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ON p-HYPONORMAL OPERATORS

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Abstract

In this paper, we will give some spectral properties of p-hyponormal operators and two operators T and S on a complex Hilbert space as follows :

(1) T is a p-hyponormal operator which is not quasi-hyponormal.

(2) S is a quasi-hyponormal operator which is not p-hyponormal.

1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$. An operator $T \in B(\mathcal{H})$ is said to be $p-$hyponormal if $(T^*T)^p \geq (TT^*)^p$. Especially, when $p = \frac{1}{2}$, T is called semi-hyponormal. Throughout this paper, let $0 < p \leq \frac{1}{2}$. It is well known that a p-hyponormal operator is q-hyponormal for $q \leq p$ by Lõwner's Theorem. An operator $T \in B(\mathcal{H})$ is said to be quasi-hyponormal if $T^{*2}T^2 \geq (T^*T)^2$. An operator $T \in B(\mathcal{H})$ is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ for all unit vectors $x \in \mathcal{H}$. For an operator T, we denote the spectrum and the approximate point spectrum by $\sigma(T)$ and $\sigma_a(T)$, respectively. A point $z \in \mathbf{C}$ in the joint approximate point spectrum $\sigma_{ja}(T)$ if there exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $(T-z)x_n \to 0$ and $(T-z)^*x_n \to 0$. For an operator $T \in B(\mathcal{H})$, we denote the polar decomposition of T by $T = U|T|$.

We need the following results.

THEOREM A (Th.4 of [6]). *Let T be p-hyponormal. If $Tx = \lambda x$, then $T^*x = \overline{\lambda}x$.*

THEOREM B (Th.8 of [6]). *Let T be p-hyponormal. Then*

$$\sigma_a(T) = \sigma_{ja}(T).$$

Next, let $\mathcal{T}$ be the set of all strictly monotone increasing continuous non-negative functions on $\mathbf{R}^+ = [0, \infty)$. Let $\mathcal{T}_o = \{\varphi \in \mathcal{T} : \varphi(0) = 0\}$. For $\varphi \in \mathcal{T}_o$, the mapping $\tilde{\varphi}$ is defined by

$$\tilde{\varphi}(re^{i\theta}) = e^{i\theta}\varphi(r) \ \text{ and } \ \tilde{\varphi}(T) = U\varphi(|T|).$$

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Then we have the following

THEOREM C (Th.3 of [7]). *Let $T = U|T|$ be p-hyponormal and U be unitary. If $\varphi \in \mathcal{T}_o$ and $\tilde{\varphi}(T)$ is p-hyponormal, then*

$$\tilde{\varphi}(\sigma(T)) = \sigma(\tilde{\varphi}(T)).$$

2. Spectral Properties. First we study the following for $T \in B(\mathcal{H})$: *If* λ is an isolated point of $\sigma(T)$, does it follow that λ is an eigenvalue of T ?

J. G. Stampfli proved that above statement is true if an operator T is hyponormal (Th.2 of [13]). S. L. Campbell and B. C. Gupta proved that it also holds if an operator T is quasi-hyponormal (Cor.7 of [4]).

THEOREM 1. *Let $T = U|T|$ be p-hyponormal and λ be an isolated point of $\sigma(T)$. If U is unitary or $\lambda \neq 0$, then λ is an eigenvalue of T.*

Proof. First we assume that U is unitary. Let $S = U|T|^p$ and $\varphi(t) = t^{\frac{1}{p}}$ for $t \geq 0$. Then S is a hyponormal operator and $\varphi \in \mathcal{T}_o$. Let $\lambda = re^{i\theta}$. Since $\tilde{\varphi}(S) = T$, by Theorem C, $r^p e^{i\theta}$ is an isolated point of σ(S). Hence by Stampfli's result it follows that $r^p e^{i\theta}$ is an eigenvalue of S. Hence there exists a nonzero eigen-vector x of λ. When $\lambda = 0$, $|T|^p x = 0$ because U is unitary. Hence 0 is an eigenvalue of T.
So we assume $\lambda \neq 0$. Then, by Theorem A, it holds that $Ux = e^{i\theta}x$ and $|T|^p x = r^p x$. Therefore this vector x is an eigen-vector of the eigenvalue λ of T.
Next, we assume that U is not unitary and $\lambda \neq 0$. Since we may assume that U is isometry, we put operators V and A on $\mathcal{H} \oplus \mathcal{H}$ as follows:

$$V = \begin{pmatrix} U & I - UU^* \\ 0 & U^* \end{pmatrix} \quad \text{and} \quad |A| = \begin{pmatrix} |T| & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $A = V|A|$, then we have $\sigma(T) \bigcup \{0\} = \sigma(A)$. Hence, $\lambda \in \sigma(A)$ and λ is an isolated point of $\sigma(A)$. Since A is p-hyponormal and V is unitary, from the above result it follows that there is a non-zero vector $x_1 \oplus x_2$ such that $A(x_1 \oplus x_2) = \lambda(x_1 \oplus x_2)$. Since $\lambda \neq 0$, it follows $x_2 = 0$. Hence $x_1 \neq 0$. Therefore, λ is an eigenvalue of T.

Next, for an operator $T \in B(\mathcal{H})$, let $C^*(T)$ be the C^*-algebra generated by T and the identity I.

