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A FORMULA FOR THE MINIMUM GAP OF OPERATORS

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Abstract. We gave two formulas (the McIntosh formula and the Horn-Li-Merino formula) for the gap of operators and their applications([5]). Furthermore we clarified relationships between the gap and the sherical gap of operators([6]). In [4], the minimum gap of subspaces is defined and investigated. For operators acting on a Hilbert space, we define the minimum gap using the graphs and give a McIntosh type formula for it. Also we give some applications of this formula.

1. Introduction.

In [3] Habibi discussed the gap of the graph of a matrix and obtained the formula

$$\theta(A,0)=\frac{\|A\|}{\sqrt{1+\|A\|^2}}. \tag{1}$$

In general, for (bounded linear) operators A and B acting on a Hilbert space H, the gap is defined as follows:

$$\theta(A,B)=\theta(G(A),G(B)),$$

where $G(C)$ is the graph of an operator C and

$$\theta(M,N)=\max\{\sup_{x\in S(N)}\operatorname{dist}(x,M),\ \sup_{y\in S(M)}\operatorname{dist}(y,N)\}$$

for subspaces M and N in H. Here $S(L)$ denotes the unit sphere of a subspace L.

It is well known that this definition is equivalent to the following:

$$\theta(A,B)=\|P_A-P_B\|,$$

where P_C is the orthogonal projection of $H\oplus H$ onto the graph $G(C)$([4]).

We realized the Habibi formula (1) as the McIntosh formula for operators A and B:

$$\begin{aligned}\theta(A,B)=\max\{&\|(1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2}\|,\\ &\|(1+AA^*)^{-1/2}(A-B)(1+B^*B)^{-1/2}\|\},\end{aligned} \tag{2}$$

see [5].

For operators A and B, the spherical gap $\tilde{\theta}(A,B)$ is defined as $\tilde{\theta}(G(A),G(B))$, where

$$\tilde{\theta}(M,N)=\max\{\sup_{x\in S(N)}\operatorname{dist}(x,S(M)),\ \sup_{y\in S(M)}\operatorname{dist}(y,S(N))\}$$

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for subspaces M and N (see [6]).

We obtained a formula for the spherical gap $\tilde{\theta}(A, B)$:

$$\tilde{\theta}(A, B) = \sqrt{2 - 2\min\{m(\Gamma(A, B)), m(\Gamma(B, A))\}},$$

where $\Gamma(A, B) = (1 + A^*A)^{-1/2}(1 + A^*B)(1 + B^*B)^{-1/2}$ and $m(C) = \inf_{\|x\|=1} \|Cx\|$ for an operator C. Consequently, a relationship between $\theta(A, B)$ and $\tilde{\theta}(A, B)$ was given by

$$\tilde{\theta}(A, B) = \sqrt{2(1 - \sqrt{1 - \theta^2(A, B)})}.$$

Now, for subspaces M and N in H, the minimum gap $\hat{\theta}(M, N)$ is defined as follows:

$$\hat{\theta}(M, N) = \min\{d(M, N), d(N, M)\}, \tag{3}$$

where

$$d(M, N) = \inf_{u \in M, u \notin N} \frac{\operatorname{dist}(u, N)}{\operatorname{dist}(u, M \cap N)}. \tag{4}$$

A pair M, N is said to be regular if $d(M, N) = d(N, M)$. Any pair M, N in a Hilbert space is regular(see [4]).

For operators A and B, we define

$$\hat{\theta}(A, B) := \hat{\theta}(G(A), G(B)) \tag{5}$$

and call this quantity the minimum gap of operators A and B.

In this paper, we give a McIntosh type formula for the minimum gap of operators. This formula is given by replacing the norm with the reduced minimum modulus in the McIntosh formula (2); precisely, for operators A and B, we have

$$\begin{aligned}\hat{\theta}(A, B) &= \gamma((1 + BB^*)^{-1/2}(A - B)(1 + A^*A)^{-1/2}) \\ &= \gamma((1 + AA^*)^{-1/2}(A - B)(1 + B^*B)^{-1/2}),\end{aligned}$$

where $\gamma(C)$ denotes the reduced minimum modulus of an operator C (For the definition, see §2 below). Also we give its applications. For an example, $G(A) + G(B)$ is closed if and only if the range $\mathrm{R}(A - B)$ of an operator $A - B$ is closed.

