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THE STABLE RANK AND CONNECTED STABLE RANK FOR CERTAIN NON SELF-ADJOINT BANACH ALGEBRAS

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ABSTRACT. We consider the stable rank and connected stable rank for certain non self-adjoint Banach algebras such as the triangular matrix algebras of all (finite or infinite) triangular matrices over a unital Banach algebra and certain nest algebras. Also, the stable rank estimate for certain crossed products of unital Banach algebras by isometries is obtained.

1. Introduction

The (left or right, topological) stable rank for Banach algebras is introduced and studied by Rieffel [5]. Especially, it is shown (by [5] and Herman-Vaserstein [4]) that the (left and right) stable rank for C^*-algebras is the same as the Bass stable rank for rings. Also, the left stable rank is the same as the right stable rank for Banach $*$-algebras with involutions ([5]). The (left or right) connected stable rank for Banach algebras is also introduced and studied by Rieffel [5]. Among many things, Rieffel [5] obtained the stable rank formula for matrix algebras over a C^*-algebra in terms of the stable rank of the C^*-algebra in coefficient, and also the stable rank formula for the crossed product of a C^*-algebra by an action of the group of integers. In [6], Rieffel obtained the connected stable rank formula for matrix algebras over a C^*-algebra as well.

On the other hand, remarkably, it is recently shown by Davidson, Levene, Marcoux, and Radjavi [1] that there exist non self-adjoint Banach algebras which have left stable rank infinity and right stable rank two, so that without involutions, the left and right stable ranks differ in general. Such algebras are provided from nest algebras.

Inspired by the papers [5] (and [6]) and [1], in this paper we consider the (left or right) stable rank and connected stable rank for certain non self-adjoint Banach

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algebras as follows. In Section 1, we compute the stable rank of the triangular matrix algebras of all (finite, or infinite and bounded) triangular matrices over unital Banach algebras in terms of the Banach algebras in coefficient, and in Section 2, we compute the connected stable rank for those triangular matrix algebras in terms of the Banach algebras in coefficient. In Section 3, we consider certain cases of nest algebras (in fact, those triangular matrix algebras in Sections 1 and 2 may be also viewed as nest algebras). In Section 4, we further consider the stable rank estimate for (non self-adjoint) crossed products of unital Banach algebras by isometries.

Notation. Let $\mathfrak{A}$ be a unital Banach algebra. Denote by $\mathrm{lsr}(\mathfrak{A})$ the left (topological) stable rank of $\mathfrak{A}$ (a positive integer or ∞). By definition, $\mathrm{lsr}(\mathfrak{A}) \leq n$ if and only if $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, where $(a_j) \in L_n(\mathfrak{A})$ if $\sum_{j=1}^n b_j a_j = 1$ for some $(b_j) \in \mathfrak{A}^n$. If no such n, then $\mathrm{lsr}(\mathfrak{A}) = \infty$. Similarly, the right stable rank $\mathrm{rsr}(\mathfrak{A})$ of $\mathfrak{A}$ is defined by just replacing $L_n(\mathfrak{A})$ with $R_n(\mathfrak{A})$ of $(a_j) \in \mathfrak{A}^n$ such that $\sum_{j=1}^n a_j b_j = 1$ for some $(b_j) \in \mathfrak{A}^n$.

Denote by $\mathrm{lcsr}(\mathfrak{A})$ the left connected stable rank of $\mathfrak{A}$. By definition, $\mathrm{lcsr}(\mathfrak{A}) \leq n$ if and only if $GL_m(\mathfrak{A})_0$ acts transitively on $L_m(\mathfrak{A})$ for any $m \geq n$, where $GL_m(\mathfrak{A})_0$ means the connected component with the identity matrix of the group $GL_m(\mathfrak{A})$ of all $m \times m$ invertible matrices over $\mathfrak{A}$, which is equivalent to say that $L_m(\mathfrak{A})$ is connected for any $m \geq n$. Similarly, the right connected stable rank $\mathrm{rcsr}(\mathfrak{A})$ of $\mathfrak{A}$ is defined by just replacing $L_m(\mathfrak{A})$ with $R_m(\mathfrak{A})$.

2. Stable rank

Let $\mathfrak{A}$ be a unital Banach algebra. Define and denote by

$$T_2(\mathfrak{A}) = \left\{ x = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \;\middle|\; a, b, c \in \mathfrak{A} \right\}$$

the Banach algebra (or the triangular matrix algebra) of all triangular 2×2 matrices over $\mathfrak{A}$.

Proposition 2.1. *Let $\mathfrak{A}$ be a unital Banach algebra. If we have $\mathrm{lsr}(\mathfrak{A}) = 1$, then $\mathrm{lsr}(T_2(\mathfrak{A})) = 1$.*

Similarly, if $\mathrm{rsr}(\mathfrak{A}) = 1$, then $\mathrm{rsr}(T_2(\mathfrak{A})) = 1$.

Proof. Let $x \in T_2(\mathfrak{A})$ as above. Since $\mathrm{lsr}(\mathfrak{A}) = 1$, the elements a and b are approximated closely by a' and b' in $\mathfrak{A}$ such that $(a')^{-1}a' = 1$ and $(b')^{-1}b' = 1$ for some $(a')^{-1}$ and $(b')^{-1}$ in $\mathfrak{A}$. Then

$$\begin{pmatrix} (a')^{-1} & 0 \\ 0 & (b')^{-1} \end{pmatrix} \begin{pmatrix} a' & c \\ 0 & b' \end{pmatrix} = \begin{pmatrix} 1 & (a')^{-1}c \\ 0 & 1 \end{pmatrix}$$

Furthermore,

$$\begin{pmatrix} 1 & -(a')^{-1}c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (a')^{-1}c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, we obtain $\mathrm{lsr}(T_2(\mathfrak{A})) = 1$.

