned}$$

Corollary 3.9. *Let* X *and* Y *be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let* Z *be a real topological vector space and* $C \subset Z$ *a solid closed convex cone. Suppose that* $f : X \times Y \to Z$ *is a vector-valued function. Assume the following conditions:*

(1) *for each* $y \in Y$, $f(\cdot, y)$ *is weakly* C*-quasiconvex on* X;

(2) *for each* $x \in X$, $f(x, \cdot)$ *is* $(-C)$*-properly quasiconvex on* Y;

(3) f *is* C*-continuous and* $(-C)$*-continuous on* $X \times Y$.

Then (P_3) *has at least one solution, i.e., there exists a pair* $(x_0, y_0) \in X \times Y$ *such that*

$$\begin{aligned} &f(u,y_0) - f(x_0,y_0) \notin -\operatorname{int} C \quad &&\text{for all } u \in X, \text{ and} \\ &f(x_0,y_0) - f(x_0,v) \in C &&\text{for all } v \in Y. \end{aligned}$$

3.4. Existence results and examples for type 4 problem

Theorem 3.4. *Let* X *and* Y *be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let* Z *be a real topological vector space and* $C \subset Z$ *a proper convex cone. Suppose that* $f : X \times Y \to Z$ *is a vector-valued function. Let* $F : X \times Y \to 2^X$ *and* $G : X \times Y \to 2^Y$ *be two set-valued mappings defined by*

$$F(x,y) = \{u \in X : f(u,y) - f(x,y) \notin C\}$$

and

$$G(x,y) = \{v \in Y : f(x,y) - f(x,v) \notin C\}.$$

Suppose the following conditions hold:

(1) $F(x,y)$ *is convex for each* $(x,y) \in X \times Y$;

(2) $G(x,y)$ *is convex for each* $(x,y) \in X \times Y$;

(3) $F^{-1}(u)$ *is open for each* $u \in X$;

(4) $G^{-1}(v)$ *is open for each* $v \in Y$.

Then (P_4) *has at least one solution, i.e., there exists a pair* $(x_0, y_0) \in X \times Y$ *such that*

$$\begin{aligned} &f(u,y_0) - f(x_0,y_0) \in C \quad &&\text{for all } u \in X, \text{ and} \\ &f(x_0,y_0) - f(x_0,v) \in C &&\text{for all } v \in Y. \end{aligned}$$

Remark 3.2. In Theorem 3.4, we don't require any topology of Z. Thus, Z can be replaced by a real vector space.

Corollary 3.10. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a proper convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Assume the following conditions:*

(1) *for each $y \in Y$, $f(\cdot, y)$ is C-properly quasiconvex on X;*
(2) *for each $x \in X$, $f(x, \cdot)$ is $(-C)$-properly quasiconvex on Y;*
(3) *$f(u, y) - f(x, y)$ is $(-C)$-continuous on $X \times Y$ for each $u \in X$;*
(4) *$f(x, y) - f(x, v)$ is $(-C)$-continuous on $X \times Y$ for each $v \in Y$.*

Then (P_4) has at least one solution, i.e., there exists a pair $(x_0, y_0) \in X \times Y$ such that

$$\begin{aligned} f(u, y_0) - f(x_0, y_0) &\in C \quad \text{for all } u \in X, \text{ and} \\ f(x_0, y_0) - f(x_0, v) &\in C \quad \text{for all } v \in Y. \end{aligned}$$

Theorems 3.4 and Corollaries 3.10, 3.11 are extended results of Corollary 2.1 in [3]. Example 2.1 in [3] is also an example satisfying every condition of Theorem 3.4 and Corollaries 3.10, 3.11. Moreover the following example does not meet conditions of their results. However, it does not mean that our results cover all existence results of Gong [3].

Example 3.5. Let $X = [-1, 1], Y = [0, 1]$, $Z = \mathbb{R}^2$, and $C = \mathbb{R}^2_+ = \{(z_1, z_2) \in \mathbb{R}^2 : z_1, z_2 \geq 0\}$. Suppose that $f : X \times Y \to Z$ is a vector-valued function defined by

$$f(x, y) = \begin{cases} (3, 0) + |x| \cdot y \cdot (1, 1) & \text{if } x > 0, \\ (0, 0) & \text{if } x = 0, \\ (0, 3) + |x| \cdot y \cdot (1, 1) & \text{if } x < 0. \end{cases}$$

Then, X and Y are nonempty, convex and compact subsets of $\mathbb{R}$ and C is a proper, closed and convex cone of Z. Moreover, we see that

(1) for each $y \in Y$, $f(\cdot, y)$ is C-properly quasiconvex on X;
(2) for each $x \in X$, $f(x, \cdot)$ is $(-C)$-properly quasiconvex on Y;
(3) for each $u \in X$ $f(u, y) - f(x, y)$ is $(-C)$-continuous on $X \times Y$; and
(4) for each $v \in Y$ $f(x, y) - f(x, v)$ is continuous on $X \times Y$.

