On the Supports of Linearly Closed Convex Sets

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ABSTRACT. A condition for the supports of linearly closed convex sets to be closed is investigated.

1. Introduction

Let A be a convex subset of a real topological vector space. The frame of A is defined as the set

$$A_f = \{x \in A: \text{there exists } y \in A \text{ such that } y + t(x-y) \in A \text{ implies } t \leqq 1\},$$

and we denote $A^i = A \setminus A_f$. There exists a bounded closed convex set A with $A^i = \emptyset$. For example, in the space $C[0, 1]$ with supnorm, let A be the set of all points f in $C[0, 1]$ such that $f(0)=0$, $f(1)=1$ and $0 \leqq f(x) \leqq 1 (x \in [0, 1])$ then A is a bounded closed convex set with $A^i = \emptyset$. A convex set A is called to be linearly closed if for any two points x and $y (x \neq y)$ of A, the interesction of A and the line through x and y has two extreme points. A set consists simply of one point is also called to be linearly closed. Bounded closed convex subsets of a Hausdorff topological vector space are linearly closed, but the converse is not true. For example, in the space $C[0, 1]$ with supnorm, the set of all points f in $C[0, 1]$ such that $f(x)=0$ on a neighbourhood of 0, $f(x)=1$ on a neighbourhood of 1 and $0 \leqq f(x) \leqq 1 (x \in [0, 1])$ is not closed but linearly closed.

A support S of A is a nonempty convex subset of A which satisfies the condition that if an interior point of a line segment $[x, y]$ in A belongs to S, then $[x, y] \subset S$. A itself is a support of A. A support of A which is not equal to A is called a non-trivial support of A. No point of A^i is contained in the non-trivial support of A.

2. Statement of theorem

LEMMA 1. *The frame A_f of a convex set A has the following property: if an interior point of a line segment $[x, y]$ in A belongs to A_f, then $[x, y] \subset A_f$.*

PROOF. It is sufficient to prove that if $x \in A^i$, then for each y in A, $[x, y)$ is contained in A^i. Let x be an element of A^i. It is easy to see that for any two points z and w of A, there exists $s>0$ such that $|t|<s$ implies that $x+t(z-w)$ belongs to A. For each y in A, let $w=\lambda x+(1-\lambda)y$ $(0 \leqq \lambda \leqq 1)$. Since

$$\lambda x+(1-\lambda)y+\lambda t(z-(\lambda x+(1-\lambda)y))$$
$$=\lambda(x+t(z-(\lambda x+(1-\lambda)y)))+(1-\lambda)y\in A,$$

it follows that if $0<\lambda\leqq 1$, then $\lambda x+(1-\lambda)y\in A^i$. Therefore we obtain that $[x, y)\subset A^i$.

LEMMA 2. *Each support of a linearly closed convex set is a linearly closed set.*

PROOF. Let A be a linearly closed convex set and S be a support of A. If S consists simply of one point, then it is clear. The intersection of A and the line L joining two points x and y of S has the extreme points x' and y'. Since S is a support of A, the two points x' and y' belong to S. Therefore we obtain that $L\cap A=L\cap S=[x', y']$, hence S is linearly closed.

Every linearly closed convex set A is the convex hull of the set A_f. For each x in A_f, let F_x be the set of all points of A_f such that x can be expressed as the finite convex combination, with non-zero coefficients, of its elements. From the next Lemma, we can see that the set F_x is a face (facette, Bourbaki [2]) of A and that any face of A can be represented as F_x using some $x\in A_f$.

LEMMA 3. *Let $x\in A_f$. For $y(y\neq x)$ in A, $y\in F_x$ if and only if x is an interior point of the line segment which is the intersection of A and the line joining x and y.*

PROOF. If $y\in F_x(y\neq x)$, then x can be expressed as

$$x=\sum_{i=1}^{n-1}\lambda_i x_i+\lambda_n y\ (x_i\in F_x,\ \lambda_i>0\ (i=1, 2 \cdots, n), \sum_{i=1}^{n}\lambda_i=1).$$

For sufficiently small $\varepsilon>0$, $(1+\varepsilon)\lambda_n-\varepsilon>0$. Therefore for $\alpha=1+\varepsilon$, we obtain that

$$\alpha x+(1-\alpha)y=(1+\varepsilon)\sum_{i=1}^{n-1}\lambda_i x_i+((1+\varepsilon)\lambda_n-\varepsilon)y\in A.$$

Conversely, from Lemma 1, y which satisfies the condition belongs to A_f. Hence, by the definition of F_x, it is clear that $y\in F_x$.

