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OPERATOR INEQUALITIES RELATED TO CAUCHY-SCHWARZ AND HÖLDER-McCARTHY INEQUALITIES

Masatoshi Fujii*, Saichi Izumino**, Ritsuo Nakamoto*** and Yuki Seo****

Abstract. We give an improvement of the Cauchy-Schwarz inequality, which is based on the covariance-variance inequality. We also give a complementary inequality of the Hölder-McCarty inequality. Furthermore we extend it to the case of two variables using the operator mean in the Kubo-Ando theory. Consequently we have a noncommutative version of the Greub-Rheinboldt inequality as an extension of the Kantrovich one. Finally we discuss about order preserving properties of increasing functions through the Kantorovich inequality.

1. Introduction. In [1], we proved the covariance-variance inequality in the noncommutative probability theory established by Umegaki[12]:

$$|\mathrm{Cov}(A,B)|^2 \le \mathrm{Var}(A)\mathrm{Var}(B), \tag{1}$$

where $\mathrm{Cov}(A,B)$ and $\mathrm{Var}(A)$ are defined as

$$\mathrm{Cov}(A,B) = (B^*Ax,x) - (B^*x,x)(Ax,x) \text{ and } \mathrm{Var}(A) = \mathrm{Cov}(A,A)$$

for (bounded linear) operators A, B acting on a Hilbert space H and a fixed unit vector $x \in H$.

The covariance-variance inequality has many applications for operator inequalities, see [1,2,6]. Among others, we pointed out that (1) implies the celebrated Kantorovich inequality: If a positive operator A on a Hilbert space H satisfies $0 < m \le A \le M$, then for each unit vector $x \in H$

$$(Ax,x)(A^{-1}x,x) \le \frac{(m+M)^2}{4mM}, \tag{2}$$

or equivalently,

$$(A^2x,x) \le \frac{(m+M)^2}{4mM}(Ax,x)^2. \tag{3}$$

Since the covariance-variance inequality is equivalent to the Cauchy-Schwarz inequality, the Kantorovich inequality lies on the line of the Cauchy-Schwarz inequality. More precisely, it is considered as an estimation of the ratio of factors appearing in the Cauchy-Schwarz inequality. Another viewpoint is to estimate the difference of the factors. Actually it has been done in the numerical case. Its operator version will be given by the covariance-variance inequality in the below.

On the other hand, the Hölder-McCarthy inequality[3,8] is a generalization of the Cauchy-Schwarz inequality. Along with our argument, we attempt to generalize the Hölder-McCarthy

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inequality and give its complementary inequality, in which the geometric mean plays an essential role, see [7].

Finally we discuss the bridge between the Kantorovich inequality and the Löwner-Heinz inequality via the condition number with the origin by Turing.

2. Cauchy-Schwarz inequality. The covariance-variance inequality is equivalent to the Cauchy-Schwarz inequality[1]. Nevertheless we can discuss an improvement of the Cauchy-Schwarz inequality lying on the line of the covariance-variance inequality .

First of all, we remark that the covariance-variance inequality (1) has a nice relation with the Gram matrix as follows. For a unit vector x, the Gram matrix

$$\begin{pmatrix} (Ax, Ax) & (Ax, Bx) & (Ax, x) \\ (Bx, Ax) & (Bx, Bx) & (Bx, x) \\ (x, Ax) & (x, Bx) & (x, x) \end{pmatrix}$$

is positive definite and its determinant $G(Ax, Bx, x)$ is just the difference of the covariance-variance inequality:

$$G(Ax, Bx, x) = \mathrm{Var}(A)\mathrm{Var}(B) - |\mathrm{Cov}(A, B)|^2 \geq 0. \tag{4}$$