By considering the right invertible elements that approximate a and b and the multiplications by the corresponding such matrices from the right, we obtain the second claim. □

Let $\mathfrak{A}$ be a unital Banach algebra. Define and denote by

$$T_n(\mathfrak{A}) = \left\{ x = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \middle| \, a_{ij} \in \mathfrak{A} \ (1 \le i \le j \le n) \right\}$$

the Banach algebra (or the triangular matrix algebra) of all triangular $n \times n$ matrices over $\mathfrak{A}$.

Proposition 2.2. *Let $\mathfrak{A}$ be a unital Banch algebra. If we have $\mathrm{lsr}(\mathfrak{A}) = 1$, then $\mathrm{lsr}(T_n(\mathfrak{A})) = 1$.*

Similarly, if $\mathrm{rsr}(\mathfrak{A}) = 1$, then $\mathrm{rsr}(T_n(\mathfrak{A})) = 1$.

Proof. Let $x \in T_n(\mathfrak{A})$ as above. Since $\mathrm{lsr}(\mathfrak{A}) = 1$, the diagonal elements a_{ii} are approximated closely by a'_{ii} in $\mathfrak{A}$ such that $b_{ii}a'_{ii} = 1$ for some b_{ii} in $\mathfrak{A}$ $(1 \le i \le n)$. Denote by x' the matirx obtained by replacing a_{ii} with a'_{ii}. Then

$$\begin{pmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix} x' = \begin{pmatrix} 1 & * & b_{11}a_{1n} \\ & \ddots & * \\ 0 & & 1 \end{pmatrix}$$

Furthermore, it is deduced from a finite number of multiplications by the invertible elementary matrices as given in the proof above that the resulting matrix extracted by these operations is the identity matrix in $T_n(\mathfrak{A})$. Therefore, we obtain $\mathrm{lsr}(T_n(\mathfrak{A})) = 1$.

By considering the right invertible elements that approximate a_{ii} and the multiplications by the corresponding such matrices from the right, we obtain the second claim. □

As shown in [5, Lemma 3.4], the following analogue holds:

Lemma 2.3. *Let $\mathfrak{A}$ be a unital Banach algebra. If the diagonal sum $t \oplus 1_n$ in $T_{n+1}(\mathfrak{A})$ for some $t \in \mathfrak{A}$ can be approximated by an element of $L_1(T_{n+1}(\mathfrak{A}))$ within ε with $0 < \varepsilon < 1$, then t can be approximated by an element of $L_1(\mathfrak{A})$ within ε, where 1_n is the $n \times n$ identity matrix.*

Proof. Let

$$\begin{pmatrix} a & B \\ 0_n^t & D \end{pmatrix} \in T_{n+1}(\mathfrak{A})$$

be an element of $L_1(T_{n+1}(\mathfrak{A}))$ that approximates $t \oplus 1_n$ within ε, where $a \in \mathfrak{A}$, $D \in T_n(\mathfrak{A})$, and $B = (b_j) \in \mathfrak{A}^n$ (a row vector), and 0_n^t is the transpose of $0_n = (0, \dots, 0)$ (a row vector). Then $\|D - 1_n\| \leq \varepsilon$. Hence $I_n - (I_n - D) = D$ is invertible and $D^{-1} = \sum (1_n - D)^n$ so that $\|D^{-1}\| \leq (1 - \varepsilon)^{-1}$. Furthermore,

$$\begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} a & B \\ 0_n^t & D \end{pmatrix} = \begin{pmatrix} a & 0_n \\ 0_n^t & D \end{pmatrix}$$

It follows that a is in $L_1(\mathfrak{A})$ and approximates t within ε.

Also, if the first matrix is in $L_1(T_{n+1}(\mathfrak{A}))$, then we have a is in $L_1(\mathfrak{A})$ by considering the matrix multiplication. $\square$

Combining the above lemma and proposition, we obtain

Theorem 2.4. *Let $\mathfrak{A}$ be a unital Banch algebra. Then* $\mathrm{lsr}(\mathfrak{A}) = 1$ *if and only if* $\mathrm{lsr}(T_n(\mathfrak{A})) = 1$.

Similarly, $\mathrm{rsr}(\mathfrak{A}) = 1$ *if and only if* $\mathrm{rsr}(T_n(\mathfrak{A})) = 1$.

Moreover,

Theorem 2.5. *Let $\mathfrak{A}$ be a unital Banach algebra. Then*

$$\mathrm{lsr}(T_m(\mathfrak{A})) = \mathrm{lsr}(\mathfrak{A}).$$

Similarly, we obtain $\mathrm{rsr}(T_m(\mathfrak{A})) = \mathrm{rsr}(\mathfrak{A})$.

Proof. Now suppose that $\mathrm{lsr}(T_m(\mathfrak{A})) = s$ finite. Then any element $(A_k)_{k=1}^s \in (T_m(\mathfrak{A}))^s$ with

$$A_j = \begin{pmatrix} a_{11}^k & & * \\ & \ddots & \\ 0 & & a_{mm}^k \end{pmatrix}$$

with $a_{ij}^k \in \mathfrak{A}$ is approximated closely by an element $(B_k)_{k=1}^s \in (T_m(\mathfrak{A}))^s$ such that $\sum_{k=1}^s C_k B_k = 1_m$ for some $(C_k)_{k=1}^s \in (T_m(\mathfrak{A}))^s$. It follows that any element $(a_{11}^k)_{k=1}^s \in \mathfrak{A}^s$ is approximated closely by an element $(b_{11}^k)_{k=1}^s \in \mathfrak{A}^s$ such that $\sum_{k=1} c_{11}^k b_{11}^k = 1$ for some $(c_{11}^k)_{k=1}^s \in \mathfrak{A}^s$. Hence, $\mathrm{lsr}(\mathfrak{A}) \leq \mathrm{lsr}(T_m(\mathfrak{A}))$.

