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On branching theorem of the pair $(G_2, SU(3))$

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Dedicated to Professor Hisao Nakagawa
on his sixtieth birthday

Let G be a compact connected Lie group and K be a closed subgroup. A finite dimensional complex irreducible representation $V^G(\lambda)$ of G with highest weight λ is decomposed into a direct sum of irreducible representations $V^K(\mu)$ of K with highest weight μ;

$$V^G(\lambda) = \sum_{\mu} m(\lambda, \mu) V^K(\mu).$$

It is an important problem to study the branching multiplicity $m(\lambda, \mu)$.

In [3], F. Sato studied the stability of branching coefficient. Roughly speaking, the branching coefficient $m(\lambda, \mu)$ satisfies $m(\lambda, \mu) = m(\lambda + \lambda_0, \mu)$ if λ_0 is a spherical representation of (G, K) and λ is sufficiently large.

In [2] the author studied the branching theorem of the pair $(G_2, SO(4))$ and obtained the following stability theorem (see section 2 for the description of the fundamental weights $\{\lambda_i\}$ of G_2).

Theorem 1 (Mashimo [2]) *Let $\lambda = m_1\lambda_1 + m_2\lambda_2$ be a dominant integral weight of G_2 and $\mu = \sum_{i=1}^{3} b_i\varepsilon_i$ be a dominant integral weight of $SO(4)$. Then*

(1) *if $m_1 \geq 2b_1 + b_2 + 4$ then $m(\lambda + 2\lambda_1, \mu) = m(\lambda, \mu)$,*

(2) *if $m_2 \geq b_1 + 1$ then $m(\lambda + 2\lambda_2, \mu) = m(\lambda, \mu)$.*

The aim of this note is to calculate the branching coefficients of the pair $(G_2, SU(3))$ and to prove the "stability" of branching coefficients.

1. Kostant's multiplicity formula. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. We assume that G and K are of the same rank. Let T be a maximal torus of K and $\mathfrak{t}$ be its Lie algebra. We denote by $\Sigma(G)$ the set of non-zero roots of $\mathfrak{g}^{\mathbf{C}}$ with respect $\mathfrak{t}^{\mathbf{C}}$ and $\Sigma^+(G)$ the set of all positive roots. We denote by $\mathcal{D}(G)$ the set of all equivalence classes of complex irreduible representations of G. Let $V^G(\lambda)$ be a representation space of an element λ of $\mathcal{D}(G)$.

We denote by $\mathfrak{k}$ the Lie algebra of K and by $\Sigma(K)$ the set of all non-zero roots of $\mathfrak{k}^{\mathbf{C}}$ with respect to $\mathfrak{t}^{\mathbf{C}}$. By our assumption $\Sigma(K)$ is contained in $\Sigma(G)$. We denote by $\Sigma^+(K)$ the set of positive roots of $\mathfrak{k}^{\mathbf{C}}$. A complex irreducible reprensentation $V^G(\lambda)$ of G is decomposed into irreducible K-modules;

$$V^G(\lambda) = \sum_{\mu \in \mathcal{D}(K)} m(\lambda, \mu) V^K(\mu).$$

Let $\gamma_1, \dots, \gamma_r \in \sqrt{-1}\mathfrak{t}$ be the set of elements of the set $\Sigma^+(G) \setminus \Sigma^+(K)$. For every $\nu \in \sqrt{-1}\mathfrak{t}$, we denote by $P(\nu)$ the number of non-negative integral r-tuples $(a_1, \dots, a_r)$ such that $\nu = \sum_{j=1}^r a_j \gamma_j$. The multiplicity $m(\lambda, \mu)$ of $V^K(\mu)$ in $V^G(\lambda)$ is expressed, by using the partition function P, as follows;

Theorem 2 (Kostant [1]) *The multiplicity $m(\lambda, \mu)$ is give by*

$$m(\lambda, \mu) = \sum_{\sigma \in W} (\det \sigma) P(\sigma(\lambda + \delta) - (\mu + \delta)),$$

where W is the Weyl group of G and δ is half the sum of positive roots of $\mathfrak{g}^{\mathbf{C}}$.

2. Root systems and Weyl groups of G_2. We denote by G_2 the compact simple Lie group of type $\mathfrak{g}_2$. We shall give a brief review on root systems $\Sigma(G_2)$.