Hence, f has at least one vectorial saddle-point in the sense of (P_4). Indeed, $(0, 1) \in X \times Y$ is a solution.

Corollary 3.11. *Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a closed convex cone. Suppose that $f : X \times Y \to Z$ is a vector-valued function. Assume the following conditions:*

(1) *for each $y \in Y$, $f(\cdot, y)$ is C-properly quasiconvex on X;*
(2) *for each $x \in X$, $f(x, \cdot)$ is $(-C)$-properly quasiconvex on Y;*

(3) *f is C-continuous and $(-C)$-continuous on $X \times Y$.*

Then (P_4) has at least one solution, i.e., there exists a pair $(x_0, y_0) \in X \times Y$ such that

$$f(u, y_0) - f(x_0, y_0) \in C \quad \textit{for all } u \in X, \textit{ and}$$
$$f(x_0, y_0) - f(x_0, v) \in C \quad \textit{for all } v \in Y.$$

Example 3.6. Let $X, Y = [0, 3]$, $Z = \mathbb{R}^3$ and $C = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 = z_2, z_3 \geq 0\}$. Suppose that $f : X \times Y \to Z$ is a vector-valued function defined by

$$f(x, y) = \begin{cases} (x - \sin y, x - \sin y, -x), & \text{if } x \in [0, 1), \\ (3 - x + y, 3 - x + y, -x), & \text{if } x \in [1, 2], \\ (x + 2, x + 2, y^2 - 4), & \text{otherwise.} \end{cases}$$

Then, X and Y are nonempty, convex and compact subsets of $\mathbb{R}$, and C is a proper, closed and convex but non-solid cone of Z. Moreover, we see that

(1) for each $y \in Y$, $f(\cdot, y)$ is C-properly quasiconvex on X;
(2) for each $x \in X$, $f(x, \cdot)$ is $(-C)$-properly quasiconvex on Y;
(3) f is C-continuous and $(-C)$-continuous on $X \times Y$.

Hence, f has at least one vectorial saddle-point in the sense of (P_4). Indeed, $(2, 2) \in X \times Y$ is a solution.

Example 3.7. Let $X = Y = [0, 1]^\infty$, $Z = l^\infty$. Suppose that C is a cone in Z defined by

$$C := \left\{ z \in Z : z_1 = 0, z_2 \geq \frac{1}{i}|z_i|, \ i = 3, 4, \dots \right\}$$

and that $f : X \times Y \to Z$ is a vector-valued function defined by

$$f(x, y) = \left(0, \sup_{i \in \mathbb{N}} \left\{ \left(1 + \sup_{j \in \mathbb{N}}\{y_j\}\right) \left(2 - \mathrm{co}\, s(ix_i)\right) \frac{1}{i} x_i \right\}, x_3, x_4, \dots \right).$$

Then, X and Y are nonempty, convex and compact subsets of $\mathbb{R}^{\mathbb{N}}$, and C is a closed convex, (but non-solid), cone of Z. Moreover, we see that

(1) for each $y \in Y$, $f(\cdot, y)$ is C-properly quasiconvex on X;
(2) for each $x \in X$, $f(x, \cdot)$ is $(-C)$-properly quasiconvex on Y;
(3) f is C-continuous and $(-C)$-continuous on $X \times Y$.

Hence, f has at least one vectorial saddle-point in the sense of (P_4). Indeed, $(\theta_X, \theta_Y) \in X \times Y$ is a solution.

4. On generalized vectorial equilibrium problems.

In the section, we consider the following generalized vectorial equilibrium problem as an application of existence results for vectorial saddle-point problems. Let X and Y be two nonempty convex and compact subsets of two real Hausdorff topological vector spaces, respectively. Let Z be a real topological vector space and $C \subset Z$ a solid closed convex cone. Suppose that $f : X \times Y \to Z$. Then, generalized vectorial equilibrium problem, (GVEP) for short, consists to find $y \in Y$ such that

$$f(x, y) \notin -\operatorname{int} C \text{ for all } x \in X.$$

We use the above assumptions throughout this section.