To show the reverse inequality we use induction on m. Suppose that $\mathrm{lsr}(\mathfrak{A}) = s$ finite and $\mathrm{lsr}(T_m(\mathfrak{A})) \leq s$. Take an element $(A_k)_{k=1}^s \in T_{m+1}(\mathfrak{A})^s$ such that

$$A_k = \begin{pmatrix} a_{11}^k & b_k \\ 0_n^t & D^k \end{pmatrix}, \quad a_{11}^k \in \mathfrak{A}, \quad D_k \in T_m(\mathfrak{A})$$

and $b_k \in \mathfrak{A}^m$ (a row vector). By the assumptions, $(a_{11}^k)_{k=1}^s \in \mathfrak{A}^s$ and $(D_k)_{k=1}^s \in T_m(\mathfrak{A})^s$ are approximated closely by elements $(c_{11}^k)_{k=1}^s \in \mathfrak{A}^s$ and $(E_k)_{k=1}^s \in T_m(\mathfrak{A})^s$ respectively, such that $\sum_{k=1}^s d_{11}^k c_{11}^k = 1$ and $\sum_{k=1}^s F_k E_k = 1_m$ for some $(d_{11}^k)_{k=1}^s \in \mathfrak{A}^s$ and $(F_k)_{k=1}^s \in T_m(\mathfrak{A})^s$. Furthermore, let

$$\begin{pmatrix} c_{11}^k & b_k \\ 0_m^t & E_k \end{pmatrix} = A_k', \quad \text{and} \quad \sum_{k=1}^s \begin{pmatrix} d_{11}^k & 0_m \\ 0_m^t & F_k \end{pmatrix} A_k' = \begin{pmatrix} 1 & \sum_{k=1}^s d_{11}^k b_k \\ 0_m^t & 1_m \end{pmatrix}$$

It follows that $(A_k')_{k=1}^s \in L_s(T_{m+1})$. Therefore, $\mathrm{lsr}(T_{m+1}(\mathfrak{A})) \leq s$. Therefore, we can conclude that $\mathrm{lsr}(T_{m+1}(\mathfrak{A})) \leq \mathrm{lsr}(\mathfrak{A})$.

Now let $\mathrm{lsr}(\mathfrak{A}) = \infty$. If $\mathrm{lsr}(T_m(\mathfrak{A}))$ is finite, the first part of this proof implies that $\mathrm{lsr}(\mathfrak{A})$ is finite. This is a contradiction. Thus, $\mathrm{lsr}(T_m(\mathfrak{A})) = \infty$.

Now let $\mathrm{lsr}(T_m(\mathfrak{A})) = \infty$. If $\mathrm{lsr}(\mathfrak{A})$ is finite, the second part of this proof implies that $\mathrm{lsr}(T_m(\mathfrak{A}))$ is finite. This is a contradiction. Thus, $\mathrm{lsr}(\mathfrak{A}) = \infty$.

Similarly, we can obtain the same for the right stable rank by exchanging left and right multiplications. □

Define $T_\infty(\mathfrak{A})$ to be the inductive limit of $T_m(\mathfrak{A})$ under the canonical inclusions $T_m(\mathfrak{A}) \subset T_{m+1}(\mathfrak{A})$ by $x \mapsto x \oplus 0$ (diagonal sum) for $x \in T_m(\mathfrak{A})$.

Theorem 2.6. *Let $\mathfrak{A}$ be a unital Banach algebra. Then*

$$\mathrm{lsr}(T_\infty(\mathfrak{A})) = \mathrm{lsr}(\mathfrak{A}).$$

Similarly, we obtain $\mathrm{rsr}(T_\infty(\mathfrak{A})) = \mathrm{rsr}(\mathfrak{A})$.

Proof. Since $T_\infty(\mathfrak{A}) = \varinjlim T_n(\mathfrak{A})$, it follows by the same way as [5, Theorem 5.1] that

$$\mathrm{lsr}(T_\infty(\mathfrak{A})) \leq \lim \mathrm{lsr}(T_m(\mathfrak{A})) = \mathrm{lsr}(\mathfrak{A}).$$

Moreover, we can show that if $\mathrm{lsr}(\mathfrak{A}) \geq n$, then $\mathrm{lsr}(T_\infty(\mathfrak{A})) \geq n$ by observing the matrix equation:

$$\begin{pmatrix} b & \cdots & * \\ & \ddots & \vdots \\ 0 & & \ddots \end{pmatrix} \begin{pmatrix} a & \cdots & * \\ & \ddots & \vdots \\ 0 & & \ddots \end{pmatrix} = \begin{pmatrix} ba & \cdots & * \\ & \ddots & \vdots \\ 0 & & \ddots \end{pmatrix}$$

Hence we obtain $\mathrm{lsr}(\mathfrak{A}) \leq \mathrm{lsr}(T_\infty(\mathfrak{A}))$.

The same also holds when $\mathrm{lsr}(\cdot)$ is replaced by $\mathrm{rsr}(\cdot)$. □

Remark 2.7. In fact, by the same way as [5, Theorem 5.1] we obtain

$$\mathrm{lsr}(\varinjlim \mathfrak{A}_n) \leq \liminf \mathrm{lsr}(\mathfrak{A}_n) \quad \text{and} \quad \mathrm{rsr}(\varinjlim \mathfrak{A}_n) \leq \liminf \mathrm{rsr}(\mathfrak{A}_n)$$

for $\varinjlim \mathfrak{A}_n$ an inductive limit of Banach algebras $\mathfrak{A}_n$.

As for tensor products $\mathfrak{A}\otimes\mathfrak{B}$ of Banach algebras $\mathfrak{A}$ and $\mathfrak{B}$, on them one can take Banach tensor product norms such as restrictions from C^*-tensor product norms and the (so called) γ norm defined by the infimum of $\sum\|a_j\|\|b_j\|$ for $\sum a_j\otimes b_j\in\mathfrak{A}\otimes\mathfrak{B}$, and more suitable weaker norms (see, for instance, [8]).

