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NONLINEAR SCALARIZATIONS AND SOME APPLICATIONS IN VECTOR OPTIMIZATION

YOUSUKE ARAYA

ABSTRACT. We first give a little improvement of nonlinear scalarizing function for vector optimization problem organized by Luc and Tammer-Weidner. As applications, we present Gordan's type alternative theorems for vector-valued function, optimality conditions for vector optimization problem and an existence theorem for vector saddle-points.

1. Introduction

This paper is concerned with nonlinear scalarization technique for vector optimization problems and some applications. The idea of the sublinear scalarizing function was dealt by Krasnosel'skij [7] in 1962 and by Rubinov [11] in 1977, and then it was applied to vector optimization with its concrete definition by Tammer (Gerstewitz) [1] in 1983, and to separation theorems for not necessary convex sets by Tammer (Gerstewitz) and Iwanow [2] in 1985. Afterwards Luc [8] and Tammer (Gerth) and Weidner [3] organized sublinear scalarizing functions for vectors. These functions, which appear in slightly different forms, have wide applications in vector optimization.

In this paper, we mixed their idea on those functions, that is, we introduce unified approach on such scalarization for vectors and investigate some properties of nonlinear scalarizing functions. Some applications for them are also given.

The organization of this paper is as follows. We first give a little improvement of nonlinear scalarizing function for vector optimization problem organized by Luc and Tammer-Weidner. Next we give some applications of the scalarizing functions. At first, we present Gordan's type alternative theorems for vector-valued function. A generalized Gordan's type alternative theorem was given for vector-valued function by Jeyakumar [5] in 1986. It relies on a certain convexity assumption like cone-subconvexlikeness and separation theorems; we relax the convexity assumption. As a corollary, we give optimality conditions for vector optimization problem and more simple proofs than that of Gerth and Weidner in [3]. Finally we give an existence theorem for vector saddle points.

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In this section, let Y be a topological vector space and 0_Y the origin of Y. For a set $A \subset Y$, $\mathrm{int}A$, $\mathrm{cor}A$ and $\mathrm{cl}A$ denote the topological interior, the algebraic interior and the topological closure of A, respectively. Let $C \subset Y$ be a closed convex cone, that is, $\mathrm{cl}C = C$, $C + C \subset C$ and $[0, \infty) \cdot C \subset C$. A cone C is called pointed if $C \cap (-C) = \{0_Y\}$, and solid if $\mathrm{int}C \neq \emptyset$.

It is well known that, given a pointed convex cone $C \subset Y$, we can induce a partial ordering $\leq_C$ in Y defined by $x \leq_C y$ when $y - x \in C$. We denote $x \leq_{\mathrm{int}C} y$ when $y - x \in \mathrm{int}C$, $x \not\leq_C y$ when $y - x \notin C$ and $x \not\leq_{\mathrm{int}C} y$ when $y - x \notin \mathrm{int}C$. This ordering is compatible with the vector structure of Y, that is, for every $x \in Y$ and $y \in Y$,

(i) $x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in Y$

(ii) $x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$.

We say that $a \in A$ is a minimal [resp. weak minimal] point of A if

$$A \cap (a - C) = \{a\} \quad [\text{resp. } A \cap (a - \mathrm{int}C) = \emptyset],$$

or equivalently, there is no $\hat{a} \in A$ such that $\hat{a} \leq_C a$ (resp. $\hat{a} \leq_{\mathrm{int}C} a$). Similarly, we say that $a \in A$ is a maximal [resp. weak maximal] point of A if

$$A \cap (a + C) = \{a\} \quad [\text{resp. } A \cap (a + \mathrm{int}C) = \emptyset],$$

or equivalently, there is no $\hat{a} \in A$ such that $a \leq_C \hat{a}$ (resp. $a \leq_{\mathrm{int}C} \hat{a}$). We denote by $\mathrm{Min}(A; C)$ [resp. $\mathrm{wMin}(A; \mathrm{int}C)$] and $\mathrm{Max}(A; C)$ [resp. $\mathrm{wMax}(A; \mathrm{int}C)$] the set of minimal [resp. weak minimal] and maximal [resp. weak maximal] points of A with respect to C [resp. $\mathrm{int}C$]. We can easily see that

$$\mathrm{Min}(A; C) \subset \mathrm{wMin}(A; \mathrm{int}C) \subset A,$$

$$\mathrm{Max}(A; C) \subset \mathrm{wMax}(A; \mathrm{int}C) \subset A.$$

2. Nonlinear scalarizing functions

2.1. Infimum type

We introduce the following two-variable infimum type of nonlinear scalarizing function for vector optimization problems; we define $h_{\inf}(y; a) := \varphi_{C,k^0}(y - a)$ for $a, y \in Y$ where $\varphi_{C,k^0}(z) = \inf\{t \in \mathbb{R} \,|\, z \leq_C tk^0\}$ $(z \in Y)$ used in Theorem 2.3.1 in [4] and [8].

