Nihonkai Math. J.
Vol.8 (1997), 37-46

A GEOMETRICAL STRUCTURE IN THE FURUTA INEQUALITY, II

MASATOSHI FUJII*, JIAN FEI JIANG**, AND EIZABURO KAMEI***

ABSTRACT. We discuss some geometrical structures in the Furuta inequality as a continuation of our preceding note. We show the monotonicity of operator functions associated with the Furuta inequality under the chaotic order. Consequently it gives us geometrical views and helps us to explain obtained operator inequalities,

1. Introduction. In what follows, a capital letter means a (bounded linear) operator acting on a Hilbert space H. An operator T is said to be positive, in symbol, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$. In particular, we denote by $T > 0$ if T is positive and invertible. The positivity of operators induces the (usual) order $A \geq B$ by $A - B \geq 0$ and moreover the operator monotonicity of $\log t$ does the weaker order $A \gg B$ by $\log A \geq \log B$ for $A, B > 0$. It is called the chaotic order, cf. [7].

It is interesting to discuss order-preserving problems for positive operators. One of the most typical examples is the Löwner-Heinz inequality [18, 22]:

Theorem LH. *If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for $0 \leq \alpha \leq 1$.*

In 1987, Furuta [11] considered a background of Theorem LH and finally proposed it as the following surprising form, which is a historical extension of the Löwner-Heinz inequality:

Theorem F. (The Furuta inequality)
If $A \geq B \geq 0$, then for each $r \geq 0$

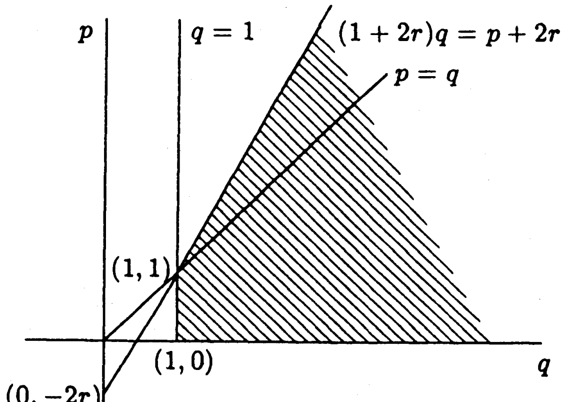
(1) $$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

and

(1') $$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

hold for $p \geq 0$ and $q \geq 1$ with

(*) $$(1 + 2r)q \geq p + 2r.$$



Figure

Related topics are discussed in [2, 3, 9, 12, 13, 14, 15, 17, 19, 20, 23, 24]; among others,an elementary and one-page proof is given in [12] and the best possibility of the conditions on p, q and r in the Furuta inequality is discussed in [23].

1991 *Mathematics Subject Classification.* 47A63 and 47B15.
Key words and phrases. Löwner-Heinz inequality and Furuta inequality..

Now the Furuta inequality can be rephrased by using operator means established by Kubo and Ando [21]. For the sake of convenience, we define the binary operation $_\alpha$ ($\alpha \in \mathbb{R}$) by

$$A \natural_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$$

for $A > 0$ and $B \geq 0$. If $0 \leq \alpha \leq 1$, then it is the α-power mean $\sharp_\alpha$. In fact, $\natural_\alpha$ is defined for $\alpha \geq 0$ in [16]. The Furuta inequality (1) should be understood in our discussion as follows: If $A \geq B \geq 0$ and A is invertible, then

$$A^t \sharp_{\frac{1-t}{p-t}} B^p \leq B \leq A \tag{2}$$

holds for $p \geq 1$ and $t \leq 0$, [20].

In our preceding note [9], a geometrical structure in the Furuta inequality is discussed. It is pointed out that for $A \geq B > 0$ and $s \geq 0$, the figure deduced by the set $\{B^{-s} \sharp_\alpha A^p; p \geq 1, \alpha \in [0,1]\}$, which is equivalent to the set $\{B^{-s} \sharp_{\frac{q+s}{p+s}} A^p; p \geq 1, -s \leq q \leq p\}$, looks like a gingko leaf. In this paper, the discussion will be continued in general setting under the weakened condition $A \gg B$. Precisely, geometric structures of the following sets are discussed under the assumption $A \gg B$:

(1) $\{B^{-s} \natural_{\frac{q+s}{p+s}} A^p; -s \leq q \leq 2p+s\}$ for a fixed $p \geq 0$.
(2) $\{A^{-s} \natural_{\frac{q+s}{p+s}} B^p; -s \leq q \leq 2p+s\}$ for a fixed $p \geq 0$.
(3) $\{B^{-s} \natural_{\frac{q+s}{p+s}} A^p; -2s-p \leq q \leq p\}$ for a fixed $s \geq 0$.
(4) $\{A^{-s} \natural_{\frac{q+s}{p+s}} B^p; -2s-p \leq q \leq p\}$ for a fixed $s \geq 0$.