The covariance-variance inequality also appears in an improvement of Cauchy's inequality (see [9]): Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be real numbers and let

$$u = n^{-1/2} \sum a_i \text{ and } v = n^{-1/2} \sum b_i.$$

Then

$$\sum a_i^2 \sum b_i^2 - (\sum a_i b_i)^2 \geq u^2 \sum b_i^2 - 2uv \sum a_i b_i + v^2 \sum a_i^2.$$

An operator version of this inequality is seemed to be as follows: If A and B are commuting hermitian operators, then

$$\begin{aligned} (A^2x, x)(B^2x, x) - (ABx, x)^2 &\geq (A^2x, x)(Bx, x)^2 - 2(ABx, x)(Ax, x)(Bx, x) \\ &\quad + (B^2x, x)(Ax, x)^2 \geq 0 \end{aligned} \tag{5}$$

for all unit vectors x. However the assumption of the commutativity on A and B is not needed; as a matter of fact, we have the following operator version of Cauchy's inequality, in which we will be able to recognize the utility of the covariance-variance inequality:

Theorem 1. *Let A and B be positive. Then*

$$\begin{aligned} (A^2x, x)(B^2x, x) - |(ABx, x)|^2 &\geq (A^2x, x)(Bx, x)^2 - 2|(ABx, x)|(Ax, x)(Bx, x) \\ &\quad + (B^2x, x)(Ax, x)^2 \geq 0 \end{aligned} \tag{6}$$

for all unit vectors x.

Proof. By the covariance-variance inequality (1), we have

$$\begin{aligned} \{(A^2x, x) - (Ax, x)^2\}\{(B^2x, x) - (Bx, x)^2\} &\geq |(ABx, x) - (Ax, x)(Bx, x)|^2 \\ &\geq \{(Ax, x)(Bx, x) - |(ABx, x)|\}^2 \end{aligned}$$

It is easily checked that this inequality can be rephrased as the first inequality of (6). The positivity of the middle term is shown as follows:

$$\begin{aligned}&(A^2x,x)(Bx,x)^2 - 2|(ABx,x)|(Ax,x)(Bx,x) + (B^2x,x)(Ax,x)^2\\ &\quad= \left\{(A^2x,x)^{1/2}(Bx,x) - (B^2x,x)^{1/2}(Ax,x)\right\}^2\\ &\qquad+ 2\left\{(A^2x,x)^{1/2}(B^2x,x)^{1/2} - |(ABx,x)|\right\}(Ax,x)(Bx,x) \geq 0.\square\end{aligned}$$

3. Hölder-McCarthy inequality. In this section we show an operator version of Hölder's inequality and its complementary inequality. Moreover we generalize it using the geometric mean in the Kubo-Ando theory[7]. The geometric mean $A\#B$ is defined by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

for positive invertible operators A and B.

We need the following useful result, which gives Jensen's inequality and a complementary inequality of it with respect to the convex function $f(x) = x^p$ $(p > 1)$.

Lemma 2([9, p.694, (11.2)]). *Let $(a_1, \cdots, a_n)$ and $(w_1, \cdots, w_n)$ be n-tuples of nonnegative numbers such that $0 < m \leq a_k \leq M$ $(k = 1, \cdots, n)$ and $\sum w_k = 1$. Then, for $p \geq 1$*

$$\left(\sum w_k a_k\right)^p \leq \sum w_k a_k^p \leq \lambda(p; m, M)\left(\sum w_k a_k\right)^p, \tag{7}$$

where $\lambda(p; m, M) = \left\{\dfrac{1}{p^{1/p}q^{1/q}}\dfrac{M^p - m^p}{(M-m)^{1/p}(mM^p - Mm^p)^{1/q}}\right\}^p$ *and* $q = \dfrac{p}{p-1}$.