Theorem 2.1. *Let Y be a topological vector space and $C \subset Y$ a closed convex cone. We take $k^0 \in C \setminus (-C)$ and define $h_{\inf} : Y \times Y \to [-\infty, \infty]$ by*

$$h_{\inf}(y; a) = \inf\{t \in \mathbb{R} \,\big|\, y \leq_C tk^0 + a\} = \inf\{t \in \mathbb{R} \,\big|\, y \in tk^0 + a - C\}.$$

Then the function $h_{\inf}$ has the following properties:

(i) $h_{\inf}$ *is proper;*

(ii) $\{y \in Y | h_{\inf}(y; a) \leq t\} = tk^0 + a - C$;

(iii) $\{a \in Y | h_{\inf}(y; a) \leq t\} = -tk^0 + y + C$;

(iv) $h_{\inf}(\cdot; a)$ *is lower semicontinuous;*

(v) $h_{\inf}(y; \cdot)$ *is lower semicontinuous;*

(vi) $h_{\inf}(\cdot; a)$ *is* C*-increasing (i.e.,* $y_1 \leq_C y_2$ *implies* $h_{\inf}(y_1; a) \leq h_{\inf}(y_2; a)$*);*

(vii) $h_{\inf}(y; \cdot)$ *is* C*-decreasing (i.e.,* $a_1 \leq_C a_2$ *implies* $h_{\inf}(y; a_1) \geq h_{\inf}(y; a_2)$*);*

(viii) $h_{\inf}(y + \lambda k^0; a) = h_{\inf}(y; a) + \lambda$ *for every* $y \in Y$ *and* $\lambda \in \mathbb{R}$;

(ix) $h_{\inf}(y; a + \lambda k^0) = h_{\inf}(y; a) - \lambda$ *for every* $y \in Y$ *and* $\lambda \in \mathbb{R}$.

(x) $h_{\inf}(\cdot; 0_Y)$ *and* $h_{\inf}(0_Y; \cdot)$ *are sublinear.*

Moreover, if $k^0 \in \mathrm{int}C$ *then* $h_{\inf}$ *has the following properties:*

(xi) $h_{\inf}$ *achieves a real value;*

(xii) $\{y \in Y | h_{\inf}(y; a) < t\} = tk^0 + a - \mathrm{int}C$;

(xiii) $\{a \in Y | h_{\inf}(y; a) < t\} = -tk^0 + y + \mathrm{int}C$;

(xiv) $h_{\inf}(\cdot; a)$ *is continuous;*

(xv) $h_{\inf}(y; \cdot)$ *is continuous;*

(xvi) $h_{\inf}(\cdot; a)$ *is strictly* $\mathrm{int}C$*-increasing (i.e.,* $y_1 \leq_{\mathrm{int}C} y_2$ *implies* $h_{\inf}(y_1; a) < h_{\inf}(y_2; a)$*);*

(xvii) $h_{\inf}(y; \cdot)$ *is strictly* $\mathrm{int}C$*-decreasing (i.e.,* $a_1 \leq_{\mathrm{int}C} a_2$ *implies* $h_{\inf}(y; a_1) > h_{\inf}(y; a_2)$*).*

Proof. The proof is similar to that of Theorem 2.3.1 in [4]. We prove statements (i), (vii), and (xi).

For (i), we have $h_{\inf}(y; a) > -\infty$. Indeed, if $h_{\inf}(y; a)$ is not bounded from below, then we have $y \in -nk^0 + a - C$ for all $n \in \mathbb{N}$. Then $-k^0 + \frac{a-y}{n} \in C$ for all $n \in \mathbb{N}$. Taking $n \to \infty$, we have $-k^0 \in C$, which is a contradiction. Since $h_{\inf}(0_Y; 0_Y) \leq 1$, $h_{\inf}$ is proper.

