Nihonkai Math. J.
Vol.8 (1997), 171-177

WEAK TYPE INEQUALITY FOR POISSON MAXIMAL OPERATORS

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ABSTRACT. A necessary and sufficient condition for a certain maximal operator to be of weak type (p, q), $1 \leq p \leq q < \infty$, is studied. This operator unifies various results about the Poisson integral operators cited in the literatures.

I. Introduction Consider the maximal operator

$$\mathcal{M}f(x,t) = \sup\{\frac{1}{|Q|}\int_Q |f(y)|\,dy : x \in Q \text{ and sidelength}(Q) \geq t\}.$$

It is well known that this maximal operator $\mathcal{M}$ controls Poisson integral defined by, for $x \in \mathbf{R}^n, t \geq 0$,

$$P(f)(x,t) = \int_{\mathbf{R}^n} f(y)P(x-y,t)dy,$$

where

$$P(x,t) = \frac{c_n t}{(|x|^2+t^2)^{\frac{n+1}{2}}}$$

is the Poisson kernel.

For a given positive measure ν on $\overline{\mathbf{R}^{n+1}_+} = \{(x,t) : x \in \mathbf{R}^n, t \geq 0\}$, the problem under what conditions $\mathcal{M}$ is bounded from $L^p(\mathbf{R}^n)$ into $L^p(\overline{\mathbf{R}^{n+1}_+}, \nu)$ and from $L^1(\mathbf{R}^n)$ into weak-$L^1(\overline{\mathbf{R}^{n+1}_+}, \nu)$ was studied by several authors: Carleson[C] showed that $\mathcal{M}$ is bounded from $L^p(\mathbf{R}^n, dx)$ into $L^p(\overline{\mathbf{R}^{n+1}_+}, d\nu)$ if and only if ν satisfies the Carleson condition

$$\sup_{x\in Q}\frac{\nu(\widetilde{Q})}{|Q|} \leq C.$$

Later, Fefferman-Stein[FS] proved that $\mathcal{M}$ is bounded from $L^p(\mathbf{R}^n, w(x)dx)$ into $L^p(\overline{\mathbf{R}^{n+1}_+}, d\nu)$ if

$$\sup_{x\in Q}\frac{\nu(\widetilde{Q})}{|Q|} \leq Cw(x) \quad a.e. \quad x,$$

where $\widetilde{Q} = Q \times (0, l(Q)]$ if we denote $l(Q)$ the sidelength of Q. More recently, Ruiz[R] and Ruiz-Torrea[RT] unified various results concerning these problems.

1991 *Mathematics Subject Classification.* 42B25.
Key words and phrases. maximal operator, A_p-weights, spaces of homogeneous type.

On the other hand, Sueiro[Su] studied a certain maximal operator and applied to study the Poisson-Szegö integral. This operator is in fact a generalization of the standard Hary-Littlewood maximal operator and works on spaces of homogeneous spaces.

In this paper, a maximal operator $\mathcal{M}_\Omega$ will be defined on spaces of homogeneous type. This is a generalization of the operator $\mathcal{M}$ given above. Finally we characterize the condition for which $\mathcal{M}_\Omega$ is of weak type (p, q). This condition will unify various results obtained before.

II. Preliminaries

Definition 2.1. Let X be a topological space and let $d : X \times X \to [0, \infty)$ is a map satisfying

(i) $d(x, x) = 0$; $d(x, y) > 0$ if $x \neq y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq K[d(x, y) + d(y, z)]$, where K is some fixed constant.

Assume further that

(iv) the balls $B(x, r) = \{y \in X : d(x, y) < r\}$ form a basis of open neighborhoods of $x \in X$ and that μ is a Borel measure on X such that

(v) $0 < \mu(B(x, 2r) \leq A\mu(B(x, r)) < \infty$, where A is some fixed constant.

Then the triple (X, d, μ) is called a space of homogeneous type[CW, S].

Remark 2.1. Properties (iii) and (v) will be referred to as *the triangle inequality* and *the doubling property* respectively.

Note that the condition (v) is equivalent that for every $c > 0$, there exists a constant $A_c < \infty$ such that $\mu(B(x, cr)) \leq A_c\mu(B(x, r))$.