Each face F_x of a convex set A has the following properties (Bourbaki [2]):

(1) F_x is a support of A;

(2) For each y in F_x, F_y is a face of F_x;

(3) Suppose that F_x contains more than one point, then $y\in(F_x)^i$ if and only if $F_y=F_x$.

LEMMA 4. *Let A be a linearly closed convex set which contains more than one point. Then for each x in A_f, F_x is a non-trivial support of A.*

PROOF. The set F_x is a support of A (property (1)). If F_x consists of one point, then it is clear. If F_x has more than one point, then from Lemma 1 and 3, it follows that $F_x\subset A_f$ and $x\in(F_x)^i$. Hence if $A^i\neq\emptyset$, then F_x is not equal to A, and if $A^i=\emptyset$, then F_x is not equal to $A_f=A$.

LEMMA 5. *Let A be a linearly closed convex set with $A^i \neq \emptyset$ and let $\mathscr{S}$ be a nonempty family of non-trivial supports of A which is totally ordered under inclusion. Let $S = \cup \{S_\alpha : S_\alpha \in \mathscr{S}\}$. Then,*

(1) S is *a non-trivial support of A,*

(2) *Suppose that the members of $\mathscr{S}$ are faces of A, and that S contains more than one point, then $S \notin \mathscr{S}$ if and only if $S = S_f$.*

PROOF. (1) For any two points x and y in S, there exist S_α and S_β in $\mathscr{S}$ such that $x \in S_\alpha$, $y \in S_\beta$. We may assume that $S_\alpha \subset S_\beta$. Then we obtain that $[x, y] \subset S_\beta \subset S$, hence S is convex. If an interior point a_0 of a line segment $[a, b]$ in A belongs to S, then there exists $S_\gamma \in \mathscr{S}$ such that $a_0 \in S_\gamma$, hence $[a, b] \subset S_\gamma \subset S$. Since $S \subset A_f$, S is a non-trivial support of A.

(2) If $S \notin \mathscr{S}$, then it is clear that $S^i = \emptyset$. Conversely if $S = S_f$, then for each F_z in $\mathscr{S}$, there exists $y \in S$ $(y \neq z)$ and $F_{z'} \in \mathscr{S}$ such that $y \in F_{z'}$ and

$$\alpha z + (1-\alpha) y \in S \Rightarrow \alpha \leqq 1.$$

If $F_{z'} \subset F_z$, then $y \in F_z$, hence, by Lemma 3, it is contradiction. Therefore we obtain that $F_z \subsetneqq F_{z'}$, and $\mathscr{S}$ dose not contain the supremum. This completes the proof.

Lemma 5 implies, by Zorn's lemma, that for each x in the frame of a linearly closed convex set A, with $A^i \neq \emptyset$, there exists a maximal non-trivial support S of A which contains x. Moreover if $S^i = \emptyset$, then S is not the union of infinitely countable faces containing x such that $F_{z_1} \subset F_{z_2} \subset \cdots$.

THEOREM 6. *Let A be a linearly closed convex set with $A^i \neq \emptyset$ and suppose that A_f is nonempty closed. Let S be a maximal non-trivial support of A. If $S^i \neq \emptyset$, then S is closed.*

PROOF. Suppose that there exists a point y in $\bar{S} \setminus S$, where $\bar{S}$ is the closure of S, then, since $\bar{S}$ is contained by A_f, y belongs to A_f. For each x in S^i, x is an extreme point of the intersection of A and the line through x and y, and no interior point of $[x, y]$ belongs to S. For z in (x, y), we obtain $x \in (F_z)_f$. It follows that $F_x \subset F_z$ by the property (2) of the faces, and since $S = F_x$, we have $S = F_z$ by the maximality of S. Then we obtain that $x \in (F_z)^i$, and this contradiction establishes the desired result.

References

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