If A is a selfadjoint operator with $m \leq A \leq M$, then for a unit vector $x \in H$, there is a spectral measure μ_x on $[m, M]$ such that

$$(A^p x, x) = \int_m^M t^p d\mu_x. \tag{8}$$

Applying the inequality (7) to the approximate sum of the integral of (8), we have:

Theorem 3. *Let A be a selfadjoint operator with $m \leq A \leq M$ and $p > 1$. Then for a unit vector $x \in H$,*

$$(Ax, x)^p \leq (A^p x, x) \leq \lambda(p; m, M)(Ax, x)^p. \tag{9}$$

Here we note that the first inequality of (9) is due to McCarthy [8] and is called the Hölder-McCarthy inequality[3].

If we replace x by $x/\|x\|$ in (9), and taking the p-th root of each term, we obtain

$$(Ax, x) \leq (A^p x, x)^{1/p}\|x\|^{2/q} \leq \lambda(p; m, M)^{1/p}(Ax, x), \tag{10}$$

for every $x \in H$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Recall the s-power mean $A\#_s B$ ($s \in [0,1]$) in the Kubo-Ando theory;

$$A\#_s B = A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2}.$$

Consequently, we have the following noncommutative version of Theorem 3.

Theorem 4. *Let A and B be positive operators satisfying $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$. Then for $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and for $x \in H$,*

$$(11) \quad (B^q \#_{1/p} A^p x, x) \le (A^p x, x)^{1/p}(B^q x, x)^{1/q} \le \lambda(p; \frac{m_1}{M_2^{q-1}}, \frac{M_1}{m_2^{q-1}})^{1/p}(B^q \#_{1/p} A^p x, x)$$

and

$$(12) \quad (A^p \#_{1/q} B^q x, x) \le (A^p x, x)^{1/p}(B^q x, x)^{1/q} \le \lambda(q; \frac{m_2}{M_1^{p-1}}, \frac{M_2}{m_1^{p-1}})^{1/q}(A^p \#_{1/q} B^q x, x).$$

Proof. Replace A by $(B^{-q/2}A^pB^{-q/2})^{1/p}$ and x by $B^{q/2}x$ in (10). Then we have

(13)
$$(B^{q/2}(B^{-q/2}A^pB^{-q/2})^{1/p}B^{q/2}x, x) \le (A^p x, x)^{1/p}(B^q x, x)^{1/q}$$
$$\le \lambda(p; \frac{m_1}{M_2^{q-1}}, \frac{M_1}{m_2^{q-1}})^{1/p}(B^{q/2}(B^{-q/2}A^pB^{-q/2})^{1/p}B^{q/2}x, x).$$

Since

$$\frac{m_1^p}{M_2^q} \le m_1^p B^{-q} \le B^{-q/2}A^pB^{-q/2} \le M_1^p B^{-q} \le \frac{M_1^p}{m_2^q},$$

we have $\frac{m_1}{M_2^{q-1}} \le \left(B^{-q/2}A^pB^{-q/2}\right)^{1/p} \le \frac{M_1}{m_2^{q-1}}$. Hence (11) holds by noting that $B^q \#_{1/p} A^p = B^{q/2}(B^{-q/2}A^pB^{-q/2})^{1/p}B^{q/2}$. The latter (12) is proved similarly.□

Thus a noncommutative variant of the Greub-Rheinboldt inequality[4] is also obtained by putting $p = q = 2$ in particular.

Corollary 5. *Under the same assumption as in Theorem 4, the following holds:*

$$(14) \qquad (A^2 \# B^2 x, x) \le (A^2 x, x)^{1/2}(B^2 x, x)^{1/2} \le \frac{m_1 m_2 + M_1 M_2}{2\sqrt{m_1 m_2 M_1 M_2}}(A^2 \# B^2 x, x).$$

Moreover, if A and B is replaced by $A^{1/2}$ and $A^{-1/2}$ respectively in (14), then the Kantorovich inequality is obtained (cf. [10]):

$$(Ax, x)^{1/2}(A^{-1}x, x)^{1/2} \le \frac{m_1 + M_1}{2\sqrt{m_1 M_1}}.$$

4. Kantorovich inequality. The Kantorovich inequality is a complementary one of the Cauchy-Schwarz inequality and gives the bound of its ratio. Also it has many generalizations (see (1) and Theorem 4).