Definition 2.2. Assume for each $x \in X$ we are given a set $\Omega_x \subset X \times [0, \infty)$. Let Ω denote the family $\{\Omega_x : x \in X\}$. For each $t \geq 0$ set

$$\Omega_{(x,t)} = \Omega_x \cap (X \times [t, \infty))$$

and

$$\mathcal{R}_\alpha(x, t) = \{(y, r) \in X \times [0, \infty) : \Omega_{(y,r)}(t) \cap B(x, \alpha t) \neq \emptyset\},$$

where $\Omega_{(y,r)}(t) = \{z \in X : (z, t) \in \Omega_{(y,r)}\}$ is the cross-section of $\Omega_{(y,r)}$ at height t.

Definition 2.3. Assume that we have a family $\Omega = \{\Omega_x : x \in X\}$. For $f \in L^1_{loc}(X, d\mu)$ and $x \in X$, $t \geq 0$ set

$$\mathcal{M}_\Omega f(x, t) = \sup_{(y,s)\in\Omega_{(x,t)}} \frac{1}{\mu(B(y, s))} \int_{B(y,s)} |f| d\mu.$$

Definition 2.4. Let $1 \leq p, q < \infty$. An operator T defined in $L^p(wd\mu)$ and having a ν-measurable function as its range is said to be of weak type (p, q) with respect to (ν, w) if there is a constant $A(p, q)$ so that

$$\nu\{(x, t) : |T(f)(x, t)| > \lambda\} \leq A(p, q) \left(\frac{\|f\|_{L^p(wd\mu)}}{\lambda}\right)^q$$

for all $\lambda > 0$.

Definition 2.5. Let $1 \le p \le q < \infty$. A pair (ν, w) is said to satisfy the condition $C_{p,q}(\Omega)$ if there are constant $C = C(K, A, \alpha, p, q)$ and $\alpha > 0$ such that

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))^q}\left(\int_{B(x,r)} w^{-\frac{1}{p-1}}\, d\mu\right)^{q(p-1)/p} \le C(K, A, \alpha, p, q),$$

if $p > 1$ and

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))^q} \le C(K, A, \alpha, p, q) w(y)^q$$

a.e. $y \in B(x,r)$ if $p = 1$.

Example 2.1. If $\Omega_x = \{(y,t) \in \mathbf{R}^n \times [0,\infty) : |x-y| < t\}$, then Ω induces the standard maximal operator $\mathcal{M}$ given in the introduction. Note that $\mathcal{R}_\alpha(x,r)$ is a truncated cone having a base $B(x,(1+\alpha)r)$, a top $B(x,\alpha r)$, and a height r.

Throughout the article, there are several constants which are not necessarily the same at each occurrence. These constants depend only on $K, A, \alpha, p,$ and q.

III. Main Results

The following lemma is given in [CW]. Also see [Su].

Lemma 3.1. *Let E be a bounded subset of X and for each $x \in X$, assign $r(x) > 0$. Then there is a sequence of disjoint balls $B(x_i, r(x_i))$, $x_i \in E$, such that the balls $B(x_i, 4Kr(x_i))$ cover E, where K is the constant in the definition 2.1. Further, every $x \in E$ is contained in some ball $B(x_i, 4Kr(x_i))$ satisfying $r(x) \le 2r(x_i)$.*

Theorem 3.1. *Assume that Ω satisfies that if $x \in X$, $(y,r) \in \Omega_x$ and $s \ge r$, then $(y,s) \in \Omega_x$. Let $1 \le p \le q < \infty$. Then $\mathcal{M}_\Omega$ is of weak type (p,q) with respect to (ν, w) if and only if (ν, w) satifies the condition $C_{p,q}(\Omega)$.*

Proof. Suppose that $\mathcal{M}_\Omega$ is of weak type (p,q) with respect to (ν, w). If $(x_0, t) \in \mathcal{R}_\alpha(x,r)$, then $\Omega_{(x_0,t)}(r) \cap B(x,\alpha r) \ne \phi$ and so we can choose $y \in \Omega_{(x_0,t)}(r) \cap B(x,\alpha r)$. From the triangle inequality,

$$B(x,r) \subset B(y, K(\alpha+1)r) \subset B(x, (K^2\alpha + K\alpha + K^2)r). \tag{1}$$

