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FORMALLY INTEGRABLE MIZOHATA SYSTEMS IN R^3

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In this article we investigate the local structure of Mizohata's systems in R^3, i.e. the local structure of systems of the form

$$\left| \begin{array}{l} \partial_1 u = \epsilon_1 i x^1 \partial_3 u + f \\ \partial_2 u = \epsilon_2 i x^2 \partial_3 u + g, \end{array} \right. \tag{1}$$

where $u(x) = u(x^1, x^2, x^3)$ is the unknown complex function of $x = (x^1, x^2, x^3) \in R^3$, $f(x)$ and $g(x)$ are given smooth complex functions, $\epsilon_j = +1$ or -1, $j = 1, 2$, and $\partial_k u = \frac{\partial u}{\partial x^k}$.

The local existence of solutions of Mizohata systems of codimension 1 in R^n was studied in the paper [1] by P. Cordaro and J. Hounie, where sufficient conditions for the local solvability were given in the case when the given functions in the right hand sides of the equations (f and g in (1)) satisfy certain conditions. The key problem for these systems, which are a special case of the complex linear systems of PDEs, is the problem of their solvability. As in the case of the classical Frobenius theorem, a necessary condition for the solvability of such systems is their *formal integrability*, i.e. the compatibility of their right hand sides; for the system (1) these conditions are

$$L_1 g = L_2 f, \tag{2}$$

where $L_1 = \partial_1 - \epsilon_1 i x^1 \partial_3$ and $L_2 = \partial_2 - \epsilon_2 i x^2 \partial_3$.

The purpose of this note is to show that any system (1), satisfying (2), can be reduced by a suitable change of the variables to a system (possibly non-linear) in R^2, i.e. to a system, involving only two independent variables and two unknown functions, which is not overdetermined. This result relates the local theory of the formally integrable Mizohata systems with the local theory of the first order systems of PDEs in R^2. More exactly, the theory of the systems of the form (1), having solutions, is a part of the theory of the systems of PDEs with two unknown real functions of two real variables.

In view of this result any formally integrable system of the form (1) is equivalent to an ordinary first order smooth complex equation of the form $\partial w = F(z, w, \frac{\partial w}{\partial \bar{z}})$, where w is the unknown complex function of the complex variable z.

Note that, given a system of partial differential equations, it is often possible to simplify its equations by increasing the number of the variables (independent and dependent). The converse, i.e. to simplify the equations with a simultaneous decreasing the number of the variables is rarely possible; in the present article we consider a class of systems of the form (1), for which the procedure suggested below leads to a simplification.

It should be noted, that the general idea of the method, which we use here, as well as the key notion of characteristic vector fields, have been introduced by S. Lie.

Let $u = u^1 + iu^2$, $f = f^1 + if^2$ and $g = g^1 + ig^2$. Then (1) can be rewritten in real variables (dependent and independent) in the following way:

$$\left|\begin{aligned} \partial_1 u^1 &= -\epsilon_1 x^1 \partial_3 u^2 + f^1 \\ \partial_1 u^2 &= \epsilon_1 x^1 \partial_3 u^1 + f^2 \\ \partial_2 u^1 &= -\epsilon_2 x^2 \partial_3 u^2 + g^1 \\ \partial_2 u^2 &= \epsilon_2 x^2 \partial_3 u^1 + g^2. \end{aligned}\right. \tag{3}$$

For the sake of convenience, we denote the right hand sides of the above equations by m_1, n_1, m_2 and n_2, respectively, and $\partial_3 u^1$, $\partial_3 u^2$, u^1 and u^2 by x^4, x^5, x^6 and x^7, respectively. Then Pfaff's system, corresponding to (3), is

$$\left|\begin{aligned} \omega^1(dx) &\equiv dx^6 - m_1 dx^1 - m_2 dx^2 - x^4 dx^3 = 0 \\ \omega^2(dx) &\equiv dx^7 - n_1 dx^1 - n_2 dx^2 - x^5 dx^3 = 0. \end{aligned}\right. \tag{4}$$

Recall that, according to the classical definition (see e.g. Schouten and van der Kulk [2]), the vector field η is called a *characteristic vector field for the system* $\omega^k(dx) = 0$, $k = 1, 2, \dots, p$, if for any ξ, satisfying $\omega^k(\xi) = 0$, $k = 1, 2, \dots, p$, the equalities

$$\omega^k(\eta) = 0, \quad \partial\omega^k(\xi, \eta) = 0, \quad k = 1, 2, \dots, p$$

hold.