They are basis of our results in this paper. As a matter of fact, the set (1) stated above corresponds to the following result: For $A \gg B$, $p \geq 0$ and $q \in \mathbb{R}$, the operator function

$$f_{p,q}(s) = B^{-s} \natural_{\frac{q+s}{p+s}} A^p \ (s \geq 0).$$

is increasing for $s \geq -q$ if $p \geq q$, and it is decreasing for $s \geq q - 2p$ if $p \leq q$. The other sets (2), (3) and (4) induce the monotonicity of the corresponding operator functions. Consequently our result [4; Theorem 3] is discussed in a general setting.

2. Results. For our purpose, we have to cite the following fundermental inequality [4] which characterizes the chaotic order and is given a simple proof in [5] recently.

Theorem C. *For $A, B > 0$, $A \gg B$ if and only if*

$$(B^r A^p B^r)^{\frac{2r}{p+2r}} \geq B^{2r}, \ \text{or} \ B^{-2r} \sharp_{\frac{2r}{p+2r}} A^p \geq 1 \tag{3}$$

for $p, r \geq 0$.

Next we state simple lemmas; Lemma 1 was used in [14, 16] and our note [6,10]. Lemma 2 follows from the fact that $\sharp_\alpha$ is a mean for $\alpha \in [0,1]$, and Lemma 3 is a consequence of it.

Lemma 1. *For $A, B > 0$ and $\alpha \in \mathbb{R}$, the following equalityies hold:*
(1) $A \natural_\alpha B = B \natural_{1-\alpha} A$.
(2) $A \natural_{\alpha\beta} B = A \natural_\alpha (A \natural_\beta B)$.
(3) $A \natural_\alpha B = B(B^{-1} \natural_{\alpha-1} A^{-1})B$.
(4) $A \natural B = A(A^{-1} \natural_{-\alpha} B^{-1})A$.

Lemma 2. *If $A \geq C \geq 0$, $B \geq D \geq 0$ and $\alpha \in [0,1]$, then*

$$A \sharp_\alpha B \geq C \sharp_\alpha D.$$

Lemma 3. *If $A \geq C > 0$, $B > 0$ and $\alpha \in [1,2]$, then*

$$A \natural_\alpha B \leq C \natural_\alpha B.$$

Proof. It follows from Lemma 1 (3) and Lemma 2 that

$$A \natural_\alpha B = B(B^{-1} \sharp_{\alpha-1} A^{-1})B \leq B(B^{-1} \sharp_{\alpha-1} C^{-1})B = C \natural_\alpha B.$$

Under this preparation, we prove our theorems on the monotonicity of some operator functions.

Theorem 4. *For $A \gg B$, $p \geq 0$ and $q \in \mathbb{R}$, define an operator function by*

$$f_{p,q}(s) = B^{-s} \natural_{\frac{q+s}{p+s}} A^p \ (s \geq 0).$$

Then it is increasing for $s \geq -q$ if $p \geq q$, and it is decreasing for $s \geq q - 2p$ if $p \leq q$.

Proof. First of all, Lemma 1 (1) and (2) imply that for $\delta > 0$

$$\begin{aligned} f_{p,q}(s+\delta) &= B^{-(s+\delta)} \natural_{\frac{q+s+\delta}{p+s+\delta}} A^p \\ &= A^p \natural_{\frac{p-q}{p+s+\delta}} B^{-(s+\delta)} \\ &= A^p \natural_{\frac{p-q}{p+s}} (A^p \sharp_{\frac{p+s}{p+s+\delta}} B^{-(s+\delta)}) \\ &= (B^{-(s+\delta)} \sharp_{\frac{\delta}{p+s+\delta}} A^p) \natural_{\frac{q+s}{p+s}} A^p. \end{aligned}$$

Next we note that for $p, s \geq 0$

$$\begin{aligned} & B^{-(s+\delta)} \sharp_{\frac{\delta}{p+s+\delta}} A^p \\ &= B^{-(s+\delta)} \sharp_{\frac{\delta}{s+\delta}} (B^{-(s+\delta)} \sharp_{\frac{s+\delta}{p+s+\delta}} A^p) \\ &\geq B^{-(s+\delta)} \sharp_{\frac{\delta}{s+\delta}} 1 \quad \text{by Theorem C and Lemma 2} \\ &= B^{-s} \quad \text{by Lemma 1 (1).} \end{aligned}$$