For a nonnegative measurable function f defined on X, we put

$$f_{B(y,r)} = \frac{1}{\mu(B(y,r))}\int_{B(y,r)} f\, d\mu$$

for simplicity. Since $(y, K(\alpha+1)r) \in \Omega_{(x_0,t)}$ by the hypothesis, it follows from (1) and the doubling property that

$$\begin{aligned} \mathcal{M}_\Omega f\chi_{B(x,r)}(x_o,t) &\ge \frac{1}{\mu(B(y,K(\alpha+1)r))}\int_{B(y,K(\alpha+1)r)} f \\ &\ge \frac{1}{\mu(B(x,(K^2\alpha+K\alpha+K^2)r))}\int_{B(x,r)} f \\ &\ge \frac{\mu(B(x,r))}{\mu(B(x,(K^2\alpha+K\alpha+K^2)r))} f_{B(x,r)} \\ &> C(K,A,\alpha) f_{B(x,r)} \end{aligned} \tag{2}$$

for some constant $C(K, A, \alpha)$.

Let λ be chosen so that $0 < \lambda < f_{B(x,r)}$. If we write

$$E_\lambda = \{\mathcal{M}_\Omega(f \cdot \chi_{B(x,r)}) > C(K, A, \alpha)\lambda\},$$

then the previous argument shows that $\mathcal{R}_\alpha(x, r) \subset E_\lambda$ and so

$$\nu(\mathcal{R}_\alpha(x,r)) \leq \frac{C(K, A, \alpha, p, q)}{\lambda^q} \left(\int_{B(x,r)} f^p w \, d\mu \right)^{\frac{q}{p}}. \tag{3}$$

Hence

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))^q} \left(\int_{B(x,r)} f d\mu \right)^q \leq C(A, K, \alpha, p, q) \left(\int_{B(x,r)} f^p w d\mu \right)^{\frac{q}{p}}. \tag{4}$$

Suppose $p > 1$ and $p' = p/(1-p)$. If we replace f by $w^{-\frac{1}{p-1}} \chi_{B(x,r)}$ so that $f = f^p w$ on $B(x, r)$, then (4) implies

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))^q} \left(\int_{B(x,r)} w^{-\frac{1}{p-1}} \, d\mu \right)^{q/p'} \leq C(K, A, \alpha, p, q). \tag{5}$$

Thus (ν, w) satisfies the condition $C_{p,q}(\Omega)$ for the case $p > 1$ and $1 \leq q < \infty$.

Suppose $p = 1$. By (4), we have

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))^q} \leq C \left(\frac{1}{\mu(S)} \int_S w d\mu \right)^q, \tag{6}$$

for any $S \subset B(x, r)$. Pick a so that $a > \text{ess.inf}_{y \in B(x,r)} w(y)$ and let $S_a = B(x, r) \cap \{w < a\}$. Replace S in (6) by S_a. Then by (6) we obtain

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))^q} \leq C a^p$$

and so

$$\frac{\nu(\mathcal{R}_\alpha(x,r))}{\mu(B(x,r))^q} \leq C w(y)^q \tag{7}$$

a.e. $y \in B(x, r)$.

Conversely, suppose (ν, w) satisfies the condition $C_{p,q}(\Omega)$. We follow the idea of Sueiro[Su]. For each $\lambda > 0$, define

$$E_\lambda = \{(x,t) \in X \times [0, \infty) : \mathcal{M}_\Omega f(x,t) > \lambda\}$$

and

$$E'_\lambda = \{x \in X : \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu > \lambda\}$$

Also for each $x \in E'_\lambda$, if we put

$$r(x) = \sup\{r > 0 : \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| d\mu > \lambda\},$$

then $r(x) > 0$ and

$$\frac{1}{\mu(B(x,r(x)))} \int_{B(x,r(x))} |f| \, d\mu \ \geq \ \lambda.$$

Assume for a moment that E'_λ is bounded. Then by the covering lemma, there exists a sequence of balls $\{B(x_i, r(x_i))\}$ so that $E'_\lambda \subset \cup_i B(x_i, 4Kr_i)$, where $r_i = r(x_i)$. Now we want to verify