2. Results.

At first we describe some properties about the reduced minimum modulus which is seemed to be essential for the investigation of the minimum gap of operators.

For an operator C, the reduced minimum modulus $\gamma(C)$ is defined as follows[1,2]:

$$\gamma(C) = \inf\{\|Cx\|;\ x \in \ker C^{\perp}, \|x\| = 1\}.$$

It is well known that $\gamma(C) > 0$ if and only if the range $R(C)$ is closed. If C is invertible then $\gamma(C) = \|C^{-1}\|^{-1}$.

Lemma 1([1]). For an operator C,

$$\begin{aligned}\gamma(C) &= \inf\{\lambda;\ \lambda \in \sigma(|C|) \setminus \{0\}\}\\ &= \inf\{\lambda;\ \lambda \in \sigma(|C^*|) \setminus \{0\}\} = \gamma(C^*).\end{aligned}$$

Lemma 2. If A is invertible, then

$$\|A\|\gamma(B) \geq \gamma(AB) \geq \gamma(A)\gamma(B)$$

for any operator B.

Proof. For $x \in \ker AB^{\perp} = \ker B^{\perp}$, we have

$$\|A\|\|Bx\| \geq \|ABx\| \geq \gamma(A)\|Bx\|.$$

Taking the infimum we get the desired result. ■

Next we mention the main result. The following theorem states that the minimum gap is given by replacing the norm with the reduced minimum modulus in the McIntosh formula (2).

Theorem 1. For operators A and B, we have

$$\begin{aligned}\hat{\theta}(A,B) &= \gamma((1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2})\\ &= \gamma((1+AA^*)^{-1/2}(A-B)(1+B^*B)^{-1/2}).\end{aligned}$$

In particular,

$$\hat{\theta}(A,0) = \frac{\gamma(A)}{\sqrt{1+\gamma(A)^2}}.$$

To prove Theorem 1, we need the following lemmas.

Lemma 3. For $u = \binom{x}{Ax} \in G(A)$, there exists a $y_x \in \ker(A-B)(1+A^*A)^{-1/2}$ such that

$$\mathrm{dist}(u, G(A) \cap G(B)) = \|(1+A^*A)^{-1/2}x - y_x\|.$$

Proof. For $u = \binom{x}{Ax} \in G(A)$ and $\binom{y}{Ay} = \binom{y}{By} \in G(A) \cap G(B)$, we have

$$\begin{aligned}\mathrm{dist}(u, G(A) \cap G(B)) &= \inf_{y \in \ker(A-B)} \{\|x-y\|^2 + \|Ax - By\|^2\}^{1/2}\\ &= \inf_{y \in \ker(A-B)} \|(1+A^*A)^{1/2}(x-y)\|\\ &= \|(1+A^*A)^{1/2}x - y_x\| \ (\exists y_x \in \ker(A-B)(1+A^*A)^{-1/2}). \blacksquare\end{aligned}$$

We remark that

$$(1+A^*A)^{1/2}x - y_x \in \ker(A-B)(1+A^*A)^{-1/2\perp}$$
$$= \ker(1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2\perp}.$$

Lemma 4. For $u = \binom{x}{Ax} \in G(A)$, we have

$$\operatorname{dist}(u, G(B)) = \|(1+BB^*)^{-1/2}(A-B)x\|.$$

Proof. For $\binom{y}{By} \in G(B)$, it follows that

$$\begin{aligned}\operatorname{dist}(u, G(B)) &= \inf_y \{\|x-y\|^2 + \|Ax - By\|^2\}^{1/2} \\ &= \inf_y \{\|(1+A^*A)^{1/2}x\|^2 + \|(1+B^*B)^{1/2}y\|^2 - 2\mathrm{Re}\ ((1+B^*A)x, y)\}^{1/2}.\end{aligned}$$

Here we have

$$\begin{aligned}&\inf_y\{\|(1+B^*B)^{1/2}y\|^2 - 2\mathrm{Re}\ ((1+B^*A)x, y)\} \\ &\quad = \inf_z\{\|z\|^2 - 2\mathrm{Re}\ (1+B^*B)^{-1/2}(1+B^*A)x, z)\} \\ &\quad = \inf\{\|z - (1+B^*B)^{-1/2}(1+B^*A)x\|^2 - \|(1+B^*B)^{-1/2}(1+B^*A)x\|^2\} \\ &\quad = -\|(1+B^*B)^{-1/2}(1+B^*A)x\|^2.\end{aligned}$$