Under a suitable choise of an orthonormal base $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathbf{R}^3$, the maximal abelian subalgebra of $\mathfrak{g}_2$ is $\sqrt{-1}\mathfrak{t} = \{\sum a_i \varepsilon_i : a_1 + a_2 + a_3 = 0\}$. The set of positive roots $\Sigma^+(G_2)$ of $\mathfrak{g}_2{}^{\mathbf{C}}$ with respect to $\mathfrak{t}^{\mathbf{C}}$ is

$$\Sigma^+(G_2) = \left\{ \begin{array}{l} \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_3, \\ 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - 2\varepsilon_2 + \varepsilon_3,\ \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3 \end{array} \right\}$$

and $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_1 - 2\varepsilon_2 + \varepsilon_3$ are simpe roots. A linear form $x = \sum_{i=1}^3 a_i \varepsilon_i$ is a dominant form if and only if $a_1 - a_2 \geq a_2 - a_3 \geq 0$ and is an integral form if and only if a_1, a_2, a_3 are integers. If $x = \sum_{i=1}^3 a_i\varepsilon_i$ is a dominant form, we have $a_1 - 2a_2 + a_3 = -3a_2 \geq 0$. The fundamental weights of G_2 are

$$\lambda_1 = \varepsilon_1 - \varepsilon_3,\ \lambda_2 = 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3.$$

We denote by S_α the reflection with respect to the hyperplanes perpendicular to α and put $S_1 = S_{\alpha_1}$, $S_2 = S_{\alpha_2}$;

$$S_1(\sum_{i=1}^3 a_i\varepsilon_i) = a_1\varepsilon_1 + a_3\varepsilon_2 + a_2\varepsilon_3,$$
$$S_2(\sum_{i=1}^3 a_i\varepsilon_i) = -a_3\varepsilon_1 - a_2\varepsilon_2 - a_1\varepsilon_3.$$

3. Branching theorem of the pair $(G_2, SU(3))$. The set of roots $\{\pm\alpha_2, \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\}$ generates a Lie subalgebra isomorphic to $\mathfrak{su}(3)$. The set of fundamental roots of $SU(3)$ is $\{3\alpha_1 + \alpha_2, \alpha_2\}$. The linear form $\sum_{i=1}^3 b_i\varepsilon_i$ is a domonant form for $\mathfrak{su}(3)$ if and only if $b_1 \geq 0 \geq \max(b_2, b_3)$ and is an integral form if and only if b_1, b_2, b_3 are integers.

Kostant's partition function for the pair $(G_2, SU(3))$ is given as follows;

Lemma 3 *For an integral weight $x = \sum_{i=1}^3 x_i\varepsilon_i$ of $\mathfrak{g}_2$ we have*

$$P(x) = \#\{k \in \mathbf{Z} : 0 \leq k \leq \min(x_1, x_1 + x_2)\}.$$

Proof. Put $\gamma_1 = \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_2$, $\gamma_2 = \alpha_1 = \varepsilon_2 - \varepsilon_3$, and $\gamma_3 = 2\alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$, which are elements of $\Sigma^+(G_2) \setminus \Sigma^+(SU(3))$

Since γ_1 and γ_2 are linearly independent and $\gamma_3 = \gamma_1 + \gamma_2$, the expressions of x as linear combinations of γ_i are $x = (x_1 - k)\gamma_1 + (x_1 + x_2 - k)\gamma_2 + k\gamma_3$ $(k \geq 0)$. Thus we obtain the lemma. Q.E.D.

Theorem 4 *Let $\lambda = \sum_{i=1}^3 a_i\varepsilon_i$ be a dominant integral weight of G_2 and $\mu = \sum_{i=1}^3 b_i\varepsilon_i$ be a dominant integral weight of $SU(3)$. Then the multiplicity $m(\lambda, \nu)$ is equal to*

$$\#\{k \in \mathbf{Z}_{\geq 0} : -a_2 - b_1 - b_2 - 1 < k \leq \min(a_1 - b_1, a_1 + a_2 - b_1 - b_2)\}$$
$$-\#\{k \in \mathbf{Z}_{\geq 0} : -a_2 - b_1 - 2 < k \leq \min(a_1 - b_1 - b_2 + 1, a_1 + a_2 - b_1 - 1)\}.$$