Corollary 2.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital Banach algebras. Then*

$$\mathrm{lsr}(T_m(\mathfrak{A})\otimes T_n(\mathfrak{B}))=\mathrm{lsr}(\mathfrak{A}\otimes\mathfrak{B}),$$

where each of m and n is finite or infinite. Also,

$$\mathrm{lsr}(\otimes^\infty\mathfrak{A})\leq\mathrm{lsr}(\otimes^\infty T_m(\mathfrak{A}))\leq\liminf\mathrm{lsr}(\otimes^k\mathfrak{A}).$$

If we have $\liminf\mathrm{lsr}(\otimes^k\mathfrak{A})=\mathrm{lsr}(\otimes^\infty\mathfrak{A})$*, then*

$$\mathrm{lsr}(\otimes^\infty T_m(\mathfrak{A}))=\mathrm{lsr}(\otimes^\infty\mathfrak{A}).$$

Furthermore, $\mathrm{lsr}(\cdot)$ *can be replaced with* $\mathrm{rsr}(\cdot)$*.*

Proof. Note that $T_m(\mathfrak{A})\cong T_m(\mathbb{C})\otimes\mathfrak{A}$. Thus,

$$\begin{aligned}\mathrm{lsr}(T_m(\mathfrak{A})\otimes T_n(\mathfrak{B}))&=\mathrm{lsr}(T_m(\mathbb{C})\otimes\mathfrak{A}\otimes T_n(\mathfrak{B}))\\&=\mathrm{lsr}(T_m(\mathfrak{A}\otimes T_n(\mathfrak{B})))\\&=\mathrm{lsr}(T_n(\mathbb{C})\otimes\mathfrak{A}\otimes\mathfrak{B})=\mathrm{lsr}(\mathfrak{A}\otimes\mathfrak{B}).\end{aligned}$$

Note that $\otimes^\infty T_m(\mathfrak{A})$ is an inductive limit of finite tensor products $\otimes^k T_m(\mathfrak{A})$ for $k\geq 1$ under the canonical inclusions $\otimes^k T_m(\mathfrak{A})\subset\otimes^{k+1}T_m(\mathfrak{A})$ by $x\mapsto x\otimes 1_m$ for $x\in\otimes^k T_m(\mathfrak{A})$. Then

$$\mathrm{lsr}(\otimes^\infty T_m(\mathfrak{A}))\leq\liminf\mathrm{lsr}(\otimes^k T_m(\mathfrak{A}))=\liminf\mathrm{lsr}(\otimes^k\mathfrak{A})$$

where we are using the equality shown above. Moreover, we can show that $\mathrm{lsr}(\otimes^\infty\mathfrak{A})\leq\mathrm{lsr}(\otimes^\infty T_m(\mathfrak{A}))$ by observing the matrix equation as given above. $\square$

Let $\mathfrak{A}$ be a unital Banach algebra. Define $T^b(\mathfrak{A})$ to be the Banach algebra of all upper triangular $(\infty\times\infty)$ matrices over $\mathfrak{A}$ which are bounded as operators. In fact, $T^b(\mathfrak{A})\cong T^b(\mathbb{C})\otimes\mathfrak{A}$ when $\mathfrak{A}$ is a complex Banach algebra. If $\mathfrak{A}$ is real, $T^b(\mathbb{C})$ is replaced by $T^b(\mathbb{R})$.

Proposition 2.9. *Let $\mathfrak{A}$ be a unital Banach algebra. Then* $\mathrm{lsr}(T^b(\mathfrak{A}))=\infty$*.*
Also, $\mathrm{rsr}(T^b(\mathfrak{A}))=\infty$

Proof. Note that since $\mathfrak{A}$ is unital, $T^b(\mathbb{C})$ is a subalgebra of $T^b(\mathfrak{A})$. There exists two orthogonal isometries of $T^b(\mathfrak{A})^*$ with orthogonal ranges, where $T^b(\mathfrak{A})^*$ just consists of all lower triangular matrices over $\mathfrak{A}$ that are bounded. For instance, define an isometry A by $Ae_n=e_{3^n}$ and another B by $Be_n=e_{2^n}$, where e_n is the n-th standard unit vector. Then $A^*,B^*\in T^b(\mathfrak{A})$ are co-isometries with orthogonal ranges. Since $\{A, BA, B^2A, \ldots, B^{n-1}A\}$ are $n+1$ isometries of $T^b(\mathfrak{A})^*$, it follows that $\mathrm{rsr}(T^b(\mathfrak{A})^*) \geq n+1$. Indeed, this follows from that such an tuple can not be in $R_{n+1}(T^b(\mathfrak{A})^*)$ (cf. [1]). Therefore, $\mathrm{rsr}(T^b(\mathfrak{A})^*) = \infty$, which is equivalent to that $\mathrm{lsr}(T^b(\mathfrak{A})) = \infty$.

Also, it is shown by [1] that $T^b(\mathbb{C})$ has left and right stable ranks ∞. Therefore, it follows that $T^b(\mathfrak{A})$ has left and right stable ranks ∞. □

3. Connected stable rank

Theorem 3.1. *Let $\mathfrak{A}$ be a unital Banach algebra. Then*

$$\mathrm{lcsr}(T_m(\mathfrak{A})) = \mathrm{lcsr}(\mathfrak{A}).$$

Similarly, we obtain $\mathrm{rcsr}(T_m(\mathfrak{A})) = \mathrm{rcsr}(\mathfrak{A})$.