By using the following operator equation

$$1 + A^*A - (1+A^*B)(1+B^*B)^{-1}(1+B^*A) = (A-B)^*(1+BB^*)^{-1}(A-B),$$

it follows that

$$\begin{aligned}\operatorname{dist}(u, G(B)) &= \{\|(1+A^*A)^{1/2}x\|^2 - \|(1+B^*B)^{-1/2}(1+B^*A)x\|^2\}^{1/2} \\ &= \|(1+BB^*)^{-1/2}(A-B)x\|. \blacksquare\end{aligned}$$

Proof of Theorem 1. By Lemmas 3 and 4, we have

$$\begin{aligned}&\inf_{u\in G(A), u\notin G(B)} \frac{\operatorname{dist}(u, G(B))}{\operatorname{dist}(u, G(A)\cap G(B))} \\ &= \inf_{x\in\ker(A-B)^\perp} \frac{\|(1+BB^*)^{-1/2}(A-B)x\|}{\|(1+A^*A)^{1/2}x - y_x\|} \\ &= \inf_{x\in\ker(A-B)^\perp} \frac{\|(1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2}\{(1+A^*A)^{1/2}x - y_x\}\|}{\|(1+A^*A)^{1/2}x - y_x\|}.\end{aligned}$$

Put $(1+A^*A)^{1/2}x - y_x = z$, then $z \in \ker((1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2})^{\perp}$ as already remarked.

Therefore it follows that

$$\begin{aligned}
&\inf_{u\in G(A), u\notin G(B)} \frac{\operatorname{dist}(u, G(B))}{\operatorname{dist}(u, G(A)\cap G(B))} \\
&= \inf_{z} \frac{\|(1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2}z\|}{\|z\|} \\
&= \gamma((1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2}).
\end{aligned}$$

By the regularity of the pair $G(A), G(B)$, we have

$$\begin{aligned}
\hat{\theta}(A,B) &= \gamma((1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2}) \\
&= \gamma((1+AA^*)^{-1/2}(A-B)(1+B^*B)^{-1/2}).
\end{aligned}$$

In particular, $\hat{\theta}(A,0) = \gamma(A(1+A^*A)^{-1/2})$. A direct computation shows that $|A(1+A^*A)^{-1/2}| = |A|(1+|A|^2)^{-1/2}$. Also the function $\dfrac{\lambda}{\sqrt{1+\lambda^2}}$ is increasing. Hence by Lemma 1 we have

$$\begin{aligned}
\hat{\theta}(A,0) &= \inf\{\frac{\lambda}{\sqrt{1+\lambda^2}}; \lambda \in \sigma(|A|)\setminus\{0\}\} \\
&= \frac{\gamma(A)}{\sqrt{1+\gamma(A)^2}}.
\end{aligned}$$

This completes the proof. ■

Using the McIntosh formula (2) for the gap of operators,

Corollary 1. For operators A and B,

$$0 \le \hat{\theta}(A,B) \le \theta(A,B) \le 1$$

Remark. By the McIntosh formula (2), for scalar α and β we have

$$\theta(\alpha,\beta) = \frac{|\alpha-\beta|}{\sqrt{1+|\alpha|^2}\sqrt{1+|\beta|^2}}.$$

This value is the chordal distance between $\pi(\alpha)$ and $\pi(\beta)$ where π is the stereographic projection of the complex plane onto the Riemann sphere with the center $(0,0,1/2)$ and the radius 1/2([5]). For the minimum gap $\hat{\theta}(\alpha,\beta)$ we have the same conclusion by Theorem 1.

By using Theorm 1, the following estimation is attained.