Consider the following vector fields:

$$\begin{aligned} \xi_1 &= \partial_4, \\ \xi_2 &= \partial_5, \\ \xi_3 &= \partial_3 + x^4 \partial_6 + x^5 \partial_7, \\ \xi_4 &= \partial_1 + m_1 \partial_6 + n_1 \partial_7, \\ \xi_5 &= \partial_2 + m_2 \partial_6 + n_2 \partial_7. \end{aligned}$$

Theorem 1. If f and g satisfy (2), then the vector field

$$\eta = (-\epsilon_2 x^2 \partial_3 f^1 + \epsilon_1 x^1 \partial_3 g^1)\xi_1 + (\partial_2 f^1 - \partial_1 g^1)\xi_2 - \epsilon_2 x^2 \xi_4 + \epsilon_1 x^1 \xi_5$$

is characteristic for the system (4).

Proof. We have to prove that for any vector field ξ, satisfying the conditions $\omega^k(\xi) = 0$, $k = 1, 2$, the equalities

$$\omega^k(\eta) = 0, \quad \partial\omega^k(\xi, \eta) = 0, \quad k = 1, 2 \tag{5}$$

hold. A straightforward verification shows that ξ_j, $j = 1, 2, 3, 4, 5$ satisfy $\omega^k(\xi_j) = 0$, $k = 1, 2$, and that they are linearly independent; hence each field ζ, satisfying the conditions $\partial\omega^k(\xi_j, \zeta) = 0$, $k = 1, 2$, $j = 1, 2, 3, 4, 5$ and $\omega^k(\zeta) = 0$, $k = 1, 2$, satisfies also (5).

Therefore, in order to prove the required result, it is sufficient to show that

$$\omega^k(\xi_j, \eta) = 0, \quad k = 1, 2, \quad j = 1, 2, 3, 4, 5 \tag{6}$$

(note that, since η is a linear combination of ξ_j, and since $\omega^k(\xi_j) = 0, \quad j = 1, 2, 3, 4, 5, \quad k = 1, 2$, then $\omega^k(\eta) = 0, \quad k = 1, 2$).

We have

$$\begin{aligned}
[\xi_1, \xi_2] &= 0, \\
[\xi_1, \xi_3] &= \partial_6, \\
[\xi_2, \xi_3] &= \partial_7, \\
[\xi_1, \xi_4] &= \epsilon_1 x^1 \partial_7, \\
[\xi_2, \xi_4] &= -\epsilon_1 x^1 \partial_6, \\
[\xi_1, \xi_5] &= \epsilon_2 x^2 \partial_7, \\
[\xi_2, \xi_5] &= -\epsilon_2 x^2 \partial_6, \\
[\xi_3, \xi_4] &= \partial_3 m_1 \partial_6 + \partial_3 n_1 \partial_7, \\
[\xi_3, \xi_5] &= \partial_3 m_2 \partial_6 + \partial_3 n_2 \partial_7, \\
[\xi_4, \xi_5] &= (\partial_1 m_2 - \partial_2 m_1)\partial_6 + (\partial_1 n_2 - \partial_2 n_1)\partial_7.
\end{aligned}$$

