Nihonkai Math. J.
Vol.8 (1997), 71-76

ON MINIMAL *CR* SUBMANIFOLDS SATISFYING A CERTAIN CONDITION ON THE RICCI CURVATURE

MASAHIRO YAMAGATA AND MASAHIRO KON

1. Introduction. We denote by $\bar{M}^m(c)$ a complex m-dimensional (real 2m-dimensional) Kaehlerian manifold of constant holomorphic sectional curvature 4c with Kaehlerian structure (J,g). Let M be a real n-dimensional Riemannian manifold isometrically immersed in $\bar{M}^m(c)$ with induced metric tensor field g. For any vector field X tangent to M, we put $JX = PX + FX$, where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$. If F vanishes identically, then M is called a *complex submanifold* of $\bar{M}^m(c)$, and if P vanishes identically, then M is callde an *anti-invariant submanifold* of $\bar{M}^m(c)$. A submanifold M of a Kaehlerian manifold $\bar{M}$ is called a *CR submanifold* of $\bar{M}$ if there exists a differentiable distribution $H : x \longrightarrow H_x \subset T_x(M)$ on M satisfying the following conditions:
(1) H is holomorphic, i.e., $JH_x = H_x$ for each $x \in M$, and
(2) the complementary orthogonal distribution $H^\perp : x \longrightarrow H_x^\perp \subset T_x(M)$ is anti-invariant, i.e., $JH_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

We denote by S the Ricci tensor of M. If M satisfies that $S(X,Y) = ag(X,Y)+bg(PX,PY)$, where a and b are constant, then M is called a *pseudo-Einstein* submanifold.

In [3] one of the present author proved that there are no Einstein real hypersurfaces of a complex projective space CP^m and classified the pseudo-Einstein real hypersurfaces of CP^m. This result was generalized by Cecil and Ryan [2] to the case that a and b are functions.

Moreover, Maeda [6] studied the Ricci tensor of a real hypersurface of a complex projective space.

On the other hand, one of the author [5] studied a compact minimal CR submanifold M of CP^m under the assumption that the Ricci tensor of M satisfies $S(X,X) \geq (n-1)g(X,X)+2g(PX,PX)$, and proved that M is a real projective space RP^n, or a complex projective space CP^n or a pseudo-Einstein real hypersurface $\pi\left(S^{(n+1)/2}\left(\sqrt{\tfrac{1}{2}}\right) \times S^{(n+1)/2}\left(\sqrt{\tfrac{1}{2}}\right)\right)$, where π denotes the projection with respect to the fibration $S^1 \longrightarrow S^{2m+1} \longrightarrow CP^m$.

The purpose of the present paper is to consider the problem on the Ricci tensor like that above without the assumption that M is compact.

Theorem 1. *Let M be an n-dimensional minimal CR submanifold of $\bar{M}^m(c)$ $(c > 0)$, which is not a complex submanifold of $\bar{M}^m(c)$. If the Ricci tensor S of M satisfies*

$$S(X,X) \geq c[(n-1)g(X,X) + 2g(PX,PX)]$$

for any vector field X tangent to M, then M is
(a) *a totally geodesic anti-invariant submanifold of $\bar{M}^m(c)$ with constant curvature c ,or*
(b) *a pseudo-Einstein submanifold of $\bar{M}^m(c)$ with* $\dim H_x^{\perp} = 1$ *and*

$$S(X,Y) = c[(n-1)g(X,Y) + 2g(PX,PY)].$$

2. Basic formulas.

In this section we prepare the basic formaulas for an n-dimensional submanifold M of $\bar{M}^m(c)$. The operator of covariant differentiation with respect to the Levi-Civita connection in $\bar{M}^m(c)$ (resp. M) will be denoted by $\bar{\nabla}$ (resp. ∇). Then the gauss and Weingarten formulas are respectively given by

$$(2.1) \qquad \bar{\nabla}_X Y = \nabla_X Y + B(X,Y) \quad \text{and} \quad \bar{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. A and B are both called the second fundamental forms of M, and are related by $g(B(X,Y),V) = g(A_V X, Y)$. For the second fundamental form A we define its covariant derivative $\nabla_X A$ by

$$(2.2) \qquad (\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V(\nabla_X Y),$$

for any vector fields X, Y tangent to M and any vector field V normal to M.

If $\mathrm{Tr} A_V = 0$ for any vector field V normal to M, then M is said to be *minimal*, where Tr denotes the trace of a operator. If the second fundamental form of M vanishes, then M is said to be *totally geodesic*. For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$, and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. For any vector field V normal to M we put

$$JV = tV + fV,$$

where tV is the tangential part of JV and fV the normal part of JV.