Now suppose that $-s \leq q \leq p$. Since $\frac{q+s}{p+s} \in [0,1]$, we have

$$f_{p,q}(s+\delta) = (B^{-(s+\delta)} \sharp_{\frac{\delta}{p+s+\delta}} A^p) \natural_{\frac{q+s}{p+s}} A^p \geq B^{-s} \sharp_{\frac{q+s}{p+s}} A^p = f_{p,q}(s),$$

which implies the former. On the other hand, if $p \leq q \leq 2p + s$, then $\frac{q+s}{p+s} \in [1,2]$, so that Lemma 3 implies

$$f_{p,q}(s+\delta) = (B^{-(s+\delta)} \sharp_{\frac{\delta}{p+s+\delta}} A^p) \natural_{\frac{q+s}{p+s}} A^p \leq B^{-s} \sharp_{\frac{q+s}{p+s}} A^p = f_{p,q}(s).$$

The following theorems are variational expressions of Theorem 4.

Theorem 5. *For $A \gg B$, $p \geq 0$ and $q \in \mathbb{R}$, define an operator function by*

$$\tilde{f}_{p,q}(s) = A^{-s} \natural_{\frac{q+s}{p+s}} B^p \ (s \geq 0).$$

Then it is decreasing for $s \geq -q$ if $p \geq q$, and it is increasing for $s \geq q - 2p$ if $p \leq q$.

Proof. This follows from Theorem 4 and the fact that $B^{-1} \gg A^{-1}$ and

$$\tilde{f}_{p,q}(s) = A^{-s} \natural_{\frac{q+s}{p+s}} B^p = [(A^{-1})^{-s} \natural_{\frac{q+s}{p+s}} (B^{-1})^p]^{-1} \ (s \geq 0).$$

Theorem 6. *For $A \gg B$, $s \geq 0$ and $q \in \mathbb{R}$, define an operator function by*

$$g_{s,q}(p) = B^{-s} \natural_{\frac{q+s}{p+s}} A^p \ (p \geq 0).$$

Then it is increasing for $p \geq q$ if $q \geq -s$, and it is decreasing for $p \geq -2s - q$ if $q \leq -s$.

Proof. We first remark that

$$g_{s,q}(p) = B^{-s} \natural_{\frac{q+s}{p+s}} A^p = A^p \natural_{\frac{p-q}{p+s}} B^{-s} = (A^{-1})^{-p} \natural_{\frac{-q+p}{s+p}} (B^{-1})^s \ (p \geq 0)$$

by Lemma 1 (1), and $-s \leq q \leq p$ (resp. $-2s - p \leq q \leq -s$) is equivalent to $-p \leq -q \leq s$ (resp. $s \leq -q \leq 2s + p$). Hence Theorem 4 implies the conclusion since $B^{-1} \gg A^{-1}$.

The following theorem is obtained by using Theorem 6 and the same way as in the proof of Theorem 5:

Theorem 7. *For $A \gg B$, $s \geq 0$ and $q \in \mathbb{R}$, define an operator function by*

$$\tilde{g}_{s,q}(p) = A^{-s} \natural_{\frac{q+s}{p+s}} B^p \ (p \geq 0).$$

Then it is decreasing for $p \geq q$ if $q \geq -s$, and it is increasing for $p \geq -2s - q$ if $q \leq -s$.

Concluding this section, we have two corollaries; Corollary 8 (resp. Corollary 9) follows from Theorems 4 and 6 (resp. Theorems 5 and 7).

Corollary 8. *If $A \gg B$ and $q \in \mathbb{R}$, then*

$$\alpha(s,p) = B^{-s} \sharp_{\frac{q+s}{p+s}} A^p \ (s, p \geq 0)$$

is increasing for both s and p with $-s \leq q \leq p$.

Corollary 9. *If $A \gg B$ and $q \in \mathbb{R}$, then*

$$\beta(s,p) = A^{-s} \sharp_{\frac{q+s}{p+s}} B^p \ (s,p \geq 0)$$

is decreasing for both s and p with $-s \leq q \leq p$.

Remark. Corollary 9 is also a variant of the result [4; Theorem 3] which is an extension of Ando's theorem [1], cf. [8].