Proof. Suppose first that $m = 2$. Let $(A_1, \ldots, A_n) \in L_n(T_2(\mathfrak{A}))$, where

$$A_j = \begin{pmatrix} a_j & b_j \\ 0 & c_j \end{pmatrix} \quad a_j, b_j, c_j \in \mathfrak{A} \quad (1 \leq j \leq n).$$

Then there exists $(B_1, \ldots, B_n) \in T_2(\mathfrak{A})^n$, where

$$B_j = \begin{pmatrix} a'_j & b'_j \\ 0 & c'_j \end{pmatrix} \quad a'_j, b'_j, c'_j \in \mathfrak{A} \quad (1 \leq j \leq n)$$

such that $\sum_{j=1}^n B_j A_j = 1_2$ the 2×2 identity matrix. It follows that

$$\sum_{j=1}^n a'_j a_j = 1, \quad \sum_{j=1}^n (a'_j b_j + b'_j c_j) = 0, \quad \sum_{j=1}^n c'_j c_j = 1.$$

For $0 \leq \varepsilon \leq 1$, let

$$A_j^\varepsilon = \begin{pmatrix} a_j & \varepsilon b_j \\ 0 & c_j \end{pmatrix}, \quad B_j^\varepsilon = \begin{pmatrix} a'_j & \varepsilon b'_j \\ 0 & c'_j \end{pmatrix}.$$

Then $\sum_{j=1}^n B_j^\varepsilon A_j^\varepsilon = 1_2$, so that $(A_j^\varepsilon) \in L_n(T_2(\mathfrak{A}))$ which gives a path between $(A_j^1) = (A_j)$ and (A_j^0) a diagonal matrix. Therefore, $L_n(T_2(\mathfrak{A}))$ is connected if and only if $L_n(\mathfrak{A})$ is connected. Hence $\mathrm{lcsr}(\mathfrak{A}) = \mathrm{lcsr}(T_2(\mathfrak{A}))$.

Next suppose that $m = 3$. Let $(A_1, \ldots, A_n) \in L_n(T_3(\mathfrak{A}))$, where

$$A_k = \begin{pmatrix} a_{11}^k & a_{12}^k & a_{13}^k \\ 0 & a_{22}^k & a_{23}^k \\ 0 & 0 & a_{33}^k \end{pmatrix} \quad a_{ij}^k \in \mathfrak{A} \quad (1 \leq i \leq j \leq 3, 1 \leq k \leq n).$$

Then there exists $(B_1, \ldots, B_n) \in T_2(\mathfrak{A})^n$, where

$$B_k = \begin{pmatrix} b_{11}^k & b_{12}^k & b_{13}^k \\ 0 & b_{22}^k & b_{23}^k \\ 0 & 0 & b_{33}^k \end{pmatrix} \quad b_{ij}^k \in \mathfrak{A} \quad (1 \le i \le j \le 3, 1 \le k \le n).$$

such that $\sum_{j=1}^n B_j A_j = 1_3$ the 3×3 identity matrix. It follows that

$$\sum_{k=1}^n b_{11}^k a_{11}^k = 1, \quad \sum_{k=1}^n (b_{11}^k a_{12}^k + b_{12}^k a_{22}^k) = 0,$$
$$\sum_{k=1}^n (b_{11}^k a_{13}^k + b_{12}^k a_{23}^k + b_{13}^k a_{33}^k) = 0, \quad \sum_{k=1}^n b_{22}^k a_{22}^k = 1,$$
$$\sum_{k=1}^n (b_{22}^k a_{23}^k + b_{23}^k a_{33}^k) = 0, \quad \sum_{k=1}^n b_{33}^k a_{33}^k = 1.$$

For $0 \le \varepsilon \le 1$, let

$$A_k^\varepsilon = \begin{pmatrix} a_{11}^k & \varepsilon a_{12}^k & \varepsilon a_{13}^k \\ 0 & a_{22}^k & \varepsilon a_{23}^k \\ 0 & 0 & a_{33}^k \end{pmatrix}, \quad B_k^\varepsilon = \begin{pmatrix} b_{11}^k & \varepsilon b_{12}^k & \varepsilon b_{13}^k \\ 0 & b_{22}^k & \varepsilon b_{23}^k \\ 0 & 0 & b_{33}^k \end{pmatrix}.$$

Then $\sum_{k=1}^n B_k^\varepsilon A_k^\varepsilon$ is 1_3 plus $\varepsilon \sum_{k=1}^n (b_{11}^k a_{13}^k + \varepsilon b_{12}^k a_{23}^k + b_{13}^k a_{33}^k)$ for the $(1,3)$-component so that the sum is invertible. Hence $(A_k^\varepsilon) \in L_n(T_3(\mathfrak{A}))$ which gives a path between $(A_k^1) = (A_k)$ and (A_k^0) a diagonal matrix. Therefore, $L_n(T_3(\mathfrak{A}))$ is connected if and only if $L_n(\mathfrak{A})$ is connected. Hence $\mathrm{lcsr}(\mathfrak{A}) = \mathrm{lcsr}(T_3(\mathfrak{A}))$.

Moreover, $\mathrm{lcsr}(\mathfrak{A}) = \mathrm{lcsr}(T_m(\mathfrak{A}))$ can be shown by the same argument using the matrix perturbation.

The same also holds when $\mathrm{lcsr}(\cdot)$ is replaced by $\mathrm{rcsr}(\cdot)$. $\square$

Theorem 3.2. *Let $\mathfrak{A}$ be a unital Banach algebra. Then*

$$\mathrm{lcsr}(T_\infty(\mathfrak{A})) = \mathrm{lcsr}(\mathfrak{A}).$$

Similarly, we obtain $\mathrm{rcsr}(T_\infty(\mathfrak{A})) = \mathrm{rcsr}(\mathfrak{A})$.

Proof. Since $T_\infty(\mathfrak{A}) = \varinjlim T_n(\mathfrak{A})$, it follows by the same way as [5, Theorem 5.1] that

$$\mathrm{lcsr}(T_\infty(\mathfrak{A})) \le \lim \mathrm{lcsr}(T_m(\mathfrak{A})) = \mathrm{lcsr}(\mathfrak{A}).$$

Moreover, we can show that if $\mathrm{lcsr}(\mathfrak{A}) \ge n$, then $\mathrm{lcsr}(T_\infty(\mathfrak{A})) \ge n$ by observing the matrix equation:

$$\begin{pmatrix} b & \cdots & * \\ & \ddots & \vdots \\ 0 & & \ddots \end{pmatrix} \begin{pmatrix} a & \cdots & * \\ & \ddots & \vdots \\ 0 & & \ddots \end{pmatrix} = \begin{pmatrix} ba & \cdots & * \\ & \ddots & \vdots \\ 0 & & \ddots \end{pmatrix}$$

Hence we obtain $\mathrm{lcsr}(\mathfrak{A}) \leq \mathrm{lcsr}(T_\infty(\mathfrak{A}))$.

