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OZEKI'S INEQUALITY AND NONCOMMUTATIVE COVARIANCE

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ABSTRACT. J.I.Fujii introduced the covariance of operators in Umegaki's theory of noncommutative probability. Very recently, it is observed that the so-called (noncommutative) covariance-variance inequality gives a unified method to prove certain operator inequalities including the celebrated Kantorovich inequality. Following after them, we shall discuss an operator version of Ozeki's inequality and consequently we show that the inequality needs a minor correction.

1. Introduction. From Umegaki's viewpoint [4] of noncommutative probability, M.Fujii, T.Furuta, R.Nakamoto and S.E.Takahashi [1] discussed the covariance and the variance of operators acting on a Hilbert space H. The covariance of two operators A and B (at a state $x \in H$) is defined by

$$\text{Cov}(A, B) = (B^*Ax, x) - (Ax, x)(B^*x, x), \tag{1}$$

and the variance of A is defined by

$$\text{Var}(A) = \|Ax\|^2 - |(Ax, x)|^2. \tag{2}$$

Their fundamental tool is the following covariance-variance inequality;

$$|\text{Cov}(A, B)|^2 \leq \text{Var}(A)\text{Var}(B). \tag{3}$$

They observed that $\text{Var}(A) \leq \frac{1}{4}(M-m)^2$ if A is a selfadjoint operator with $m \leq A \leq M$, and consequently they gave an estimation of the covariance by using (3): If $0 \leq m_1 \leq A \leq M_1$ and $0 \leq m_2 \leq B \leq M_2$, then

$$|\text{Cov}(A, B)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2), \tag{4}$$

by which they unified proofs of many operator inequalities including the celebrated Kantorovich inequality.

Ozeki's inequality in [2] is the Kantorovich like inequality: Let a_i and b_i be two positive n-tuples, with $0 < m_1 \leq a_i \leq M_1$ and $0 < m_2 \leq b_i \leq M_2$ $(i = 1, \cdots, n)$ for some constants m_1, m_2, M_1, and M_2. Then the following inequality holds

$$(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - (\sum_{k=1}^{n} a_k b_k)^2 \leq \frac{n^2}{4}(M_1M_2 - m_1m_2)^2. \tag{5}$$

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We here put $A = \mathrm{diag}(a_i)$ and $B = \mathrm{diag}(b_i)$ as diagonal matrices and $x = \frac{1}{\sqrt{n}}(1, \cdots, 1)^t$. Then $0 < m_1 \le A \le M_1$, $0 < m_2 \le B \le M_2$ and $\|x\| = 1$. Moreover (5) becomes

$$(A^2x, x)(B^2x, x) - |(ABx, x)|^2 \le \frac{1}{4}(M_1M_2 - m_1m_2)^2. \tag{6}$$

As a continuation of [1], we shall attempt to consider the operator version of Ozeki's inequality by virtue of the covariance-variance inequality. However we are resisted by the following counterexample for (6): If

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad x = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

then $M_1 = M_2 = 1$, $m_1 = m_2 = 0$. Consequently we have

$$(A^2x, x)(B^2x, x) - (ABx, x)^2 = \frac{1}{3} > \frac{1}{4}(M_1M_2 - m_1m_2)^2 = \frac{1}{4},$$

whereas

$$(A^2x, x)(B^2x, x) - (ABx, x)^2 = \frac{1}{3} < \frac{1}{2}(M_1M_2 - m_1m_2)^2 = \frac{1}{2}.$$

Surprisingly enough, the example above is not only a counterexample of (6), but that of

(5), that is, $a = (1, 1, 0)$ and $b = (0, 1, 1)$. Making a demand that all entry of it is positive, we prepare 3-dimensional vectors as the another counterexample of (5):

$$a = (\frac{1}{4}, 1, 1) \qquad \text{and} \qquad b = (1, 1, \frac{1}{4}).$$

Anyway (5) and (6) should be corrected.

In this note, we shall give an operator version of a corrected Ozeki's inequality, which has a simple proof by (4); more precisely we prove that if two selfadjoint operators A and B commutes, then

$$(A^2x, x)(B^2x, x) - (ABx, x)^2 \le \frac{1}{2}(M_1M_2 - m_1m_2)^2, \tag{7}$$

under the assumption $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$.

In finite dimensinal case, we can sharpen the bound of the right hand side of (7) as follows: If $0 < m_1 \le a_i \le M_1$, and $0 < m_2 \le b_i \le M_2 \quad (i = 1, 2, \cdots, n)$, then

$$(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - (\sum_{k=1}^{n} a_kb_k)^2 \le \frac{n(n-1)}{2}(M_1M_2 - m_1m_2)^2. \tag{8}$$

2. An operator version. The inequality (7) is an operator version of Ozeki's inequality (5). By virture of the covariance-variance inequality in [1], we can prove it:

Theorem 1. *If A and B are commutative selfadjoint operators satisfying $0 \leq m_1 \leq A \leq M_1$ and $0 \leq m_2 \leq B \leq M_2$, then they satisfy the inequality (7).*

Proof. Since A and B are commutative, the left hand side of (7) is difference of $\mathrm{Var}(AB)$ and $\mathrm{Cov}(A^2, B^2)$. We also remark that $\mathrm{Var}(AB) = \mathrm{Cov}(AB, AB)$. Since $0 < m_1 m_2 \leq AB \leq M_1 M_2$, it immediately follows from a formula (4) that

$$\mathrm{Var}(AB) = \mathrm{Cov}(AB, AB) \leq \frac{1}{4}(M_1 M_2 - m_1 m_2)^2.$$

Therefore we have

$$\begin{aligned}(A^2x, x)(B^2x, x) - (ABx, x)^2 &= \mathrm{Var}(AB) - \mathrm{Cov}(A^2, B^2)\\ &\leq \mathrm{Var}(AB) + |\mathrm{Cov}(A^2, B^2)|\\ &\leq \frac{1}{4}(M_1 M_2 - m_1 m_2)^2 + \frac{1}{4}(M_1^2 - m_1^2)(M_2^2 - m_2^2)\\ &\leq \frac{1}{2}(M_1 M_2 - m_1 m_2)^2,\end{aligned}$$

which completes the proof.

3. Ozeki's inequality. In finite dimensional case, we sharpen the bounds of Theorem 1 to some extent and give a simple and computational proof of it.

Theorem 2. *If a_i and b_i are positive n-tuples which satisfy $0 \leq m_1 \leq a_i \leq M_1$, and $0 \leq m_2 \leq b_i \leq M_2 \quad (i = 1, 2, \cdots, n)$, then the following inequality holds*

$$(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - (\sum_{k=1}^{n} a_k b_k)^2 \leq \frac{n(n-1)}{2}(M_1 M_2 - m_1 m_2)^2. \tag{9}$$

Proof. We note that the left hand side of (9) is expressed as $\sum_{i<j}(a_i b_j - a_j b_i)^2$, which has $\frac{n(n-1)}{2}$ terms. Since each term $(a_i b_j - a_j b_i)^2$ is not greater than $(M_1 M_2 - m_1 m_2)^2$, we have

$$(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - (\sum_{k=1}^{n} a_k b_k)^2 = \sum_{i<j}(a_i b_j - a_j b_i)^2 \leq \frac{n(n-1)}{2}(M_1 M_2 - m_1 m_2)^2.$$

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