Let R be the Riemannian curvature tensor of M. Then the Gauss equation is given by

$$\begin{aligned} R(X,Y)Z = c[&g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX \\ &- g(PX,Z)PY + 2g(X,PY)PZ] \\ &+ A_{B(Y,Z)}X - A_{B(X,Z)}Y. \end{aligned} \tag{2.3}$$

The Codazzi equatin of M is given by

$$\begin{aligned} &g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z) \\ &= c[g(PY,Z)g(FX,V) - g(PX,Z)g(FY,V) \\ &\quad +2g(X,PY)g(FZ,V)]. \end{aligned} \tag{2.4}$$

In the CR submanifold M, we put $\dim H_x = h$, $\dim H_x^{\perp} = q$ and codimension of $M = 2m - n = p$. If $q = 0$ (resp.$h = 0$) for any $x \in M$, then the CR submanifold is called a *complex submanifold* (resp. *anti-invariant submanifold*) of $\bar{M}$. If $p = q$ for any $x \in M$, then the CR submanifold is called a *generic submanifold.* It is obvious that every real hypersurface of a Kaehlerian manifold is automatically a generic submanifold.

On the CR submanifold M we obtain $FPX = 0, fFX = 0$ for any vector X tangent to M and $tfV = 0, PtV = 0$ for any vector V normal to M. Moreover, we have $P^2X = -X - tFX$ for any vector X tangent to M and $f^2V = -V - FtV$ for any vector V normal to M. We define the covariant defferentiations of P, F, t and f by

$$\begin{aligned} (\nabla_X P)Y &= \nabla_X(PY) - P\nabla_X Y, & (\nabla_X F)Y &= D_X(FY) - F\nabla_X Y, \\ (\nabla_X t)V &= \nabla_X(tV) - tD_X V, & (\nabla_X f)V &= D_X(fV) - fD_X V, \end{aligned}$$

respectively. We then have

$$\begin{aligned} (\nabla_X P)Y &= A_{FY}X + tB(X,Y), & (\nabla_X F)Y &= -B(X,PY) + fB(X,Y), \\ (\nabla_X t)V &= A_{fV}X - PA_V X, & (\nabla_X f)V &= -FA_V X - B(X,tV). \end{aligned}$$

We also have

$$A_{FX}Y = A_{FY}X$$

for any $X, Y \in H^{\perp}$.

3. Proof of the theorem. We use the convention that the range of indices are

$$\begin{aligned} i &= 1,2,\ldots,n; & a &= 1,2,\ldots,p; \\ \lambda &= 1,2,\ldots,q; & u &= q+1, q+2,\ldots,p. \end{aligned}$$

From the Gauss equation the Ricci tensor S of M is given by

$$(3.1) \qquad S(X,Y) = c[(n-1)g(X,Y) + 3g(PX,PY)] - \sum_a g(A_aX, A_aY),$$

for any vector fields X and Y tangent to M, where we have put $A_a = A_{v_a}$, $\{v_a\}$ being an orthonormal basis of the normal space of M. In accordance with the assumption on the Ricci tensor, we find

$$(3.2) \qquad \begin{aligned} S(X,X) - c[(n-1)g(X,X) + 2g(PX,PX)] \\ = cg(PX,PX) - \sum_a g(A_aX, A_aX) \geq 0. \end{aligned}$$

Hence we obtain, for any vector field V normal to M, $A_a tV = 0$ for all a. This means that $A_U tV = 0$ for any vector fields U and V normal to M. Moreover by (3.2), we have

$$(3.3) \qquad \sum_a \mathrm{Tr} A_a^2 \leq c(n-q) = ch,$$

where we have put $h = \dim H_x$ and $q = \dim H_x^\perp$.

Let us suppose that $h = 0$. Then M is an anti-invariant submanifold of $\bar{M}^m(c)$ $(c > 0)$. In this case, M is totally geodesic in $\bar{M}^m(c)$, and M is of sectional curvature c.

In the following we suppose that $h \neq 0$. From $A_U tV = 0$ and (2.2) we have

$$(\nabla_X A)_U tV + A_U A_{fV} X - A_U P A_V X = 0$$

for any vector field X tangent to M, and hence

$$(3.4) \qquad \begin{aligned} g((\nabla_X A)_U Y, tV) &= g((\nabla_X A)_U tV, Y) \\ &= g(A_U P A_V X, Y) - g(A_U A_{fV} X, Y) \end{aligned}$$

for any vector fields X and Y tangent to M.

In the CR submanifold we hold that

$$g(PX,Y) + g(X,PY) = 0, \qquad g(FX,V) + g(X,tV) = 0$$

for any vector fields X, Y in tangent to M and for any vector field V in normal to M. From the Codazzi equation we obtain

$$g((\nabla_X A)_U Y, tV) - g((\nabla_Y A)_U X, tV) = 2cg(PX,Y)g(tV,tU).$$

Therefore from (3.4) we have

$$(3.5) \qquad \begin{aligned} 2cg(PX,Y)g(tV,tU) &= g(A_U P A_V X, Y) + g(A_V P A_U X, Y) \\ &\quad - g(A_U A_{fV} X, Y) + g(A_U A_{fV} Y, X). \end{aligned}$$