Hence, using the above results and the identity $\partial\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$, we obtain

$$\begin{aligned}
\partial\omega^1(\xi_1, \eta) &= (-\epsilon_2 x^2 \partial_3 f^1 + \epsilon_1 x^1 \partial_3 g^1)\partial\omega^1(\xi_1, \xi_1) \\
&+(\partial_2 f^1 - \partial_1 g^2)\partial\omega^1(\xi_1, \xi_2) - \epsilon_2 x^2 \partial\omega^1(\xi_1, \xi_4) + \epsilon_1 x^1 \partial\omega^1(\xi_1, \xi_5) \\
&= -(\partial_2 f^1 - \partial_1 g^1)\omega^1([\xi_1, \xi_2]) + \epsilon_2 x^2 \omega^1([\xi_1, \xi_4]) - \epsilon_1 x^1 \omega^1([\xi_1, \xi_5]) \\
&= 0,
\end{aligned} \tag{7}$$

and similarly

$$\begin{aligned}
&\partial\omega^2(\xi_1, \eta) = 0, \\
&\partial\omega^1(\xi_2, \eta) = 0, \quad \partial\omega^1(\xi_3, \eta) = 0, \\
&\partial\omega^1(\xi_4, \eta) = 0, \quad \partial\omega^1(\xi_5, \eta) = 0.
\end{aligned} \tag{8}$$

Note that the condition (2) is equivalent to the equalities

$$\partial_1 g^1 + \epsilon_1 x^1 \partial_3 g^2 = \partial_2 f^1 + \epsilon_2 x^2 \partial_3 f^2 \tag{9}$$

and

$$\partial_1 g^2 - \epsilon_1 x^1 \partial_3 g^1 = \partial_2 f^2 - \epsilon_2 x^2 \partial_3 f^1. \tag{10}$$

Hence η can be represented also in the form

$$\eta = (-\partial_2 f^2 + \partial_1 g^2)\xi_1 + (-\epsilon_2 x^2 \partial_3 f^2 + \epsilon_1 x^1 \partial_3 g^2)\xi_2 - \epsilon_2 x^2 \xi_4 + \epsilon_1 x^1 \xi_5.$$

Then

$$\begin{aligned} \partial\omega^2(\xi_3, \eta) &= (-\partial_2 f^2 + \partial_1 g^1)\partial\omega^2(\xi_3, \xi_1) \\ &+(-\epsilon_2 x^2 \partial_3 f^2 + \epsilon_1 x^1 \partial_3 g^2)\partial\omega^2(\xi_3, \xi_2) \\ &-\epsilon_2 x^2 \partial\omega^2(\xi_3, \xi_4) + \epsilon_1 x^1 \partial\omega^2(\xi_1, \xi_5) \\ &= -(-\partial_2 f^2 + \partial_1 f^1)\omega^2([\xi_3, \xi_1]) + \epsilon_2 x^2 \omega^2([\xi_1, \xi_4]) - \epsilon_1 x^1 \omega^2([\xi_1, \xi_5]) \\ &= 0, \end{aligned} \tag{11}$$

and similarly

$$\partial\omega^2(\xi_4, \eta) = 0, \quad \partial\omega^2(\xi_5, \eta) = 0. \tag{12}$$

Combining (7), (8), (11) and (16), we get (5). Theorem 1 is proved.

For Pfaff's systems, having a characteristic vector field, there exists a standard method for reducing the number of the variables (see e.g. Schouten and van der Kulk [2]). Here for the case of system (4) the total number of the variables can be reduced from 7 to 6. We will apply a slightly modified version of the classical method in order to deduce a respective system of PDEs, equivalent to (1), with two unknown functions and two independent variables.

Consider the system

$$\xi_1 \Phi = 0, \quad \xi_2 \Phi = 0, \quad \eta\Phi = 0. \tag{13}$$

Lemma 1.

$$[\xi_1, \eta] = 0, \quad [\xi_2, \eta] = 0, \quad [\xi_1, \xi_2] = 0. \tag{14}$$