3. Geometrical structure. In this section, we investigate the geometrical view of our results. By regarding $B^{-s} \natural_{\frac{q+s}{p+s}} A^p$ as $(1-\alpha)b^{-s} + \alpha a^p$ for $\alpha = \frac{q+s}{p+s}$, we have a line $\{(q, (1-\alpha)b^{-s} + \alpha a^p); q\}$ through both points $(-s, b^{-s})$ and (p, a^p). Similarly we can regard $A^{-s} \sharp_{\frac{q+s}{p+s}} B^p$ as a line through both points $(-s, a^{-s})$ and (p, b^p). From the viewpoint of this, we can draw the following figure corresponding to Theorem 4. Suppose that $0 < b < 1 < a$ and put $f_q(s) = f_{p,q}(s)$ for a fixed p. Then one can see $f_{q_1}(s) \leq f_{q_1}(s+\delta)$ for $-s < q_1 < p$ and $f_{q_2}(s+\delta) \leq f_{q_2}(s)$ for $p < q_2 < 2p+s$ in the figure.

Figure 1. Corresponding to Theorem 4.

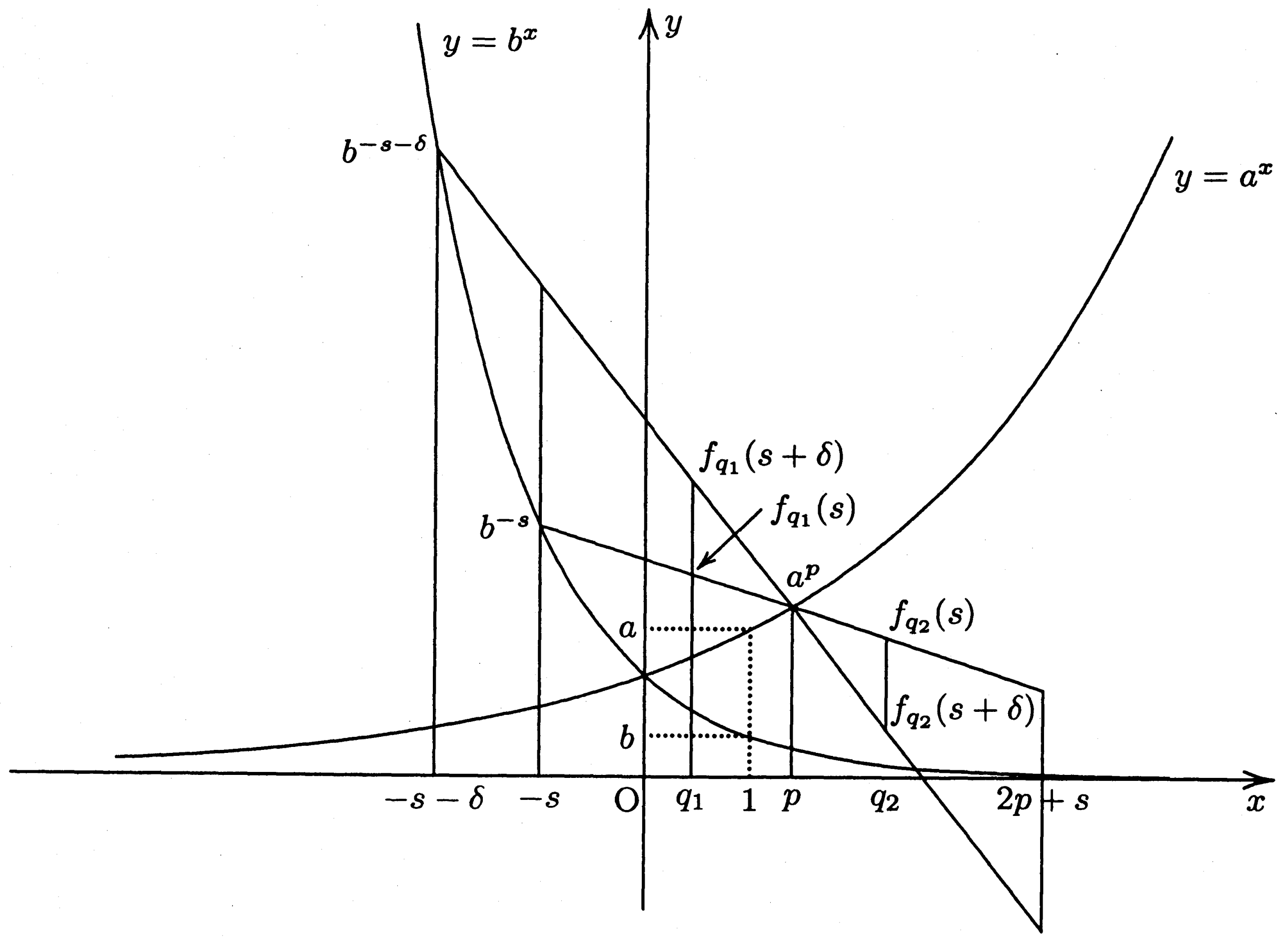


Figure 2. Corresponding to Theorem 5.