From this we have

$$
\begin{aligned}
\sum_{a,i} g(A_a P A_a e_i, Pe_i) &= c\sum_{a,i} g(Pe_i, Pe_i) g(tv_a, tv_a) \\
&+\frac{1}{2}\sum_{a,i}[g(A_a A_{fa} e_i, Pe_i) - g(A_a A_{fa} Pe_i, e_i)] \\
&= chq - \sum_a \mathrm{Tr} P A_a A_{fa},
\end{aligned} \tag{3.6}
$$

where we have put $A_{fa} = A_{fv_a}$,$\{e_a\}$ being an orthonormal basis of $T(M)^\perp$.
Using (3.1), we obtain

$$
\sum_{a,i} g(A_a Pe_i, A_a Pe_i) = \sum_i [c(n+2) g(Pe_i, Pe_i) - S(Pe_i, Pe_i)]. \tag{3.7}
$$

This implies

$$
\begin{aligned}
&\frac{1}{2}\sum_a |[P, A_a]|^2 \\
&= c(n+2-q)h - \sum_i S(Pe_i, Pe_i) + \sum_a \mathrm{Tr} P A_a A_{fa}.
\end{aligned} \tag{3.8}
$$

Therefore by (3.3), we obtain

$$
\begin{aligned}
\frac{1}{2}\sum_a |[P, A_a]|^2 &= c(n+2-q)h - c(n+2)h + \sum_a \mathrm{Tr} A_a^2 + \sum_a \mathrm{Tr} P A_a A_{fa} \\
&\le ch(1-q) + \sum_a \mathrm{Tr} P A_a A_{fa}.
\end{aligned}
$$

On the other hand, by (3.5), we can see

$$
\sum_\lambda \mathrm{Tr} P A_\lambda A_{f\lambda} = \sum_\lambda \mathrm{Tr} A_\lambda P A_\lambda P,
$$

where we have put $A_{f\lambda} = A_{fv_\lambda}$, $\{v_\lambda\}$ being an orthonormal basis of the complementary orthogonal subbundle of $FT(M)$ in $T(M)^\perp$. Hence we have

$$
\begin{aligned}
0 &\le \frac{1}{2}\sum_u |[P, A_u]|^2 + \sum_\lambda \mathrm{Tr} P A_\lambda P A_\lambda - \sum_\lambda \mathrm{Tr} P^2 A_\lambda^2 \\
&\le ch(1-q) + \sum_\lambda \mathrm{Tr} P A_\lambda P A_\lambda,
\end{aligned}
$$

from which

$$
0 \le \frac{1}{2}\sum_u |[P, A_u]|^2 + \sum_{\lambda,i} g(A_\lambda Pe_i, A_\lambda Pe_i) \le ch(1-q),
$$

where we have put $A_u = A_{v_u}$, $\{v_u\}$ being an orthonormal basis of $FT(M)$ in $T(M)^\perp$. Consequently, we have $q = 1$ and $PA_u = A_uP$, $A_\lambda = 0$ for all λ. We also have, by (3.5), $A_uPA_uX = cPX$. Hence we have

$$\sum_a g(A_aX, A_aY) = g(A_uX, A_uY) = -g(A_uP^2X, A_uY)$$
$$= -g(A_uPA_uPX, Y) = cg(PX, PY).$$

Substituting this equation into (3.2),we find that the Ricci tensor S of M is given by $S(X,Y) = c[(n-1)g(X,Y)+2g(PX,PY)]$, and M is a pseudo-Einstein submanifold of $\bar{M}^m(c)$. This proves the theorem 1.

In case of a generic submanifold, we obtain the following theorem.

Theorem 2. *Let M be an n-dimensional minimal generic submanifold of $\bar{M}^m(c)(c > 0)$. If the Ricci tensor S of M satisfies*

$$S(X,X) \geq c[(n-1)g(X,X) + 2g(PX,PX)]$$

for any vector field X tangent to M, then M is
(a) *a totally geodesic anti-invariant submanifold with constant curvature c,*
or
(b) *a pseudo-Einstein real hypersurface of $\bar{M}^m(c)$ with $2m - n = 1$ and*

$$S(X,Y) = c[(n-1)g(X,Y) + 2g(PX,PY)].$$

References

[1] A. Bejancu, *CR submanifolds of a Kaehler manifold*, Proc. Amer. Math. Soc. **69** (1978), 135–142.
[2] T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 482–499.
[3] M. Kon, *Pseudo-Einstein real hypersurfaces in complex space forms*, J. Differential Geometry **14** (1979), 339–354.
[4] M. Kon, *Generic minimal submanifolds of a complex projective space*, Bull. London Math. Soc. **12** (1980), 355–360.
[5] M. Kon, *Minimal CR submanifolds immersed in a complex projective space*, Geometriae Dedicata **31** (1989), 357–368.
[6] S. Maeda, *Real hypersurface of a complex projective space II*, Bull. Austral. Math. Soc. **29** (1984), 123–127.

Department of Mathematics, Faculty of Education, Hirosaki University, Hirosaki, 036 Japan

Received October 23, 1996 Revised February 19, 1997