$$E_\lambda \subset \cup_i \mathcal{R}_\alpha(x_i, 4Kr_i/\alpha) \tag{8}$$

To do this, let $(x,t) \in E_\lambda$. Then

$$\frac{1}{\mu(B(y,r))} \int_{B(y,r)} |f| \, d\mu > \lambda$$

for some $(y,r) \in \Omega_{(x,t)}$. So $y \in E'_\lambda$ and $t \leq r \leq r(y)$. By the last part of the covering lemma, $y \in B(x_i, 4Kr_i)$ for some i such that $r(y) \leq 2r_i$. Here we may assume $\alpha < 2K$. Consequently, $t \leq r \leq r(y) \leq 2r_i < \frac{4K}{\alpha} r_i$ and so $(y, 4Kr_i/\alpha) \in \Omega_{(x,t)}$. Since $y \in B(x_i, \alpha(4K/\alpha)r_i)$, it follows that $y \in \Omega_{(x,t)}(4Kr_i/\alpha) \cap B(x_i, \alpha(4K/\alpha)r_i)$, and thus $(x,t) \in \mathcal{R}_\alpha(x_i, 4Kr_i/\alpha)$, and so (8) holds.

Now suppose $1 < p \leq q < \infty$. Since

$$\begin{aligned}
\mu(B(x_i, r_i)) &\leq \frac{1}{\lambda} \int_{B(x_i,r_i)} |f| d\mu \\
&\leq \frac{1}{\lambda} \left(\int_{B(x_i,r_i)} |f|^p w d\mu \right)^{\frac{1}{p}} \left(\int_{B(x_i,r_i)} w^{-\frac{1}{p-1}} \, d\mu \right)^{\frac{p-1}{p}},
\end{aligned}$$

by the Hölder's inequality, it follows from the disjointness of $\{B(x_i, r_i)\}$ that

$$\begin{aligned}
\nu(E_\lambda) &\leq \sum_i \nu(\mathcal{R}_\alpha(x_i, 4Kr_i/\alpha)) \\
&\leq C(K, A, \alpha, p, q) \sum_i \mu(B(x_i, 4Kr_i/\alpha))^q \left(\int_{B(x_i, 4Kr_i/\alpha)} w^{-\frac{1}{p-1}} \, d\mu \right)^{-\frac{q}{p'}} \\
&\leq C(K, A, \alpha, p, q) \sum_i \mu(B(x_i, r_i))^q \left(\int_{B(x_i,r_i)} w^{-\frac{1}{p-1}} \, d\mu \right)^{-\frac{q}{p'}} \\
&\leq \frac{C(K, A, \alpha, p, q)}{\lambda^q} \sum_i \left(\int_{B(x_i,r_i)} |f|^p w \, d\mu \right)^{\frac{q}{p}} \left(\int_{B(x_i,r_i)} w^{-\frac{1}{p-1}} \, d\mu \right)^{\frac{q(p-1)}{p} - \frac{q}{p'}} \\
&= \frac{C(K, A, \alpha, p, q)}{\lambda^q} \sum_i \left(\int_{B(x_i,r_i)} |f|^p w \, d\mu \right)^{\frac{q}{p}} \\
&\leq \frac{C(K, A, \alpha, p, q)}{\lambda^q} \|f\|^q_{L^p(w)}.
\end{aligned}$$

Next suppose $p=1$ and $1 \le q < \infty$. Since $4K/\alpha > 1$, we have

$$\begin{aligned}
\nu(E_\lambda) &\le \sum_i \nu(\mathcal{R}_\alpha(x_i, 4Kr_i/\alpha)) \\
&\le C\sum_i [\mu(B(x_i, 4Kr_i/\alpha))]^q ess.inf_{y\in B(x_i,4Kr_i/\alpha)} w(y)^q \\
&\le C\sum_i [\mu(B(x_i, r_i))]^q ess.inf_{y\in B(x_i,r_i)} w(y)^q \\
&\le C\sum_i \left(\frac{1}{\lambda}\int_{B(x_i,r_i)} |f|w d\mu\right)^q .
\end{aligned}$$

Hence $\mathcal{M}_\Omega$ is of weak type $(1,q)$ with respect to (ν, w).