The same also holds when $\mathrm{lcsr}(\cdot)$ is replaced by $\mathrm{rcsr}(\cdot)$. $\square$

Remark 3.3. In fact, by the same way as [5, Theorem 5.1] we obtain

$$\mathrm{lcsr}(\varinjlim \mathfrak{A}_n) \leq \liminf \mathrm{lcsr}(\mathfrak{A}_n) \quad \text{and} \quad \mathrm{rcsr}(\varinjlim \mathfrak{A}_n) \leq \liminf \mathrm{rcsr}(\mathfrak{A}_n)$$

for $\varinjlim \mathfrak{A}_n$ an inductive limit of Banach algebras $\mathfrak{A}_n$.

Corollary 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital Banach algebras. Then*

$$\mathrm{lcsr}(T_m(\mathfrak{A}) \otimes T_n(\mathfrak{B})) = \mathrm{lcsr}(\mathfrak{A} \otimes \mathfrak{B}),$$

where each of m and n is finite or infinite. Also,

$$\mathrm{lcsr}(\otimes^\infty \mathfrak{A}) \leq \mathrm{lcsr}(\otimes^\infty T_m(\mathfrak{A})) \leq \liminf \mathrm{lcsr}(\otimes^k \mathfrak{A}).$$

If we have $\liminf \mathrm{lcsr}(\otimes^k \mathfrak{A}) = \mathrm{lcsr}(\otimes^\infty \mathfrak{A})$, *then*

$$\mathrm{lcsr}(\otimes^\infty T_m(\mathfrak{A})) = \mathrm{lcsr}(\otimes^\infty \mathfrak{A}).$$

Furthermore, $\mathrm{lcsr}(\cdot)$ *can be replaced with* $\mathrm{rcsr}(\cdot)$.

Proof. The proof is the same as that for the stable rank case in the previous section. $\square$

Theorem 3.5. *Let $\mathfrak{A}$ be a unital Banach algebra. Then*

$$\mathrm{lcsr}(T^b(\mathfrak{A})) = \mathrm{lcsr}(l^\infty(\mathfrak{A}))$$

where $l^\infty(\mathfrak{A})$ is the unital Banach algebra of all bounded sequences over $\mathfrak{A}$. Moreover, we obtain

$$\mathrm{lcsr}(l^\infty(\mathfrak{A})) = \mathrm{lcsr}(\mathfrak{A}).$$

Furthermore, $\mathrm{lcsr}(\cdot)$ *can be replaced with* $\mathrm{rcsr}(\cdot)$.

Proof. Let $(A_j) \in T^b(\mathfrak{A})^n$ with $A_j = D_j + N_j$, where D_j is the diagonal part of A_j and $N_j = A_j - D_j$. If $(A_j) \in L_n(T^b(\mathfrak{A}))$, then $\sum_{j=1}^n B_j A_j = 1_\infty$ for some $(B_j) \in T^b(\mathfrak{A})^n$. Set $B_j = D_j' + N_j'$ as above. Then $\sum_{j=1}^n D_j' D_j = 1_\infty$ and $\sum_{j=1}^n (D_j' N_j + B_j N_j + N_j' N_j) = 0_\infty$. Therefore, the equation

$$\sum_{j=1}^n (D_j' + \varepsilon N_j')(D_j + \varepsilon N_j) = \sum_{j=1}^n (D_j' D_j + \varepsilon(D_j' N_j + N_j' D_j + \varepsilon N_j' N_j))$$

implies that $L_n(T^b(\mathfrak{A}))$ is connected if and only if $L_n(l^\infty(\mathfrak{A}))$ is connected, where the subalgebra of all diagonal matrices D of $T^b(\mathfrak{A})$ is identified with $l^\infty(\mathfrak{A})$ by $D = (D(k))_{k=1}^\infty \in l^\infty(\mathfrak{A})$, where $D(k)$ is the (k,k)-component of D.

On the other hand, note that

$$L_n(l^\infty(\mathfrak{A})) = \Pi^\infty L_n(\mathfrak{A})$$

where the right hand side means the infinite product space of $L_n(\mathfrak{A})$. Indeed, let $(D_j)_{j=1}^n \in L_n(l^\infty(\mathfrak{A}))$ with each $D_j = (D_j(k))_{k=1}^\infty$ such that $\sum_{j=1}^n D_j' D_j = 1_\infty$ for some $D_j' = (D_j'(k))_{k=1}^\infty \in l^\infty(\mathfrak{A})$, if and only if $\sum_{j=1}^n D_j'(k) D_j(k) = 1$ for every $k \geq 1$. Furthermore, it follows from a fact of general topology that $\Pi^\infty L_n(\mathfrak{A})$ is connected if and only if $L_n(\mathfrak{A})$ is connected. $\square$

4. Nest algebras

Now recall nest algebras from [1]. A nest $\mathfrak{N}$ on a Hilbert space H is a chain of closed subspaces of H such that $\{0\}, H \in \mathfrak{N}$, and the nest is closed under taking intersections and closed linear spans of its elements.

For each $N \in \mathfrak{N}$, let $N_+ = \inf\{M \in \mathfrak{N} : M > N\}$. If $N_+ \neq N$, then $N_+ \ominus N$ is called an atom of $\mathfrak{N}$.

If H is spanned by atoms of $\mathfrak{N}$, then $\mathfrak{N}$ is called atomic. If $\mathfrak{N}$ has no atoms, then $\mathfrak{N}$ is called continuous.