Proof. It is easily verified that for $\lambda \in \mathcal{D}(G_2)$, $\mu \in \mathcal{D}(SU(3))$, $P(\sigma(\lambda+\delta)-(\mu+\delta))=0$ if $\sigma \notin \{1, S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_1+\alpha_2}\circ S_{\alpha_2}\}$. Put $S_0=1$, $S_3=S_{\alpha_1+\alpha_2}\circ S_{\alpha_2}$ and $P_i=P(S_i(\lambda+\delta)-(\mu+\delta))\quad(0\le i\le 3)$. Denote by $n(a,b)$ the number of elements of $\{k\in\mathbf{Z}: 0\le k\le \min(a,b)\}$. We have

$$(1)\qquad \begin{cases} P_0 &= n(a_1-b_1, a_1+a_2-b_1-b_2),\\ P_1 &= n(a_1-b_1, -a_2-b_1-b_2-1),\\ P_2 &= n(a_1+a_2-b_1-1, a_1-b_1-b_2+1),\\ P_3 &= n(-a_2-b_1-2, a_1-b_1-b_2+1).\end{cases}$$

Put $\alpha=a_1-b_1$, $\beta=-a_2-b_1-b_2-1$ and $\gamma=a_1+a_2-b_1-b_2$. Since $\gamma>\beta$ we consider 3 cases (i) $\alpha\le\beta<\gamma$, (ii) $\beta<\alpha\le\gamma$ and (iii) $\beta<\gamma\le\alpha$. If $\alpha\le\beta$ then $P_0=P_1$. If $\beta<\alpha\le\gamma$ then $P_0-P_1=\{k\in\mathbf{Z}_{\ge0}:\beta<k\le\alpha\}$. If $\beta<\gamma\le\alpha$ then $P_0-P_1=\{k\in\mathbf{Z}_{\ge0}:\beta<k\le\gamma\}$. In any case we have

$$(2)\qquad P_0-P_1=\#\left\{k\in\mathbf{Z}_{\ge0}: \begin{array}{l} -a_2-b_1-b_2-1<k\\ k\le\min(a_1-b_1, a_1+a_2-b_1-b_2)\end{array}\right\}.$$

Similarly we have

$$P_2-P_3=\#\left\{k\in\mathbf{Z}_{\ge0}: \begin{array}{l} -a_2-b_1-2<k\\ k\le\min(a_1-b_1-b_2+1, a_1+a_2-b_1-1)\end{array}\right\}.$$

From theorem 2 we obtain the theorem. Q.E.D.

Using the above theorem we have the following stability theorem.

Theorem 5 *Let $\lambda=m_1\lambda_1+m_2\lambda_2$ be a dominant integral weight of G_2 and $\mu=\sum_{i=1}^{3}b_i\varepsilon_i$ be a dominant integral weight of $SU(3)$. Then*

(1) *if $m_2\ge b_1+1$ then $m(\lambda,\mu)=0$,*

(2) *if $m_1+m_2\ge b_1+1$ then $m(\lambda+\lambda_1,\mu)=m(\lambda,\mu)$.*

Proof. (1) From $m_2-b_1-1=-a_2-b_1-1\ge0$ we have $b_2\ge-b_1>a_2$. Thus we have $\min(a_1-b_1, a_1+a_2-b_1-b_2)=a_1+a_2-b_1-b_2$. Since $-a_2-b_1-b_2-1\ge0$, we have

$$P_0-P_1=a_1+a_2-b_1-b_2-(-a_2-b_1-b_2-1)=a_1+2a_2+1.$$

Similarly we have $P_2-P_3=a_1+2a_2+1$. Therefore $m(\lambda,\nu)=(P_0-P_1)-(P_2-P_3)=0$.

(2) Put $\lambda+\lambda_1 = \sum_{i=1}^{3} a'_i\varepsilon_i$ and denote $P'_i = P(S_i(\lambda+\lambda_1,\delta)-(\mu+\delta))$ $(0 \leq i \leq 3)$. From $\lambda > \mu$ we have $a_1 \geq b_1$. It is easily verified that $\min(a_1 - b_1, a_1 + a_2 - b_1 - b_2) \geq 0$ and $\min(a_1 - b_1, a_1 + a_2 - b_1 - b_2) > -a_2 - b_1 - b_2 - 1$ holds. Thus $P_0 - P_1$ is non-zero. From (2) and

$$P'_0 - P'_1 = \#\left\{k \in \mathbf{Z}_{\geq 0} : \begin{array}{l} -a_2 - b_1 - b_2 - 1 < k \\ k \leq \min(a_1 - b_1 + 1, a_1 + a_2 - b_1 - b_2 + 1) \end{array}\right\}.$$

it is easily seen that $P'_0 - P'_1 = P_0 - P_1 + 1$. Similarly we have $P'_2 - P'_3 = P_2 - P_3 + 1$. Thus we have $m(\lambda + \lambda_1, \nu) = m(\lambda, \nu)$. Q.E.D.