Next, we prove (vii). Let $a_1, a_2 \in Y$ be such that $a_1 \leq_C a_2$. For any $y \in Y$, $h_{\inf}(y; a_1) > -\infty$ by (i), and hence we consider the case of $h_{\inf}(y; a_1) \in \mathbb{R}$. We have $y \in h_{\inf}(y; a_1)k^0 + a_1 - C$. Then we obtain

$$y \in h_{\inf}(y; a_1)k^0 + (a_2 - C) - C \subset h_{\inf}(y; a_1)k^0 + a_2 - C$$

and hence by (ii), we obtain $h_{\inf}(y;a_1) \geq h_{\inf}(y;a_2)$.

Finally, we prove (xi). We first prove $Y = \mathbb{R}k^0 - C$ and $Y = \mathbb{R}k^0 + C$ when $k^0 \in \mathrm{cor}C (= \mathrm{int}C)$. If $k^0 \in \mathrm{cor}C$, we have $0 \in \mathrm{cor}(k^0 - C)$. By the definition of the core, for any $y \in Y$ there exists $\bar{\lambda} > 0$ such that for all $\lambda \in (0, \bar{\lambda}]$

$$0 + \lambda y \in k^0 - C$$

and hence

$$y \in \frac{1}{\lambda}k^0 - \frac{1}{\lambda}C \subset \mathbb{R}k^0 - C.$$

In a similar way, we can show $Y = \mathbb{R}k^0 + C$. We can also show $\mathrm{dom}(h_{\inf}) \supset (\mathbb{R}k^0 - C) \times (\mathbb{R}k^0 + C) = Y \times Y$. Hence, $h_{\inf}(y;a) \in \mathbb{R}$ for any $y, a \in Y$. □

As a corollary of the above lemma, we present a simple proof for Luc's nonconvex separation theorem.

Theorem 2.2 (Luc [8])**.** *Let Y be a topological vector space, $C \subset Y$ a solid closed convex cone and $k^0 \in \mathrm{int}C$. We assume $B \subset Y$ a nonempty set and $a \in Y$ such that $B \cap (a - \mathrm{int}C) = \emptyset$ [resp. $B \cap (a - C) = \emptyset$]. Then $h_{\inf}$ is a finite-valued continuous function such that*

$$h_{\inf}(y;a) < 0 \leq h_{\inf}(x;a) \quad \forall x \in B,\ y \in a - \mathrm{int}C$$

$$[\textit{resp. } h_{\inf}(y;a) \leq 0 < h_{\inf}(x;a) \quad \forall x \in B,\ y \in a - C].$$

Proof. By (xii) [resp. (ii)] of Theorem 2.1, we have

$$\{y \in Y | h_{\inf}(y;a) < 0\} = a - \mathrm{int}C \qquad [\text{resp. } \{y \in Y | h_{\inf}(y;a) \leq 0\} = a - C].$$

Moreover, we also have

$$B \subset (a - \mathrm{int}C)^c = \{y \in Y | h_{\inf}(y;a) \geq 0\}$$

$$[\text{resp. } B \subset (a - C)^c = \{y \in Y | h_{\inf}(y;a) > 0\}].$$

□

2.2. Supremum type

We introduce the following two-variable supremum type of nonlinear scalarizing function for vector optimization problems. We can easily prove the following properties by remarking $h_{\sup}(y;a) = -h_{\inf}(-y;-a)$; see also [10].

Theorem 2.3. *Let Y be a topological vector space and $C \subset Y$ be a closed convex cone. We take $k^0 \in C \setminus (-C)$ and define $h_{\inf} : Y \times Y \to [-\infty, \infty]$ by*

$$h_{\sup}(y;a) = \sup\{t \in \mathbb{R} \,\big|\, tk^0 + a \leq_C y\} = \sup\{t \in \mathbb{R} \,\big|\, y \in tk^0 + a + C\}.$$

Then the function $h_{\sup}$ has the following properties:

(i) $h_{\sup}$ *is proper;*

(ii) $\{y \in Y | h_{\sup}(y; a) \geq t\} = tk^0 + a + C$*;*

(iii) $\{a \in Y | h_{\sup}(y; a) \geq t\} = -tk^0 + y - C$*;*

(iv) $h_{\sup}(\cdot; a)$ *is upper semicontinuous;*

(v) $h_{\sup}(y; \cdot)$ *is upper semicontinuous;*

(vi) $h_{\sup}(\cdot; a)$ *is* C*-increasing (i.e.,* $y_1 \leq_C y_2$ *implies* $h_{\sup}(y_1; a) \leq h_{\sup}(y_2; a)$*);*

(vii) $h_{\sup}(y; \cdot)$ *is* C*-decreasing (i.e.,* $a_1 \leq_C a_2$ *implies* $h_{\sup}(y; a_1) \geq h_{\sup}(y; a_2)$*);*

(viii) $h_{\sup}(y + \lambda k^0; a) = h_{\sup}(y; a) + \lambda$ *for every* $y \in Y$ *and* $\lambda \in \mathbb{R}$*.*

(ix) $h_{\sup}(y; a + \lambda k^0) = h_{\sup}(y; a) - \lambda$ *for every* $y \in Y$ *and* $\lambda \in \mathbb{R}$*.*

(x) $h_{\sup}(\cdot; 0_Y)$ *and* $h_{\sup}(0_Y; \cdot)$ *is positively homogeneous and superadditive.*

Moreover, if $k^0 \in \mathrm{int}C$ *then* $h_{\sup}$ *has the following properties:*

(xi) $h_{\sup}$ *achieves a real value;*

(xii) $\{y \in Y | h_{\sup}(y; a) > t\} = tk^0 + a + \mathrm{int}C$*;*

(xiii) $\{a \in Y | h_{\sup}(y; a) > t\} = -tk^0 + y - \mathrm{int}C$*;*

(xiv) $h_{\sup}(\cdot; a)$ *is continuous;*

(xv) $h_{\sup}(y; \cdot)$ *is continuous;*

(xvi) $h_{\sup}(\cdot; a)$ *is strictly* $\mathrm{int}C$*-increasing (i.e.,* $y_1 \leq_{\mathrm{int}C} y_2$ *implies* $h_{\sup}(y_1; a) < h_{\sup}(y_2; a)$*).*

(xvii) $h_{\sup}(y; \cdot)$ *is strictly* $\mathrm{int}C$*-decreasing (i.e.,* $a_1 \leq_{\mathrm{int}C} a_2$ *implies* $h_{\sup}(y; a_1) > h_{\sup}(y; a_2)$*);*

As a corollary of the above lemma, we obtain the following nonconvex separation theorem in a similar way as Theorem 2.2.

Theorem 2.4. *Let* Y *be a topological vector space,* $C \subset Y$ *a solid closed convex cone and* $k^0 \in \mathrm{int}C$*. We assume* $B \subset Y$ *a nonempty set and* $a \in Y$ *such that* $B \cap (a + \mathrm{int}C) = \emptyset$ *[resp.* $B \cap (a + C) = \emptyset$*]. Then* $h_{\sup}$ *is a finite-valued continuous function such that*

$$h_{\sup}(x; a) \leq 0 < h_{\sup}(y; a) \quad \forall x \in B,\ y \in a + \mathrm{int}C$$

$$[\textit{resp. } h_{\sup}(x; a) < 0 \leq h_{\sup}(y; a) \quad \forall x \in B,\ y \in a + C].$$

3. Applications

Firstly, we consider the following vector optimization problem:

$$\text{(VP)}\begin{cases} C\text{−minimize} & f(x) \\ \text{subject to} & x \in X \end{cases}$$

where $f : X \to Y$ and C is a solid closed convex cone.

In this section, we show Gordan's type alternative theorems for vector-valued function, optimality conditions for the above vector optimization problem, and existence results for vector-valued saddle-point problems as an application of scalarizing functions defined in the previous section.

3.1. Alternative theorems

We first present Gordan's type alternative theorems for vector-valued function. Jeyakumar [5] in 1986 assumed a certain convexity assumption like cone subconvexlikeness and used separation theorems. We use nonlinear scalarizing functions defined in Section 2 and relax the convexity assumption.

Theorem 3.1 (inf type). *Let Y be a topological vector space, $C \subset Y$ a solid closed convex cone, $B \subset Y$ a nonempty set and $a \in Y$. Then exactly one of the following systems hold:*

(i) $B \subset a - \text{int}C$ [*resp.* $B \subset a - C$],

(ii) *there exists* $k^0 \in \text{int}C$ *such that* $h_{\inf}(x; a) \geq 0$ [*resp.* $h_{\inf}(x; a) > 0$] *for all* $x \in B$.