Finally if E'_λ is not bounded, then fix $a \in X$ and $r > 0$. If we consider

$$E''_\lambda = \{(x,t) : \mathcal{M}_\Omega f(x,t) > \lambda \text{ and } y \in E'_\lambda \cap B(a,r) \text{ for some } y \in \Omega_{(x,t)}(r)\}$$

and letting $r \to \infty$, then we obtain the same estimate. This completes the proof. □

Remark 3.1. If $p=q$, then the condition $C_{p,p}(\Omega)$ reduces to the condition $C_p(w)$ given by Ruiz[R]:

$$\sup_Q \frac{\nu(\widetilde{Q})}{|Q|}\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \le C$$

if $p > 1$, and

$$\sup_{x\in Q} \frac{\nu(\widetilde{Q})}{|Q|} \le Cw(x) \text{ a.e.},$$

if $p=1$, where the supremum is taken over all cubes Q in $\mathbf{R}^n$.

Corollary. (Ruiz[R]) *Let $p \ge 1$. The maximal operator $\mathcal{M}$ is weak type (p,p) with respect to (ν, w) if and only if (ν, w) satisfies the condition $C_p(w)$.*

Remark 3.2. Let $d\nu = d\mu \otimes d\delta$, where $d\delta$ is the Dirac mass on $[0,\infty)$, concentrated on 0. Set

$$\mathcal{S}_\alpha(x,r) = \{y \in X : \Omega_y(r) \cap B(x, \alpha r) \ne \emptyset\}.$$

Then

$$\nu(\mathcal{R}_\alpha(x,r)) = \mu(\mathcal{S}_\alpha(x,r)).$$

The inequality (4), with $f \equiv 1$ and $w \equiv 1$, gives

$$\frac{\nu(\mathcal{S}_\alpha(x,r))}{\mu(B(x,r))} \le C,$$

which is obtained by Sueiro[Su].

Let u be a nonnegative measurable function on X. If $d\nu = ud\mu \otimes d\delta$ and $p = q \ge 1$, then the inequality (4) also gives the condition obtained by Wenjie[W], which generalizes the Muckenhoupt's A_p condition[M].

Definition 3.1. Set $\widehat{\Omega} = \{\widehat{\Omega}_{(x_o,t)} : (x_o,t) \in X \times [0,\infty)\}$, where

$$\widehat{\Omega}_{(x_o,t)} = \{(x,r) \in X \times [t,\infty) : (x,s) \in \Omega_{x_o} \text{for some } s \leq r\}$$

and

$$\widehat{\mathcal{R}}_\alpha(x,r) = \{(x_o,t) \in X \times [0,\infty) : \widehat{\Omega}_{(x_o,t)} \cap B(x,\alpha r) \neq \phi\}.$$

Following theorem 3.1 with this definition, we obtain

Theorem 3.2. *Let $1 \leq p \leq q < \infty$. Then $\mathcal{M}_{\widehat{\Omega}}$ is of weak type (p,q) with respect to (ν, w) if and only if (ν, w) satifies the condition $C_{p,q}(\widehat{\Omega})$.*

Acknowledgements. The author would like to thank the refree for pointing out the errors.

References

[C] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. Math. **76** (1962), 547-559.

[CW] R.R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certain espaces homogènes*, Lecture Notes in Math. **242** (1971), Springer-Verlag, Berlin.

[FS] C. Fefferman C. and E. M. Stein, *Some maximal inequalities*, Amer.J. Math. **93** (1971), 107-115.

[M] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 115-121.

[R] F.J. Ruiz, *A Unified approach to Carleson measures and A_p weights*, Pacific J.Math. **117** (1985), 397-404.

[RT] F. J. Ruiz and J.L. Torrea, *A Unified approach to Carleson measures and A_p weights II*, Pacific J.Math. **120** (1985), 189-197.

[GcRf] J. Garcia-Cuerva and J.L. Rubio De Francia, *Weighted Norm Inequalities and Related Topic*, Elsevier Science, North-Holland, 1985.

[S] E.M.Stein, *Harmonic Analysis: Real variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N.J., 1993.

[Su] J. Sueiro, *On maximal functions and Poisson-Szegö integrals*, Trans. Amer.Math.Soc. **298** (1986), 653-669.

[W] P. Wenjie, *Weghted Norm Inequaities for Certain Maximal Operator with Approach Region*, Lecture Notes in Math. **1494** (1988), Springer Verlag, Berlin, 169-175.

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Received March 28, 1997 **Revised July 8, 1997**