Theorem 2. For operators A and B,
$$\frac{1}{\sqrt{1+\|A\|^2}\sqrt{1+\|B\|^2}}\gamma(A-B) \le \hat{\theta}(A,B) \le \|(1+A^*A)^{-1/2}\|\|(1+B^*B)^{-1/2}\|\gamma(A-B).$$

Proof. By Lemmas 1 and 2, we have
$$\begin{aligned}\hat{\theta}(A,B) &= \gamma((1+BB^*)^{-1/2}(A-B)(1+A^*A)^{-1/2})\\ &\ge \|(1+BB^*)^{1/2}\|^{-1}\gamma(A-B)(1+A^*A)^{-1/2})\\ &\ge \|(1+BB^*)^{1/2}\|^{-1}\|(1+A^*A)^{1/2}\|^{-1}\gamma(A-B)\\ &= (1+\|B\|^2)^{-1/2}(1+\|A\|^2)^{-1/2}\gamma(A-B).\end{aligned}$$
By the same methods, the other inequality is proved.■

Theorem 2 shows that $\hat{\theta}(A,B) > 0$ if and only if $\gamma(A-B) > 0$. In [4: Theorem 4.2], it is showed that $\hat{\theta}(M,N) > 0$ if and only if $M+N$ is closed. So we have

Corollary 2. For operators A and B, $G(A)+G(B)$ is closed if and only if R $(A-B)$ is closed. Particularly, $G(A)+H$ is closed if and only if R(A) is closed.

For the relationship between the gap and minimum gap, the following theorem is known

Theorem 3([4: Theorem 5.17]). Let A be a semi Fredholm operator. If $\theta(A,B) < \dfrac{\gamma(A)}{\sqrt{1+\gamma^2(A)}}$ is satisfied, then B is a semi Fredholm operator and

(1) $\dim\ker B \le \dim\ker A$;

(2) $\dim\ker B^* \le \dim\ker A^*$.

We give an another proof of Theorem 3.

Lemma 5([1: Theorem 3.20]). Let A be a semi Fredholm operator. If $\|B\| < \gamma(A)$ is satisfied, then $A+B$ is a semi Fredholm operator and

(1) $\dim\ker(A+B) \le \dim\ker A$;

(2) $\dim\ker(A+B)^* \le \dim\ker A^*$.

Lemma 6. For operators A and B, the following oprator equation holds:
$$(1+BB^*)^{1/2}B(1+B^*B)^{-1}(1+B^*A)(1+A^*A)^{-1/2} - (1+BB^*)^{1/2}A(1+A^*A)^{-1/2}$$
$$= (1+BB^*)^{1/2}(B-A)(1+A^*A)^{-1/2}.$$

Proof. Put $S_C = (1+C^*C)^{-1/2}$ for an operator C. Then
$$\begin{aligned}&BS_B^2(1+B^*A)S_A - AS_A\\ &= B(1+B^*B)^{-1}S_A + B(1+B^*B)^{-1}B^*AS_A - AS_A\\ &= (1+BB^*)^{-1}BS_A + (1+BB^*)^{-1}\{BB^* - (1+BB^*)\}AS_A\\ &= (1+BB^*)^{-1}(B-A)S_A.\end{aligned}$$

By multiplying $(1+BB^*)^{1/2}$ on the both sides, the desired result is obtained.■

Proof of Theorem 3. By Lemma 2 we have

$$\|S_{B^*}^{-1}BS_B^2(1+B^*A)S_A - S_{B^*}^{-1}AS_A\| \geq \gamma(S_{B^*}^{-1})\|BS_B^2(1+B^*A)S_A - AS_A\|.$$

By using Lemma 6 and the McIntosh formula, the above inequality becomes

$$\gamma(S_{B^*}^{-1})\|BS_B^2(1+B^*A)S_A - AS_A\| \leq \theta(A,B).$$

By the assumption of Theorem 3 and $\gamma(S_{B^*}^{-1}) = \|S_{B^*}\|^{-1}$, it follows that

$$\|BS_B^2(1+B^*A)S_A - AS_A\| < \|S_{B^*}\|\frac{\gamma(A)}{\sqrt{1+\gamma^2(A)}}.$$

As $\|S_{B^*}\| \leq 1$ and $\hat{\theta}(A,0) = \gamma(AS_A) = \dfrac{\gamma(A)}{\sqrt{1+\gamma^2(A)}}$, we have

$$\|BS_B^2(1+B^*A)S_A - AS_A\| < \gamma(AS_A).$$

Hence by Lemma 5, $BS_B^2(1+B^*A)S_A$ is a semi Fredholm operator. Here as $\theta(A,B) < 1$, $1+B^*A$ is invertible ([5: Corollary 4.3]). Therefore B is a semi Fredholm operator. The other parts about the dimension are easy to see by Lemma 5.■

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