Proof. The last of these three equalities was mentioned above and is trivial; in order to prove the first two it is sufficient to rewrite η, using (9) and (10), in the following way:

$$\begin{aligned} \eta &= (-\epsilon_2 x^2 \partial_3 f^1 + \epsilon_1 x^1 \partial_3 g^1)\xi_1 + (\partial_2 f^1 - \partial_1 g^1)\xi_2 - \epsilon_2 x^2 \xi_4 + \epsilon_1 x^1 \xi_5 \\ &= -\epsilon_2 x^2 \partial_1 - \epsilon_2 x^2 m_1 \partial_6 - \epsilon_2 x^2 n_1 \partial_7 + \epsilon_1 x^1 \partial_2 + \epsilon_1 x^1 m_2 \partial_6 + \epsilon_1 x^1 n_2 \partial_7 \\ &\quad +(-\partial_2 f^2 + \partial_1 g^2)\partial_4 + (\partial_2 f^1 - \partial_1 g^1)\partial_5 \\ &= -\epsilon_2 x^2 \partial_1 + \epsilon_1 x^1 \partial_2 - (\partial_2 f^2 + \partial_1 g^2)\partial_4 + (\partial_2 f^1 - \partial_1 g^1)\partial_5 \\ &\quad +(-\epsilon_2 x^2 f^1 + \epsilon_1 x^1 g_1)\partial_6 - (\epsilon_2 x^2 f^2 + \epsilon_1 x^1 g_2)\partial_7; \end{aligned}$$

now the required result is clear.

It follows from Lemma 1 that (13) has four functionally independent solutions $\Phi^k = \Phi^k(x), \quad k = 1, 2, 3, 4, \quad x \in R^7$. Consider also the equation $\eta\Phi = 0$. The above four functions are its solutions, and it has two more functionally independent solutions; denote these two solutions by Φ^5 and Φ^6. And finally, let Φ^7 be any function, which is functionally independent with the previous six $\Phi^k, \quad k = 1, 2, \ldots, 6$.

Consider the local system of coordinates with coordinate functions $y^k = \Phi^k(x)$. Since

$$\omega^k(\xi_1) = \omega^k(\xi_2) = \omega^k(\eta) = 0,$$

and since Φ^k, $k = 1, 2, 3, 4$ satisfy (13), then in the new coordinates (4) has the form

$$\left| \begin{array}{l} \omega^1(dy) \equiv \omega_1^1(y)dy^1 + \omega_2^1(y)dy^2 + \omega_3^1(y)dy^3 + \omega_4^1(y)dy^4 = 0 \\ \omega^2(y) \equiv \omega_1^2(dy)dy^1 + \omega_2^2(y)dy^2 + \omega_3^2(y)dy^3 + \omega_4^2(y)dy^4 = 0. \end{array} \right. \tag{15}$$

Since the rank of (4) equals 2, (15) can be resolved with respect to two of dy^1, dy^2, dy^3 and dy^4; without loss of generality we can assume, that (15) can be resolved with respect to dy^3 and dy^4 :

$$\left| \begin{array}{l} \varpi^1(dy) \equiv dy^3 - F(y)dy^1 - G(y)dy^2 = 0 \\ \varpi^2(dy) \equiv dy^4 - H(y)dy^1 - K(y)dy^2 = 0. \end{array} \right. \tag{16}$$

This new Pfaff system is equivalent to (15) and hence to (4).

Lemma 2. $F(y)$, $G(y)$, $H(y)$ and $K(y)$ do not depend on y^7 : $\partial_7 F = \partial_7 G = \partial_7 H = \partial_7 K = 0$.

Proof. Since $\eta\Phi^k = 0$, $k = 1, 2, \ldots, 6$, then $\eta = a(y)\frac{\partial}{\partial y^7}$ for a suitable function $a(y)$. And since η is characteristic for (4) and hence for (16), in view of (5) the rank of the system

$$\omega^k(dy) = 0, \partial\omega^k(\eta, dy) = 0, \quad k = 1, 2$$

equals 2, i.e the equations $\partial\omega^k(\eta, dy) = 0$, $k = 1, 2$ are linear combinations of the equations of (16). It remains to note that

$$\partial\omega^1(\eta, dy) = \partial\omega^1(a(y)\partial_7, dy) = a(y)(-\partial_7 F dy^1 - \partial_7 G dy^2),$$

$$\partial\omega^2(\eta, dy) = \partial\omega^2(a(y)\partial_7, dy) = a(y)(-\partial_7 H dy^1 - \partial_7 K dy^2),$$

and the required result follows.