Proof. First, we assume that (i) holds. System (i) states that $B \leq_{\text{int}C} a$ [resp. $B \leq_C a$]. Using (xvi) [resp. (vi)] of Theorem 2.1, we have

$$h_{\inf}(B; a) < h_{\inf}(a; a) = 0 \quad [\text{resp. } h_{\inf}(B; a) \leq h_{\inf}(a; a) = 0]$$

for every $k^0 \in \text{int}C$, which shows that system (ii) does not hold. Next, we assume that system (i) does not hold. Then we have $B \not\subset a - \text{int}C$ [resp. $B \not\subset a - C$], which is equivalent to

$$B \cap (a - \text{int}C) = \emptyset \quad [\text{resp. } B \cap (a - C) = \emptyset].$$

Using Theorem 2.2, there exists $k^0 \in \text{int}C$ such that

$$h_{\inf}(x; a) \geq 0 \quad [\text{resp. } h_{\inf}(x; a) > 0]$$

for all $x \in B$, which shows that system (ii) holds. □

Theorem 3.2 (sup type). *Let Y be a topological vector space, $C \subset Y$ a solid closed convex cone, $B \subset Y$ a nonempty set and $a \in Y$. Then exactly one of the following systems hold:*

(i) $B \subset a + \mathrm{int}C$ [*resp.* $B \subset a + C$],

(ii) *there exists* $k^0 \in \mathrm{int}C$ *such that* $h_{\sup}(x; a) \leq 0$ [*resp.* $h_{\sup}(x; a) < 0$] *for all* $x \in B$.

Proof. The proof is similar to that of Theorem 3.1, by using Theorems 2.3 and 2.4 instead of Theorems 2.1 and 2.2. □

Combinig Theorem 3.1 to Theorem 3.2, we obtain the following alternative theorems for vector-valued function.

Theorem 3.3 (inf-sup type). *Let* Y *be a topological vector space,* $C \subset Y$ *a solid closed convex cone,* $B \subset Y$ *a nonempty set and* $a \in Y$. *Then exactly one of the following systems hold:*

(i) $B \subset a - \mathrm{int}C$ *or* $B \subset a + \mathrm{int}C$ [*resp.* $B \subset a - C$ *or* $B \subset a + C$],

(ii) *there exists* $k^0 \in \mathrm{int}C$ *such that* $h_{\inf}(x; a) \geq 0$ *and* $h_{\sup}(x; a) \leq 0$ [*resp.* $h_{\inf}(x; a) > 0$ *and* $h_{\sup}(x; a) < 0$] *for all* $x \in B$.

Remark 1. There are some alternative theorems for set-valued functions ([9] and its references therein). When we take $a = 0_Y$ in the above theorems, they are generalized for set-valued functions in Nishizawa-Onodsuka-Tanaka [9]. Theorem 3.1 is generalized to Theorems 3.1 and 3.2 [resp. Theorems 3.6 and 3.7] in [9], Theorem 3.2 is generalized to Theorems 3.3 and 3.4 [resp. Theorems 3.8 and 3.9] in [9] and Theorem 3.3 is generalized to Theorem 3.5 [resp. Theorem 3.10] in [9].

3.2. Optimality conditions

As an application of the above theorems, we present optimality conditions for vector optimization problem.

Theorem 3.4 (Infimum type). *Let* X *be a vector space,* Y *a topological vector space,* $C \subset Y$ *a solid closed convex cone,* $f : X \to Y$ *a vector-valued function and* $\bar{x} \in X$. *Then* $f(\bar{x}) \in \mathrm{wMin}(f(X); \mathrm{int}C)$ *if and only if there exists* $k^0 \in \mathrm{int}C$ *such that* $h_{\inf}(f(X); f(\bar{x})) \geq 0$.