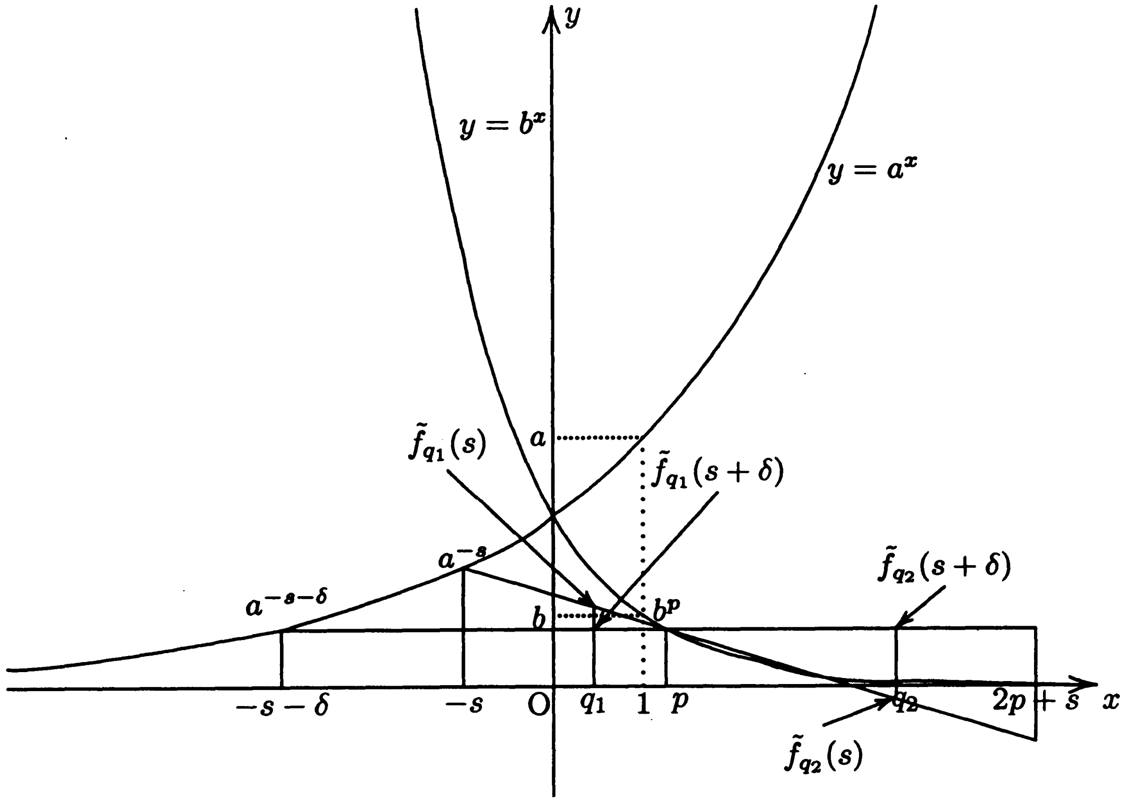


Figure 3. Corresponding to Theorem 6.