Proposition 4.1. *Let $a \notin (-C)$. Then*

$$(a + C) \cap (-C) = \emptyset.$$

Proof. Suppose to the contrary that there exists $x \in (a+C)\cap(-C)$. Then there exist $c_1, c_2 \in C$ such that $x = -c_2 = a + c_1$. However it follows $a = -(c_1 + c_2) \in (-C)$. This contradicts to the assumption. □

Theorem 4.1. *Suppose*

$$SP_1 = \left\{ (x, y) \in X \times Y : \begin{array}{ll} f(u, y) - f(x, y) \notin -\operatorname{int} C & \forall u \in X, \ \textit{and} \\ f(x, y) - f(x, v) \notin -\operatorname{int} C & \forall v \in Y \end{array} \right\} \neq \emptyset$$

and $f(x_0, y_0) \in C$ for some $(x_0, y_0) \in SP_1$. Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that

$$f(x, y) \notin -\operatorname{int} C \textit{ for all } x \in X.$$

Proof. Since $(x_0, y_0) \in \mathrm{SP}_1$, it follows

$$f(x, y_0) - f(x_0, y_0) \notin -\operatorname{int} C \text{ for all } x \in X.$$

Since $f(x_0, y_0) \in C$ by Proposition 4.1, we have

$$f(x, y_0) \notin -\operatorname{int} C \text{ for all } x \in X.$$

Hence $y_0 \in X$ is a solution of (GVEP). Therefore (GVEP) has at least one solution. □

Examples 3.1, 3.2 and 3.3 are also examples of Theorem 4.1.

Theorem 4.2. *Suppose*

$$SP_2 = \left\{ (x, y) \in X \times Y : \begin{array}{ll} f(u, y) - f(x, y) \notin -\operatorname{int} C & \forall u \in X, \ \textit{and} \\ f(x, y) - f(x, v) \in C & \forall v \in Y \end{array} \right\} \neq \emptyset$$

and $f(x_0, y_0) \in C$ for some $(x_0, y_0) \in SP_2$. Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that

$$f(x, y) \notin -\operatorname{int} C \textit{ for all } x \in X.$$

Proof. Since $(x_0, y_0) \in \mathrm{SP}_2$, it follows

$$f(x, y_0) - f(x_0, y_0) \notin -\mathrm{int}\, C \text{ for all } x \in X.$$

Since $f(x_0, y_0) \in C$ by Proposition 4.1, we have

$$f(x, y_0) \notin -\mathrm{int}\, C \text{ for all } x \in X.$$

Hence $y_0 \in X$ is a solution of (GVEP). Therefore (GVEP) has at least one solution. □

Example 3.4 is also an example of Theorem 4.2.

Theorem 4.3. *Suppose that $SP_2 \neq \emptyset$ and for each $x \in X$ there exists $y \in Y$ such that $f(x, y) \in C$. Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that*

$$f(x, y) \notin -\mathrm{int}\, C \textit{ for all } x \in X.$$

Proof. Let $(x_0, y_0) \in \mathrm{SP}_2$. Then there exists $\hat{y} \in Y$ such that $f(x_0, \hat{y}) \in C$. Since $f(x_0, y_0) - f(x_0, \hat{y}) \in C$, $f(x_0, y_0) \in C$. Hence by Theorem 4.2, $y_0 \in Y$ is a solution of (GVEP). □

Theorem 4.4. *Suppose*

$$SP_3 = \left\{ (x, y) \in X \times Y : \begin{array}{ll} f(u, y) - f(x, y) \in C & \forall u \in X, \textit{ and} \\ f(x, y) - f(x, v) \notin -\mathrm{int}\, C & \forall v \in Y \end{array} \right\} \neq \emptyset$$

and $f(x_0, y_0) \notin -\mathrm{int}\, C$ for some $(x_0, y_0) \in SP_3$. Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that

$$f(x, y) \notin -\mathrm{int}\, C \textit{ for all } x \in X.$$

Proof. Since $(x_0, y_0) \in \mathrm{SP}_3$, it follows

$$f(x, y_0) - f(x_0, y_0) \in C \text{ for all } x \in X.$$

Since $f(x_0, y_0) \notin (-C)$ by Proposition 4.1, we have

$$f(x, y_0) \notin -\mathrm{int}\, C \text{ for all } x \in X.$$

Hence $y_0 \in X$ is a solution of (GVEP). Therefore (GVEP) has at least one solution. □