Proof. This is straightforward from the fact that $f(\bar{x})$ being a weakly minimal point imples $f(x) \notin f(\bar{x}) - \mathrm{int}C$ for all $x \in X$, that is, system (i) of Theorem 3.1 is not satisfied. □

Theorem 3.5 (Infimum type). *Let* X *be a vector space,* Y *a topological vector space,* $C \subset Y$ *a solid closed convex cone,* $f : X \to Y$ *a vector-valued function and* $\bar{x} \in X$. *Then* $f(\bar{x}) \in \mathrm{Min}(f(X); C)$ *if and only if there exists* $k^0 \in \mathrm{int}C$ *such that* $h_{\inf}(f(X); f(\bar{x})) > 0$.

Theorem 3.6 (**Supremum type**). *Let X be a vector space, Y a topological vector space, $C \subset Y$ a solid closed convex cone, $f : X \to Y$ a vector-valued function and $\bar{x} \in X$. Then $f(\bar{x}) \in \mathrm{wMax}(f(X); \mathrm{int}C)$ if and only if there exists $k^0 \in \mathrm{int}C$ such that $h_{\sup}(f(X); f(\bar{x})) \leq 0$.*

Theorem 3.7 (**Supremum type**). *Let X be a vector space, Y a topological vector space, $C \subset Y$ a solid closed convex cone, $f : X \to Y$ a vector-valued function and $\bar{x} \in X$. Then $f(\bar{x}) \in \mathrm{Max}(f(X); C)$ if and only if there exists $k^0 \in \mathrm{int}C$ such that $h_{\sup}(f(X); f(\bar{x})) < 0$.*

3.3. Vector-valued saddle-point problem

In this subsection, we consider the vector-valued saddle-point problem, and we show an existence of weak C-saddle-points as an application of the scalarizations. Let $f : X \times Y \to Z$ be a vector-valued function. The vector-valued saddle-point problem is to find a pair $x \in X$ and $y \in Y$ such that

$$(\mathrm{P}) \begin{cases} f(x,y) - f(u,y) \notin \mathrm{int}C & \text{for all } u \in X, \\ f(x,v) - f(x,y) \notin \mathrm{int}C & \text{for all } v \in Y. \end{cases}$$

A point $(x, y) \in X \times Y$ is said to be a weak C-saddle point of function f on $X \times Y$, if it is a solution of the problem.

Kimura and Tanaka [6] consider applying the inf-type scalarization to existence of cone-saddle point. We consider the sup-type scalarization technique in a similar way as [6].

Definition 3.8. Let X be a topological space and Z a normed space with the partial ordering by a solid pointed convex cone $C \subset Z$. A vector-valued function $f : X \to Z$ is said to be C-continuous at X if the set $\{x \in X | f(x) \leq_C z\}$ is closed for all $z \in Z$.

Definition 3.9. Let K be a convex set in a real vector space X, Z a normed space with the partial ordering by a solid pointed convex cone $C \subset Z$. A vector-valued function $f : X \to Z$ is said to be C-quasiconvex on K if for each $x_1, x_2 \in K$, $\lambda \in [0, 1]$ and $z \in Z$, we have that

$$f(x_1), f(x_2) \in z - C \quad \text{implies} \quad f(\lambda x_1 + (1 - \lambda)x_2) \in z - C.$$

Definition 3.10. Let K be a convex set in a real vector space X, Z a normed space with the partial ordering by a solid pointed convex cone $C \subset Z$. A vector-valued function $f : X \to Z$ is said to be C-properly quasiconvex on K if either

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_1) - C,$$

or

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_2) - C,$$

for every $x_1, x_2 \in K$ and $\lambda \in [0, 1]$.

Lemma 3.11. *Let X be a topological space, Z a normed space with the partial ordering by a solid pointed convex cone $C \subset Z$ and $k^0 \in \mathrm{int}C$. Let $h_{\sup}(\cdot; 0_Y)$ be the scalarizing function on Z and $f : X \to Z$ a vector-valued function.*

(i) *If f is C-continuous at $x \in X$, then $(h_{\sup}(\cdot; 0_Y) \circ f)$ is lower semicoutinuous at $x \in X$.*

(ii) *If f is $(-C)$-continuous at $x \in X$, then $(h_{\sup}(\cdot; 0_Y) \circ f)$ is upper semicoutinuous at $x \in X$.*

Proof. It is clear from the monotonicity of the scalarizing function $h_{\sup}(\cdot; 0_Y)$. $\square$