THEOREM 2. *Let T be p-hyponormal. Then $\lambda \in \sigma_a(T)$ if and only if there exists a *-homomorphism $\phi : C^*(T) \to \mathbf{C}$ such that $\phi(T) = \lambda$.*

Proof. M. Enomoto, M. Fujii and K. Tamaki proved that the following conditions are equivalent(Th.1 of [8]):
(1) $\lambda \in \sigma_{ja}(T)$.

(2) There is a *-homomorphism ϕ on $C^*(T)$ such that $\phi(T) = \lambda$.
Hence this theorem follows from Theorem B.

3. Example.

It is well-known that the inclusive relations of these classes of non-normal operators are as follows :

$$\begin{array}{ccccc} & & p-hyponormal & & (*) \\ & \nearrow & & \searrow & \\ hyponormal & & & & paranormal, \\ & \searrow & & \nearrow & \\ & & quasi-hyponormal & & \end{array}$$

(cf. [2], [9], [10], [12]). Recently, M. Fujii, R. Nakamoto and H. Watanabe in [10] gave a nice generalization of (*). The inclusive relation of the p-hyponormality and the quasi-hyponormality is not known. In this section, we give counter-examples for the inclusive relation of these classes. The idea of operators below is due to P. R. Halmos [11] (Problem 164). Let V be a two-dimensional complex vector space ($V = \mathbf{C}^2$) and let $\mathcal{H}$ be the direct sum of countably many copies of V. Explicitly, $\mathcal{H}$ is the set of all sequences

$$x = < \cdots, x_{-1}, x_0, x_1, \cdots >$$

of vectors in V such that $\sum_n \|x_n\|^2 < \infty$; the inner product of x and y is defined by $(x, y) = \sum_n (x_n, y_n)$. Let $\{P_n\ ;\ n = 0, \pm 1, \pm 2, \cdots\}$ be a sequence of positive operators on V such that the sequence $\{\|P_n\|\}$ of norms is bounded, then the equations $(Px)_n := P_n x_n$ define an operator P on $\mathcal{H}$. If U is the shift defined by $(Ux)_n := x_{n-1}$, then U is an operator on $\mathcal{H}$.

If $A = UP$, then

$$(A^*Ax)_n = P_n^2 x_n,\ (AA^*x)_n = P_{n-1}^2 x_{n-1},$$

$$(A^{*2}A^2x)_n = P_n P_{n+1}^2 P_n x_n \quad \text{and} \quad ((A^*A)^2x)_n = P_n^4 x_n.$$

Example 1. Let positive 2×2-matrices C and D be

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let $\{P_n\}$ be a sequence of positive 2×2-matrices defined with

$$P_n = \begin{cases} C & (n \le 0) \\ D & (n \ge 1). \end{cases}$$

And an operator T on $\mathcal{H}$ is defined by

$$T = UP.$$

Then $\left((T^*T)^{\frac{1}{2}}x\right)_n = P_n x_n$ and $\left((TT^*)^{\frac{1}{2}}x\right)_n = P_{n-1}x_n$.

Since $D \geq C$, T is a semi-hyponormal operator. But since

$$(T^{*2}T^2x)_n = P_nP_{n+1}^2P_nx_n \text{ and } \left((T^*T)^2x\right)_n = P_n^4x_n,$$

if $n = 0$, then

$$(T^{*2}T^2x)_0 = CD^2Cx_0 = \begin{pmatrix} 40 & 10 \\ 10 & 5 \end{pmatrix} x_0 \text{ and } \left((T^*T)^2x\right)_0 = C^4x_0 = \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix} x_0.$$

Since

$$\begin{pmatrix} 40 & 10 \\ 10 & 5 \end{pmatrix} - \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix} \not\geq 0,$$

T is not a quasi-hyponormal operator.

Automatically, this example is a semi-hyponormal operator which is not hyponormal. Using singular integral operator techniques, D. Xia gave such an operator (Cor.1.4 of [14] of p.54).

Example 2. Next, we will give a quasi-hyponormal operator which is not p-hyponormal for every p. This example is due to S. L. Campbell and B. C. Gupta (Ex.1 of [4]). For the completeness, we will show it.
Let positive 2×2-matrices E and F be

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}.$$

Let $\{P_n\}$ be a sequence of positive 2×2-matrices defined with

$$P_n = \begin{cases} E & (n \leq 0) \\ F & (n \geq 1). \end{cases}$$

An operator S on $\mathcal{H}$ is defined by $S = UP$. Then

if $n \geq 1$, then $(S^{*2}S^2x)_n = ((S^*S)^2x)_n = F^4x_n,$

if $n \leq -1$, then $(S^{*2}S^2x)_n = ((S^*S)^2x)_n = E^4x_n$,

if $n = 0$, then

$$(S^{*2}S^2x)_0 = EF^2Ex_0 = \begin{pmatrix} 20 & 0 \\ 0 & 0 \end{pmatrix} x_0 \text{ and } \left((S^*S)^2x\right)_0 = E^4x_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_0.$$

Hence $S^{*2}S^2 \geq (S^*S)^2$. Therefore, S is a quasi-hyponormal operator.
But let $x := < x_n >_{n=-\infty}^{\infty}$ where $x_n = 0$ if $n \neq 1$ and $x_1 = (-2, 1)$. Then $Sx = 0$, but $S^*x \neq 0$. Hence, by Theorem A, S is not p-hyponormal for every p.

Addendum. Theorem 1 holds for any isolated points of $\sigma(T)$ of any p-hyponormal operator T. It is Theorem 1 of the paper "Weyl's theorem for p-hyponormal operators." by M. Chō, S. Ōshiro and H. Segawa.

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