The nest algebra $\mathfrak{T}(\mathfrak{N})$ of a nest $\mathfrak{N}$ is defined to be the algebra of all bounded operators on a Hilbert space such that every subspace of $\mathfrak{N}$ is invariant under elements of $\mathfrak{T}(\mathfrak{N})$:

$$\mathfrak{T}(\mathfrak{N}) = \{T \in \mathbb{B}(H) : T(N) \subset N \quad \text{for any } N \in \mathfrak{N}\}.$$

Theorem 4.1. *Let $\mathfrak{N}$ be an atomic nest with finite dimensional atoms and $\mathfrak{T}(\mathfrak{N})$ its nest algebra. Then*

$$\mathrm{lcsr}(\mathfrak{T}(\mathfrak{N})) = \mathrm{lcsr}(l^\infty(\mathbb{C})) = 1.$$

Furthermore, $\mathrm{lcsr}(\cdot)$ *can be replaced with* $\mathrm{rcsr}(\cdot)$.

Proof. Let $A \in \mathfrak{T}(\mathfrak{N})$. By the assumption, it has the following block decomposition:

$$A = \begin{pmatrix} A_1 & N_{12} & N_{13} & \cdots \\ & A_2 & N_{23} & \cdots \\ & & A_3 & \cdots \\ & & & \ddots \end{pmatrix}$$

where each $A_i \in M_{n_i}(\mathbb{C})$ for some $n_i \geq 1$ which corresponds to the i-th finite dimensional atom, and set $D = A_1 \oplus A_2 \oplus \cdots$ the diagonal part of A, and set $N_i = N_{i,i+1} + N_{i,i+2} + \cdots$ the i-th nilpotent part of A, and $N = A - D$ the nilpotent part of A. Let $B \in \mathfrak{T}(\mathfrak{N})$ with the same block decomposition:

$$B = \begin{pmatrix} B_1 & N_{12}' & N_{13}' & \cdots \\ & B_2 & N_{23}' & \cdots \\ & & B_3 & \cdots \\ & & & \ddots \end{pmatrix}$$

Then

$$BA = \sum_{i=1}^{\infty} B_i A_i + (B_1 N_{12} + N'_{12} A_2) + \cdots$$

where the first term diagonal part of BA and other terms are in the nilpotent part of BA. Thus, if $BA = 1_\infty$, then $\sum_{i=1}^{\infty} B_i A_i = 1_\infty$. Therefore, the same argument using deformation implies that $\mathrm{lcsr}(\mathfrak{T}(\mathfrak{N})) = \mathrm{lcsr}(l^\infty(M_{n_i}(\mathbb{C})))$, where $l^\infty(M_{n_i}(\mathbb{C}))$ means the unital Banach algebra of bounded sequences of matrices in $M_{n_i}(\mathbb{C})$. Furthermore, it follows that

$$\mathrm{lcsr}(l^\infty(M_{n_i}(\mathbb{C}))) = \mathrm{lcsr}(l^\infty(\mathbb{C})) = \mathrm{lcsr}(\mathbb{C}) = 1,$$

which is shown in the previous section. □

Theorem 4.2. *Let $\mathfrak{N}$ be a non-atomic nest with an infinite dimensional atom and $\mathfrak{T}(\mathfrak{N})$ its nest algebra. Then*

$$\mathrm{lcsr}(\mathfrak{T}(\mathfrak{N})) = \infty.$$

It follows from this that

$$\mathrm{Bsr}(\mathfrak{T}(\mathfrak{N})) = \infty = \mathrm{lsr}(\mathfrak{T}(\mathfrak{N})).$$

Similarly, we obtain $\mathrm{rcsr}(\mathfrak{T}(\mathfrak{N})) = \infty$ *so that* $\mathrm{rsr}(\mathfrak{T}(\mathfrak{N})) = \infty = \mathrm{Bsr}(\mathfrak{T}(\mathfrak{N}))$.

Proof. It follows from the assumption that $\mathfrak{T}(\mathfrak{N})$ contains a unital copy of the C^*-algebra $\mathbb{B}(H)$ of all bounded operators on an infinite dimensional Hilbert space H. In particular, $\mathfrak{T}(\mathfrak{N})$ contains a unital copy of Cuntz algebra O_n for any $n \geq 2$. Therefore, $\mathrm{lcsr}(\mathfrak{T}(\mathfrak{N})) = \infty$ by Elhage Hassan [3, Proposition 1.4] (and see also [7]).

On the other hand, it is shown in [5] that

$$\mathrm{lcsr}(\mathfrak{A}) \leq \mathrm{Bsr}(\mathfrak{A}) + 1 \leq \mathrm{lsr}(\mathfrak{A}) + 1$$

for any Banach algebra $\mathfrak{A}$. Since these estimates are applicable to $\mathbb{B}(H)$ and extendible to estimate the stable ranks of $\mathfrak{T}(\mathfrak{N})$, the second claim also follows □

Remark 4.3. This settles a question of [1, Q.3] partly since the Bass stable rank is determined in that case.

As shown by Davidson and Ji [2] (an item added later), the possible values of the left and right stable ranks for any next algebra have been completely determined to answer questions [1, Q.1 and Q.2]. Their results do not contradict to our results obtained and do not imply our results on Bass stable rank and connected stable rank in those theorems.

5. Crossed products

Following a method of [5] to estimate the stable rank of a crossed product of a unital C^*-algebra by an action of $\mathbb{Z}$ of integers, we obtain the following:

Theorem 5.1. *Let $\mathfrak{A}$ be a unital Banach algebra. Let $\mathfrak{A} \rtimes \mathbb{N}$ denote and define a (non self-adjoint) crossed product that is the unital Banach algebra generated by $\mathfrak{A}$ and a proper isometry s, where we assume that $s\mathfrak{A} = \mathfrak{A}s$ (a covariance property) so that the elements of the form $a_0 + a_1 s + \cdots + a_k s^k$ for some $k \geq 0$ and $a_j \in \mathfrak{A}$ are dense in $\mathfrak{A} \rtimes \mathbb{N}$. Then*

$$\mathrm{lsr}(\mathfrak{A} \rtimes \mathbb{N}) \leq \mathrm{lsr}(\mathfrak{A}) + 1.$$

Furthermore, $\mathrm{lsr}(\cdot)$ *can be replaced with* $\mathrm{rsr}(\cdot)$.