Lemma 3.12. *Let K be a convex set in a real vector space X, Z a normed space with the partial ordering by a solid pointed convex cone $C \subset Z$ and $k^0 \in \mathrm{int}C$. Let $h_{\sup}(\cdot; 0_Y)$ be the scalarizing function on Z and $f : X \to Z$ a vector-valued function. If f is $(-C)$-quasiconvex on K, then $(h_{\sup}(\cdot; 0_Y) \circ f)$ is quasiconcave on K.*

Proof. The proof is similar to that of Theorem 2 in [6]. $\square$

Lemma 3.13. *Let K be a convex set in a real vector space X, Z a normed space with the partial ordering by a solid pointed convex cone $C \subset Z$ and $k^0 \in \mathrm{int}C$. Let $h_{\sup}(\cdot; 0_Y)$ be the scalarizing function on Z and $f : X \to Z$ a vector-valued function. If f is C-properly quasiconvex on K, then $(h_{\sup}(\cdot; 0_Y) \circ f)$ is quasiconvex on K.*

Proof. Let

$$\mathrm{Lev}((h_{\sup}(\cdot; 0_Y) \circ f); \alpha) := \{x \in K | (h_{\sup}(\cdot; 0_Y) \circ f)(x) \leq \alpha\}.$$

Let $\lambda \in [0, 1]$ and $x_1, x_2 \in \mathrm{Lev}((h_{\sup}(\cdot; 0_Y) \circ f); \alpha)$. By (xii) of Theorem 2.3, we have

$$f(x_1), f(x_2) \notin \alpha k^0 + \mathrm{int}C$$

and hence

$$(f(x_1) - C) \cap (\alpha k^0 + \mathrm{int}C) = \emptyset \quad \text{and} \quad (f(x_2) - C) \cap (\alpha k^0 + \mathrm{int}C) = \emptyset.$$

By the C-properly quasiconvexty of f, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \notin (\alpha k^0 + \mathrm{int}C),$$

which implies that $\lambda x_1 + (1 - \lambda)x_2 \in \mathrm{Lev}((h_{\sup}(\cdot; 0_Y) \circ f); \alpha)$. $\square$

Kimura and Tanaka presented an existence theorem of cone saddle-point by using scalarizing function $h_{\inf}(\cdot; 0_Y)$.

Theorem 3.14 (Kimura and Tanaka [6]). *Let X and Y be nonempty compact convex sets in two normed spaces, respectively, and Z a normed space with a partial ordering induced by a solid pointed convex cone $C \subset Z$. If a vector-valued function $f : X \times Y \to Z$ satisfies that*

(i) $x \mapsto f(x, y)$ *is C-continuous and C-quasiconvex on X for every $y \in Y$,*

(ii) $y \mapsto f(x, y)$ *is $(-C)$-continuous and $(-C)$-properly quasiconvex on Y for every $x \in X$,*

then f has at least one weak C-saddle point.

We obtain another type of existence theorem of cone saddle-point by using scalarizing function $h_{\sup}(\cdot; 0_Y)$.

Theorem 3.15. *Let X and Y be nonempty compact convex sets in two normed spaces, respectively, and Z a normed space with a partial ordering induced by a solid pointed convex cone $C \subset Z$. If a vector-valued function $f : X \times Y \to Z$ satisfies that*

(i) $x \mapsto f(x, y)$ *is C-continuous and C-properly quasiconvex on X for every $y \in Y$,*

(ii) $y \mapsto f(x, y)$ *is $(-C)$-continuous and $(-C)$ quasiconvex on Y for every $x \in X$,*

then f has at least one weak C-saddle point.

Proof. We see that, by Lemmas 3.11 and 3.13, the map $x \mapsto (h_{\sup}(\cdot; 0_Y) \circ f)(x, y)$ is lower semicontinuous and quasiconvex on X. Moreover, we see that, by Lemmas 3.11 and 3.12, the map $y \mapsto (h_{\sup}(\cdot; 0_Y) \circ f)(x, y)$ is upper semicontinuous and quasiconcave on Y. By Sions's minimax theorem [12], $h_{\sup}(\cdot; 0_Y) \circ f$ has a saddle point and by Theorem 3.2, f has at least one weak C-saddle point. $\square$

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(Yousuke Araya) College of Science and Engineering, Ritsumeikan University, 1–1–1 Noji Higashi, Kusatsu, Shiga 525–8577, Japan

E-mail address: `y-araya@fc.ritsumei-u.ac.jp`

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