Since (4) and (16) are equivalent, the existence of functions $u^1 = x^6(x^1, x^2, x^3)$ and $u^2 = x^7(x^1, x^2, x^3)$, for which $\partial_1 x^6 = m_1$, $\partial_2 x^6 = m_2$, $\partial_1 x^7 = n_1$ and $\partial_2 x^7 = n_2$, is equivalent to the existence of functions $y^3 = v^1(y^1, y^2)$ and $y^4 = v^2(y^1, y^2)$, which reduce the equations of (16) to identities, or, in other words, which satisfy the system of PDEs

$$\frac{\partial v^1}{\partial y^1} = F, \quad \frac{\partial v^1}{\partial y^2} = G, \quad \frac{\partial v^2}{\partial y^1} = H, \quad \frac{\partial v^2}{\partial y^2} = K,$$

where F, G, H and K depend on y^1, y^2, $v^1 = y^3$, $v^2 = y^4$, y^5 and y^6 and, according to Lemma 2, do not depend on y^7. Eliminating y^5 and y^6, we obtain an equivalent system of the form

$$\left| \begin{array}{l} P(y^1, y^2, v^1, v^2, \partial_1 v^1, \partial_2 v^1, \partial_1 v^2, \partial_2 v^2) = 0 \\ Q(y^1, y^2, v^1, v^2, \partial_1 v^1, \partial_2 v^1, \partial_1 v^2, \partial_2 v^2) = 0, \end{array} \right. \tag{17}$$

where $\partial_1 = \frac{\partial}{\partial y^1}$ and $\partial_2 = \frac{\partial}{\partial y^2}$.

Let $y^3 = v^1(y^1, y^2)$, $y^4 = v^2(y^1, y^2)$ be a solution of (17). Then the system

$$\Phi^3 = v^1(\Phi^1, \Phi^2), \quad \Phi^4 = v^2(\Phi^1, \Phi^2) \tag{18}$$

determines the implicit functions $u^1 = x^6(x^1, x^2, x^3)$, $u^2 = x^7(x^1, x^2, x^3)$, which form a solution of (1); note that, since the functions Φ^k, $k = 1, 2, 3, 4$ satisfy (14), then they do not depend on x^4 and x^5, and hence (18) involves x^1, x^2, x^3, x^6 and x^7 only.

Thus we proved

Theorem 2. The change of the variables $x \to y$ reduces the Mizohata system (1) to the system (17).

Now we will apply the above results to the system

$$\left|\begin{array}{l} \partial_1 u = ix^1\partial_3 u + x^1(\phi((x^1)^2 + (x^2)^2, x^3) + ax^2 + ibx^1) - bx^3 \\ \partial_2 u = ix^2\partial_3 u + x^2(\phi((x^1)^2 + (x^2)^2, x^3) + ax^2 + ibx^1) + aix^3, \end{array}\right. \tag{19}$$

where $u(x) = u(x^1, x^2, x^3)$ is the unknown complex function of $x = (x^1, x^2, x^3) \in R^3$, $\phi = \phi^1 + i\phi^2$ is a given smooth complex function of two real variables, and a and b are given constants.

This system is a particular case of (1) with $\epsilon_1 = \epsilon_2 = 1$, $f = x^1(\phi + ax^2 + ibx^1) - bx^3$ and $g = x^2(\phi + ax^2 + ibx^1) + iax^3$.