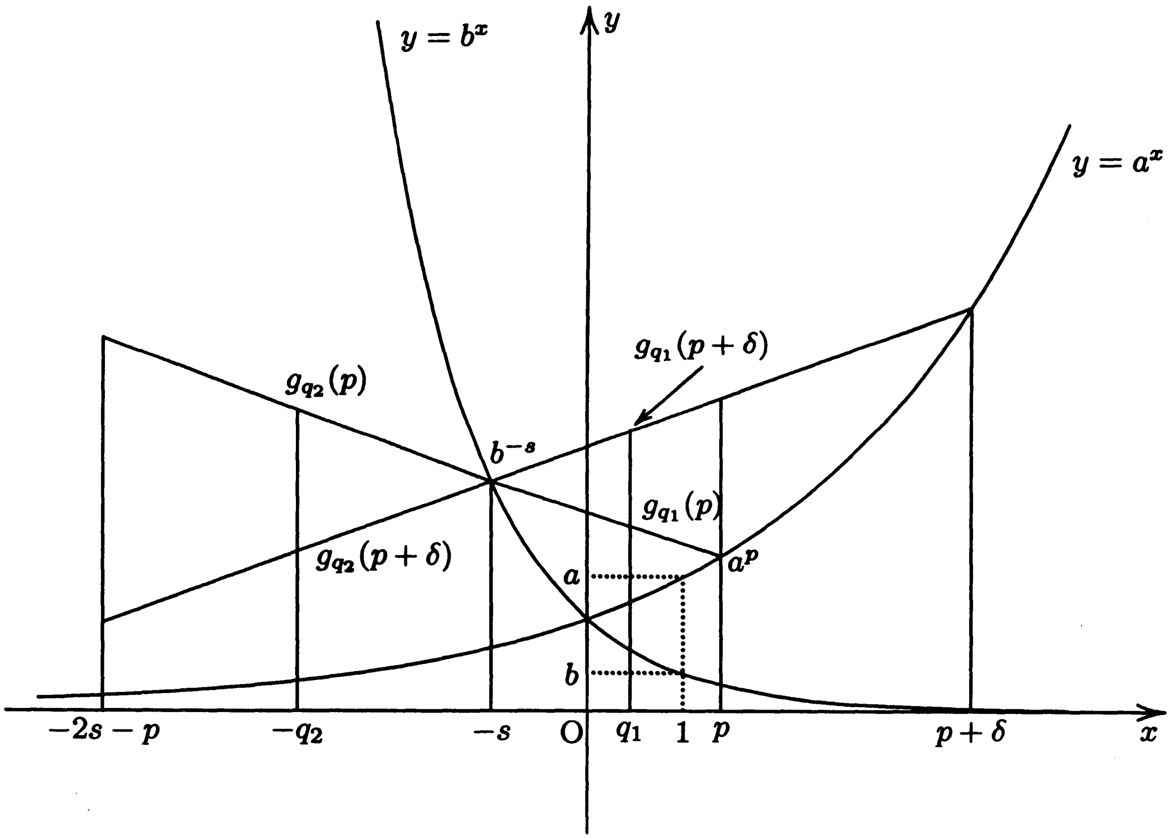
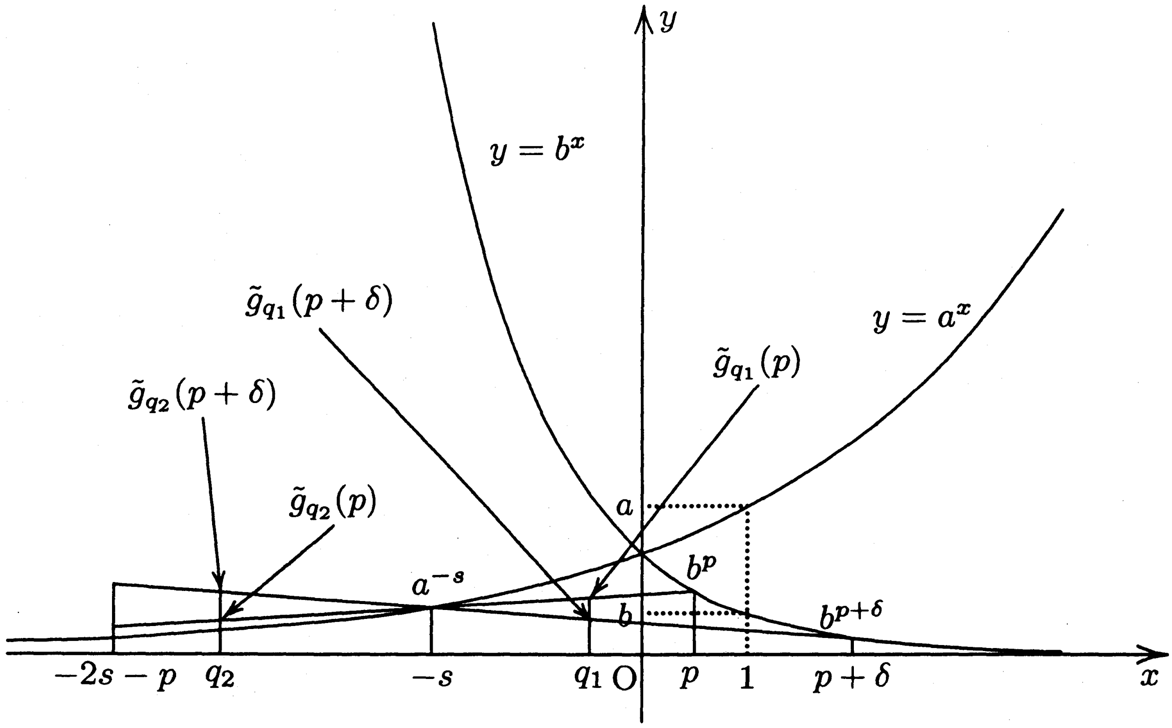


Figure 4. Corresponding to Theorem 7.