Now it is well known that t^s $(0 \le s \le 1)$ is an operator monotone function ([5]) and not so is t^2. However, by the Kantorovich inequality, we can say that t^2 is order preserving in the following sense.

Theorem 6. *Let $0 \le A \le B$ and $0 < m \le A \le M$. Then*

$$A^2 \le \frac{(m+M)^2}{4mM} B^2.$$

Proof. By the Kantorovich inequality (3), we have

$$(A^2x, x) \le \frac{(m+M)^2}{4mM}(Ax, x)^2 \le \frac{(m+M)^2}{4mM}(Bx, x)^2 \le \frac{(m+M)^2}{4mM}(B^2x, x)$$

for all unit vectors x.□

Similarly, if $0 < n \le B \le N$, we have, by Theorem 6,

$$B^{-2} \le \frac{(\frac{1}{n} + \frac{1}{N})^2}{4\frac{1}{n}\frac{1}{N}} A^{-2} = \frac{(n+N)^2}{4nN} A^{-2}.$$

So, as a variant of Theorem 6, we have

Theorem 6′. *Let $0 < A \le B$ and $0 < n \le B \le N$. Then*

$$A^2 \le \frac{(n+N)^2}{4nN} B^2.$$

Following after Turing[11], the condition number $\kappa(A)$ of an invertible operator A is defined by $\kappa(A) = \|A\| \, \|A^{-1}\|$. If a positive operator A satisfies the condition $0 < m \le A \le M$, then it may be thought as $M = \|A\|$ and $m = \|A^{-1}\|^{-1}$, so that $\kappa(A) = \frac{M}{m}$.

From the same viewpoint as Theorems 6 and 6′, we estimate the function t^p $(p \ge 1)$ using the condition number $\kappa(A) = \frac{M}{m}$.

Theorem 7. *Let $0 < A \le B$ and $0 < m \le A \le M$. Then*

$$A^p \le \left(\frac{M}{m}\right)^p B^p \quad (p \ge 1).$$

Proof. We have

$$A^{2p} = B^p B^{-p} A^{2p} B^{-p} B^p \le \|B^{-p} A^{2p} B^{-p}\| B^{2p} \le \|A\|^{2p} \, \|B^{-1}\|^{2p} B^{2p},$$

so that this implies $A^p \le \|A\|^p \|B^{-1}\|^p B^p \le M^p (\frac{1}{m})^p B^p = (\frac{M}{m})^p B^p$. □

Though the function e^t is not operator monotone, we have the following result as a consequence of Theorem 7:

Corollary 8. *Let $0 < A \le B$ and $0 < m \le A \le M$. Then*

$$e^A \le e^{\frac{M}{m} B}.$$

Proof. By Theorem 7, we have

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \leq \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{M}{m})^n B^n = e^{\frac{M}{m}B}. \square$$

Remark. Finally we remark that Theorem 7 is extended to every increasing function f as follows: If $0 < m \leq A \leq M$ and $A \leq B$ are satisfied, then we obtain

$$f(A) \leq f(\frac{M}{m} B)$$

and

$$f(A) \leq \frac{f(M)}{f(m)} f(B) \quad (f(m)f(M) > 0),$$

because $f(A) \leq f(M) \leq f(\frac{M}{m}B)$ and $f(A) \leq f(M) = \frac{f(M)}{f(m)} f(m) \leq \frac{f(M)}{f(m)} f(B)$.

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* Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582, Japan

** Faculty of Education, Toyama University, Gofuku, Toyama-shi 930, Japan

* * * Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316, Japan

* * ** Tennoji Branchi, Senior Highschool, Osaka Kyoiku University, Tennoji, Osaka 543, Japan

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