Remark 6 *Since every complex irreducible representation of G_2 is self-conjugate, we have*

$$m(\sum_{i=1}^{2} m_i\lambda_i, n_1\mu_1 + n_2\mu_2) = m(\sum_{i=1}^{2} m_i\lambda_i, n_2\mu_1 + n_1\mu_2).$$

4. Examples.

We give here tables of branching multiplicities $m(\sum_{i=1}^{2} m_i\lambda_i, \sum_{j=1}^{2} n_j\mu_j)$ with $n_1 + n_2 \leq 5$, $n_1 \geq n_2 \geq 0$.

$m_1\backslash m_2$	0	1
0	1	0
1	1	0

$(n_1, n_2) = (0, 0)$

$m_1\backslash m_2$	0	1	2
0	0	1	0
1	1	1	0
2	1	1	0

$(n_1, n_2) = (1, 0)$

$m_1\backslash m_2$	0	1	2	3
0	0	0	1	0
1	0	1	1	0
2	1	1	1	0
3	1	1	1	0

$(n_1, n_2) = (2, 0)$

$m_1\backslash m_2$	0	1	2	3
0	0	1	1	0
1	0	2	1	0
2	1	2	1	0
3	1	2	1	0

$(n_1, n_2) = (1, 1)$

$m_1 \backslash m_2$	0	1	2	3	4
0	0	0	0	1	0
1	0	0	1	1	0
2	0	1	1	1	0
3	1	1	1	1	0
4	1	1	1	1	0

$(n_1, n_2) = (3, 0)$

$m_1 \backslash m_2$	0	1	2	3	4
0	0	0	1	1	0
1	0	1	2	1	0
2	0	2	2	1	0
3	1	2	2	1	0
4	1	2	2	1	0

$(n_1, n_2) = (2, 1)$

$m_1 \backslash m_2$	0	1	2	3	4	5
0	0	0	0	0	1	0
1	0	0	0	1	1	0
2	0	0	1	1	1	0
3	0	1	1	1	1	0
4	1	1	1	1	1	0
5	1	1	1	1	1	0

$(n_1, n_2) = (4, 0)$

$m_1 \backslash m_2$	0	1	2	3	4	5
0	0	0	0	1	1	0
1	0	0	1	2	1	0
2	0	1	2	2	1	0
3	0	2	2	2	1	0
4	1	2	2	2	1	0
5	1	2	2	2	1	0

$(n_1, n_2) = (3, 1)$

$m_1 \backslash m_2$	0	1	2	3	4	5
0	0	0	1	1	1	0
1	0	0	2	2	1	0
2	0	1	3	2	1	0
3	0	2	3	2	1	0
4	1	2	3	2	1	0
5	1	2	3	2	1	0

$(n_1, n_2) = (2, 2)$

$m_1 \backslash m_2$	0	1	2	3	4	5	6
0	0	0	0	0	0	1	0
1	0	0	0	0	1	1	0
2	0	0	0	1	1	1	0
3	0	0	1	1	1	1	0
4	0	1	1	1	1	1	0
5	1	1	1	1	1	1	0
6	1	1	1	1	1	1	0

$(n_1, n_2) = (5, 0)$

$m_1\backslash m_2$	0	1	2	3	4	5	6
0	0	0	0	0	1	1	0
1	0	0	0	1	2	1	0
2	0	0	1	2	2	1	0
3	0	1	2	2	2	1	0
4	0	2	2	2	2	1	0
5	1	2	2	2	2	1	0
6	1	2	2	2	2	1	0

$(n_1, n_2) = (4, 1)$

$m_1\backslash m_2$	0	1	2	3	4	5	6
0	0	0	0	1	1	1	0
1	0	0	1	2	2	1	0
2	0	0	2	3	2	1	0
3	0	1	3	3	2	1	0
4	0	2	3	3	2	1	0
5	1	2	3	3	2	1	0
6	1	2	3	3	2	1	0

$(n_1, n_2) = (3, 2)$

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