Proof. Let $\mathfrak{B} = \mathfrak{A} \rtimes \mathbb{N}$. Let $m \geq \mathrm{lsr}(\mathfrak{A}) + 1$. Let $(b_i) \in \mathfrak{B}^m$ and U its open neighbourhood.

For $x = a_k s^k + a_{k+1} s^{k+1} + \cdots + a_l s^l$ for some $0 \leq k \leq l$ and $a_j \in \mathfrak{A}$, define the length of x to be $L(x) = l - k + 1$. For $(b_i) \in \mathfrak{B}^m$ where each b_i has the form as x, let $L(b_i) = \sum_i L(b_i)$. Denote by $EL_m(\mathfrak{D})$ the set of $m \times m$ elementary matrices over $\mathfrak{D}$, where $\mathfrak{D}$ is a dense subalgebra of $\mathfrak{B}$ generated by such elements x. Also consider the following operation by (s^{n_i}): $(b_i) \mapsto (s^{n_i} b_i)$ for some $n_i \geq 0$. This operation may not preserve $L_m(\mathfrak{B})$ since $s^* \notin \mathfrak{B}$ in general. However, its reverse operation as well as the matrix multilication by $EL_m(\mathfrak{D})$ do preserve $L_m(\mathfrak{B})$. As for the reverse operation, indeed, if $(s^{n_i} b_i) \in L_m(\mathfrak{B})$ so that $\sum_{i=1}^m t_i (s^{n_i} b_i) = 1$ for some $(t_i) \in \mathfrak{B}^m$, then $(b_i) \in L_m(\mathfrak{B})$ because $\sum_{i=1}^m (t_i s^{n_i}) b_i = 1$. Define the size of (b_i) to be the least among $L((b_i''))$ where b_i'' is obtained by iterating finitely the matrix multiplication by elements of $EL_m(\mathfrak{D})$ or the operation by (s^{n_i}) to an element $(b_i') \in U$ (which may be taken to be arbitrary near to (b_i)).

Now assume that $(c_i) \in \mathfrak{D}^m$ attains the size of (b_i). Suppose that each c_i is nonzero so that $L(c_i) \geq m \geq 2$. We may assume that $L(c_m)$ is the maximum among $L(c_i)$ if necessary applying a permutation matrix of $EL_m(\mathfrak{D})$ to (c_i). For each i, let $a_{k_i} s^{k_i}$ be the lowest term of c_i. Now if necessary, by replacing c_m with $s^p c_m$ for some $p \geq 0$, we assume that k_m is the maximum among k_i. Let $a_{k_j}' \in \mathfrak{A}$ such that $s^{k_m - k_j} a_{k_j} = a_{k_j}' s^{k_m - k_j}$ for $1 \leq j \leq m-1$. Since $m - 1 \geq \mathrm{lsr}(\mathfrak{A})$, the element $(a_{k_1}', \ldots, a_{k_{m-1}}')$ can be approximated closely by $(a_{k_1}'', \ldots, a_{k_{m-1}}'')$ such that $\sum_{j=1}^{m-1} d_j a_{k_j}'' = a_{k_m}$ for some $d_j \in \mathfrak{A}$. Let $(c_i^{\sim})$ be obtained from replacing a_{k_j} with $a_{k_j}^{\sim}$ such that $s^{k_m - k_j} a_{k_j}^{\sim} = a_{k_j}'' s^{k_m - k_j}$ for $1 \leq j \leq m-1$. We consider the following operation:

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ -d_1 s^{k_m-k_1} & \cdots & -d_{m-1}s^{k_m-s_{m-1}} & 1 \end{pmatrix} \begin{pmatrix} c_1^\sim \\ \vdots \\ c_{m-1}^\sim \\ c_m \end{pmatrix} = \begin{pmatrix} c_1^\sim \\ \vdots \\ c_{m-1}^\sim \\ c_m^\sim \end{pmatrix}$$

It follows that

$$L(c_1^\sim, \dots, c_{m-1}^\sim, c_m) > L(c_1^\sim, \dots, c_{m-1}^\sim, c_m^\sim),$$

which is a contradiction. Therefore, if $(c_1, \dots, c_{m-1}, c_m)$ attains the size, then $c_i = 0$ for some i. We may assume that $c_1 = 0$, and replace it by $\varepsilon 1$. Then

$$\begin{pmatrix} 1 & & & 0 \\ -\varepsilon^{-1}c_2 & 1 & & \\ \vdots & & \ddots & \\ -\varepsilon^{-1}c_m & & & 1 \end{pmatrix} \begin{pmatrix} \varepsilon 1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} \varepsilon 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

It follows that $(\varepsilon 1, c_2, \dots, c_{m-1}, c_m) \in L_m(\mathfrak{B})$. Thus, $(c_1, \dots, c_{m-1}, c_m) \in L_m(\mathfrak{B})$. An element close to (b_i) obtained by applying the reverse operations by $EL_m(\mathfrak{B})$ or the reverse operaionts by (s^{n_i}) to (c_m) is also in $L_m(\mathfrak{B})$ and in U, as desired. $\square$

Corollary 5.2. *Under the same assumptions as above, we obtain*

$$\mathrm{lcsr}(\mathfrak{A} \rtimes \mathbb{N}) \leq \mathrm{lsr}(\mathfrak{A}) + 1.$$

Similarly, we obtain $\mathrm{rcsr}(\mathfrak{A} \rtimes \mathbb{N}) \leq \mathrm{rsr}(\mathfrak{A}) + 1$.

Proof. Indeed, the proof of the theorem above implies that $GL_m(\mathfrak{B})_0$ acts transitively on $L_m(\mathfrak{B})$ for any $m \geq \mathrm{lsr}(\mathfrak{A}) + 1$. Hence the estimate holds. $\square$

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