Theorem 4.5. *Suppose that $SP_3 \neq \emptyset$ and for each $y \in Y$ there exists $x_y \in T(y)$ such that $f(x_y, y) \notin -\mathrm{int}\, C$, where*

$$T(y) = \{x \in X : f(u, y) - f(x, y) \in C \textit{ for all } u \in X\}.$$

Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that

$$f(x, y) \notin -\mathrm{int}\, C \textit{ for all } x \in X.$$

Proof. Let $(x_0, y_0) \in \mathrm{SP}_3$. Then there exists $\hat{x} \in T(y_0)$ such that $f(\hat{x}, y_0) \notin -\operatorname{int} C$. Since $\hat{x} \in T(y_0)$, $f(u, y_0) - f(\hat{x}, y_0) \in C$ for all $u \in X$, i.e., $f(x_0, y_0) = f(\hat{x}, y_0)$. Therefore $f(x_0, y_0) \notin -\operatorname{int} C$. Hence by Theorem 4.4, $y_0 \in Y$ is a solution of (GVEP). $\square$

Theorem 4.6. *Suppose that $SP_3 \neq \emptyset$ and for each $x \in X$ there exists $y_x \in Y$ such that $f(x, y_x) \in C$. Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that*

$$f(x, y) \notin -\operatorname{int} C \text{ for all } x \in X.$$

Proof. Let $(x_0, y_0) \in \mathrm{SP}_3$. Then there exists $\hat{y} \in Y$ such that $f(x_0, \hat{y}) \in C$. Since $(x_0, y_0) \in \mathrm{SP}_3$, $f(x_0, y_0) - f(x_0, \hat{y}) \notin -\operatorname{int} C$. Therefore $f(x_0, y_0) \notin -\operatorname{int} C$. Hence by Theorem 4.4, $y_0 \in Y$ is a solution of (GVEP). $\square$

Theorem 4.7. *Suppose*

$$SP_4 = \left\{ (x, y) \in X \times Y : \begin{array}{ll} f(u, y) - f(x, y) \in C & \forall u \in X, \text{ and} \\ f(x, y) - f(x, v) \in C & \forall v \in Y \end{array} \right\} \neq \emptyset$$

and $f(x_0, y_0) \notin -\operatorname{int} C$ for some $(x_0, y_0) \in SP_4$. Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that

$$f(x, y) \notin -\operatorname{int} C \text{ for all } x \in X.$$

Proof. Since $(x_0, y_0) \in \mathrm{SP}_4$, it follows

$$f(x, y_0) - f(x_0, y_0) \in C \text{ for all } x \in X.$$

Since $f(x_0, y_0) \notin (-C)$ by Proposition 4.1, we have

$$f(x, y_0) \notin -\operatorname{int} C \text{ for all } x \in X.$$

Hence $y_0 \in X$ is a solution of (GVEP). Therefore (GVEP) has at least one solution. $\square$

Examples 3.5, 3.6 and 3.7 are also examples of Theorem 4.7.

Theorem 4.8. *Suppose that $SP_4 \neq \emptyset$ and for each $y \in Y$ there exists $x_y \in T(y)$ such that $f(x_y, y) \notin -\operatorname{int} C$, where*

$$T(y) = \{x \in X : f(u, y) - f(x, y) \in C \text{ for all } u \in X\}.$$

Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that

$$f(x, y) \notin -\operatorname{int} C \text{ for all } x \in X.$$

Proof. Let $(x_0, y_0) \in \mathrm{SP}_4$. Then there exists $\hat{x} \in T(y_0)$ such that $f(\hat{x}, y_0) \notin -\operatorname{int} C$. Since $\hat{x} \in T(y_0)$, $f(u, y_0) - f(\hat{x}, y_0) \in C$ for all $u \in X$, i.e., $f(x_0, y_0) = f(\hat{x}, y_0)$. Therefore $f(x_0, y_0) \notin -\operatorname{int} C$. Hence by Theorem 4.7, $y_0 \in Y$ is a solution of (GVEP). $\square$

Theorem 4.9. *Suppose that $SP_4 \neq \emptyset$ and for each $x \in X$ there exists $y_x \in Y$ such that $f(x, y_x) \notin -\operatorname{int} C$. Then,(GVEP) has at least one solution, i.e., there exists $y \in Y$ such that*

$$f(x, y) \notin -\operatorname{int} C \text{ for all } x \in X.$$

Proof. Let $(x_0, y_0) \in \mathrm{SP}_4$. Then there exists $\hat{y} \in Y$ such that $f(x_0, \hat{y}) \notin -\operatorname{int} C$. Since $(x_0, y_0) \in \mathrm{SP}_4$, $f(x_0, y_0) - f(x_0, \hat{y}) \in C$, i.e., $f(x_0, y_0) = f(x_0, \hat{y})$. Therefore $f(x_0, y_0) \notin -\operatorname{int} C$. Hence by Theorem 4.7, $y_0 \in Y$ is a solution of (GVEP). □

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