4. Some interesting corollaries and counterexamples. In this section, we mention some corollaries of results in §2, which are based on the corresponding figures in the preceding section.

Corollary 10. *If $A \gg B$, $p \geq 0$ and $t \leq 0$, then*

$$B^t \sharp_{\frac{q-t}{p-t}} A^p \geq A^q \text{ for } p \geq q \geq t$$

and

$$B^t \natural_{\frac{q-t}{p-t}} A^p \leq A^q \text{ for } 2p \geq q \geq p.$$

In particular, if $A \gg B$, then for $t \leq 0$

(1) $B^t \sharp_{\frac{1-t}{p-t}} A^p \geq A$ for $p \geq 1$

(2) $B^t \natural_{\frac{1-t}{p-t}} A^p \leq A$ for $1 \geq p \geq \frac{1}{2}$

(3) $B^t \natural_{\frac{2p-t}{p-t}} A^p \leq A^{2p}$ for $p \geq 0$.

Proof. Taking $s = 0$ in Theorem 4, we have $f_{p,q}(0) = A^q$. If $t \leq q \leq p$, then for $t \leq 0$

$$A^q = f_{p,q}(0) \leq f_{p,q}(-t) = B^t \sharp_{\frac{q-t}{p-t}} A^p.$$

If $p \leq q \leq 2p$, then for $t \leq 0$

$$A^q = f_{p,q}(0) \geq f_{p,q}(-t) = B^t \sharp_{\frac{q-t}{p-t}} A^p.$$

The reminder is easily checked.

The following corollary is an equivalent expression to Corollary 10 because $A \gg B$ if and only if $B^{-1} \gg A^{-1}$.

Corollary 11. *If $A \gg B$, $p \geq 0$ and $t \leq 0$, then*

$$A^t \sharp_{\frac{q-t}{p-t}} B^p \leq B^q \text{ for } p \geq q \geq t$$

and

$$A^t \natural_{\frac{q-t}{p-t}} B^p \geq B^q \text{ for } 2p \geq q \geq p.$$

In particular, if $A \gg B$, then for $t \leq 0$
(1) $A^t \sharp_{\frac{1-t}{p-t}} B^p \leq B$ for $p \geq 1$
(2) $A^t \natural_{\frac{1-t}{p-t}} B^p \geq B$ for $1 \geq p \geq \frac{1}{2}$
(3) $A^t \natural_{\frac{2p-t}{p-t}} B^p \geq B^{2p}$ for $p \geq 0$.

Remark. (1) in Corollary 11 is just the left hand side of the Furuta inequality (2) in the first section, in which the assumption is weakened from the usual order to the chaotic order, cf. also [4].

It is known in [23] that the Furuta inequality (1) is not true for $t > 0$ and $p < 1$, by which we have counterexamples to Theorems 5, 6 and Corollary 10. For simplicity, A and B are said to be well-ordered if either $A \geq B$ or $A \leq B$ holds.

Counterexample 1. We take $p = 1$ and $q = 5$ in Theorem 5; we consider the function

$$\tilde{f}_{1,5}(s) = A^{-s} \natural_{\frac{5+s}{1+s}} B^p \ (s \geq 0).$$

Then it is increasing for $s \geq 3$ by the theorem, but it is not monotone for $0 \leq s < 3$. As a matter of fact, we have

$$\tilde{f}_{1,5}(0) = 1 \natural_5 B = B^5$$

and by Lemma 1 (3)

$$\tilde{f}_{1,5}(1) = A^{-1} \natural_3 B = B(B^{-1} \natural_2 A)B = B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^2 B^{\frac{1}{2}}.$$

Clearly they are well-ordered if and only if so are B^4 and $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^2$. In addition, if $A \geq B > 0$ and $A \neq B$, then

$$B^4 \leq (B^{\frac{1}{2}}AB^{\frac{1}{2}})^2$$

must be true by Theorem LH, but it is not true in general by Tanahashi's consideration stated above [23]. This contradiction shows that the function is not monotone in $0 \leq s < 3$.