We have

$$f^1 = x^1(\phi^1((x^1)^2 + (x^2)^2, x^3) + ax^2) - bx^3, \quad f^2 = x^2(\phi^2((x^1)^2 + (x^2)^2, x^3) + bx^1),$$

$$g^1 = x^2(\phi^1((x^1)^2 + (x^2)^2, x^3) + ax^2), \quad g^2 = x^2(\phi^2((x^1)^2 + (x^2)^2, x^3) + bx^1) + ax^3,$$

and hence

$$\eta = -x^2\partial_1 + x^1\partial_2 + bx^2\partial_4 + ax^1\partial_5 + bx^2x^3\partial_6 + ax^1x^3\partial_7.$$

Further we calculate

$$\Phi_1 = (x^1)^2 + (x^2)^2, \quad \Phi_2 = x^3, \quad \Phi_3 = x^6 + bx^1x^3,$$

$$\Phi_4 = x^7 - ax^2x^3, \quad \Phi_5 = bx^1 + x^4,$$

$$\Phi_6 = ax^2 - x^5, \quad \Phi_7 = x^1.$$

These functions imply the following formulas for the change of the variables:

$$x^1 = y^7, \quad x^2 = \sqrt{y^1 - (y^7)^2}, \quad x^3 = y^2,$$

$$x^4 = y^5 - by^7, \quad x^5 = ax^2 - y^6 = a\sqrt{y^1 - (y^7)^2} - y^6,$$

$$x^6 = y^3 - by^2y^7, \quad x^7 = y^4 + ay^2x^2 = y^4 + ay^2\sqrt{y^1 - (y^7)^2}.$$

In the new variables the system (15) becomes

$$\left| \begin{aligned} &\omega^1(dy) \equiv dy^3 - \frac{1}{2}(y^6 + \phi^1)dy^1 - y^5 dy^2 = 0 \\ &\omega^2(dy) \equiv dy^4 - \frac{1}{2}(y^5 + \phi^2)dy^1 + y^6 dy^2 = 0. \end{aligned} \right.$$

Consequently (1) is equivalent to the system

$$\left| \begin{aligned} \partial_1 v^1 &= \frac{1}{2}(y^6 + \phi^1) \\ \partial_2 v^1 &= y^5 \\ \partial_1 v^2 &= \frac{1}{2}(y^5 + \phi^2) \\ \partial_2 v^2 &= -y^6, \end{aligned} \right.$$

which after eliminating y^5 and y^6 reduces to

$$\left| \begin{aligned} 2\partial_1 v^1 + \partial_2 v^2 &= \phi^1 \\ 2\partial_1 v^2 - \partial_2 v^1 &= \phi^2. \end{aligned} \right.$$

This is a system of a well known type and can be treated by standard methods. From any of its solutions $v^1 = Y^3(y^1, y^2)$, $v^2 = Y^4(y^1, y^2)$ we obtain the respective solution $u^1 = Y^3((x^1)^2 + (x^2)^2, x^3) - bx^1x^3$, $u^2 = Y^4((x^1)^2 + (x^2)^2, x^3) + ax^2x^3$ of (19).

In particular, taking $\phi = a = b = 0$, we obtain the following result.

Theorem 3. The homogeneous Mizohata system

$$\left| \begin{aligned} \partial_1 u &= ix^1 \partial_3 u \\ \partial_2 u &= ix^2 \partial_3 u \end{aligned} \right. \tag{20}$$

reduces by the change of the variables described above to the real system

$$\left| \begin{aligned} 2\partial_1 v^1 + \partial_2 v^2 &= 0 \\ 2\partial_1 v^2 - \partial_2 v^1 &= 0. \end{aligned} \right. \tag{21}$$

Theorem 3 expresses the following well known fact: the Mizohata system (20) possesses the solution $Z(x^1, x^2, x^3) = [(x^1)^2 + (x^2)^2]/2 - ix^3$ and any other local solution u of (20), say of class C^1, can be written as $u = V \circ Z$, where V is a C^1 function in a neighbourhood of the origin in C, holomorphic for $Re\ z > 0$. This follows from the Baouendi-Treves approximation theorem ([3], [4]). Writing $V = V_1 - iV_2$ and $v_1(y^1, y^2) = V_1(2y^1, y^2)$, $v_2(y^1, y^2) = V_2(2y^1, y^2)$, we obtain - from the Cauchy-Riemann equation satisfied by V - a pair of solutions of (21).

Similar results can be derived for the homogeneous Mizohata systems in R^3 with other combinations of the signs of ϵ_1 and ϵ_2.

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