Counterexample 2. Next we consider the function

$$g_{q,q}(p) = B^{-q} \natural_{\frac{2q}{p+q}} A^p \ (p \geq 0)$$

by putting $s = q$ in Theorem 6. Then $g(p) = g_{q,q}(p)$ is increasing for $p \geq q(>0)$. On the other hand, $g(p)$ is not monotone for $p \in [0, q]$. Since

$$g(0) = B^q \quad \text{and} \quad g(q) = A^q,$$

they are not well-ordered for $q > 1$ in general even if $A \geq B > 0$.

Counterexample 3. Finally we discuss a counterexample related to Corollary 10 (2) (and (1));

$$B^t \natural_{\frac{1-t}{p-t}} A^p \leq A \quad \text{for} \quad \frac{1}{2} \leq p \leq 1.$$

On the other hand, take $p \in [0, \frac{1}{2})$. Then we choose $t \leq 0$ such that $p < \frac{1+t}{2} \leq \frac{1}{2}$ and put $q = \frac{p-t}{1-p}$. Since $q \in (0,1)$, there exist A and B such that $A \geq B > 0$ and

$$(A^{\frac{p}{2}} B^{-t} A^{\frac{p}{2}})^{\frac{1}{q}} \not\leq A^{\frac{p-t}{q}}$$

by Tanahashi's result again. Namely we have

$$B^t \natural_{\frac{1-t}{p-t}} A^p = A^{\frac{p}{2}} (A^{\frac{p}{2}} B^{-t} A^{\frac{p}{2}})^{\frac{1-p}{p-t}} A^{\frac{p}{2}} \not\leq A$$

by Lemma 1 (3).

Acknowledgement. The authors would like to express their thanks to Prof. E.Nakai and Mr.H.Mori for drawing the figures in §3 and to the referee for his kind suggestions.

References

1. T.Ando, *On some operator inequalities*, Math. Ann., **279** (1987), 157-159.
2. M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator theory **23** (1990), 67-72.
3. M.Fujii,T.Furuta and E.Kamei, *Furuta's inequality and its application to Ando's Theorem*, Linear Alg. and its Appl. **149** (1991), 91-96.
4. M.Fujii,T.Furuta and E.Kamei, *Operator functions associated with Furuta's inequality*, Linear Alg. and its Appl. **179** (1993), 161-169.
5. M.Fujii, J.-F. Jiang and E.Kamei, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc., to appear.
6. M.Fujii, J.-F. Jiang and E.Kamei, *A characterization of orderd defined by $A^\delta \geq B^\delta$ via Furuta inequality*, Math. Japon., to appear.
7. M.Fujii and E.Kamei, *Furuta's inequality for the chaotic order, I and II*, Math. Japon., **36** (1991), 603-606 and 717-722.
8. M.Fujii and E.Kamei, *Furuta's inequality and a generaalization of Ando's Theorem*, Proc. Amer. Math. Soc., **115** (1992), 409-413.
9. M.Fujii and E.Kamei, *A geometrical structure in the Furuta inequality*, Math. Japon., **43** (1996), 83-90.
10. M.Fujii and E.Kamei, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 2751-2756.
11. T.Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc. **101** (1987), 85-88.

12. T.Furuta, *Elementary proof of an order preserving inequality*, Proc. Japan Acad. **65** (1989), 126.
13. T.Furuta, *A proof via operator means of an order preserving inequality*, Linear Alg. and its Appl. **113** (1989), 12-130.
14. T.Furuta, *Two operator functions with monotone property*, Proc. Amer. Math. Soc. **111** (1991), 511-516.
15. T.Furuta, *Application of an order preserving inequalities*, Operator theory : Advances and Applications, Birkhouser Verlag Basel **59** (1992), 180-190.
16. T.Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Alg. and its Appl. **219** (1995), 139-155.
17. J.-F.Jiang, E.Kamei and M.Fujii, *The monotonicity of operator functions associated with the Furuta inequality*, Math. Japon., to appear.
18. E.Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415-438.
19. E.Kamei, *Furuta's inequality via operator means*, Math. Japon. **33** (1988), 737-739.
20. E.Kamei, *A satellite to Furuta's inequality*, Math. Japon. **33** (1988), 883-886.
21. F.Kubo and T.Ando, *Means of positive linear operators*, Math. Ann. **246** (1980), 205-224.
22. K.Löwner, *Über monotone Matrixfunktionen*, Math. Z. **38** (1934), 177-216.
23. K.Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141-146.
24. K.Tanahashi, *The Furuta inequality with negative powers*, preprint.

*) Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582, Japan

**) Department of Basic Science and Technology, China Textile University, Shanghai, China, Postal code 200051.

***) Momodani Senior Highschool, Ikuno, Osaka 544, Japan

Received September 17, 1996 **Revised October 8, 1996**