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All solutions of the Diophantine equation $2^aX^r + 2^bY^s = 2^cZ^t$ where r, s and t are 2 or 4

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1. Introduction

We shall determine all solutions of the equation $2^aX^r+2^bY^s = 2^cZ^t$ in nonzero integers X, Y, Z, where a, b, c are non-negative integers, and r, s, t are 2 or 4, and X, Y, Z are pairwise relatively prime. To discuss the solutions of this equation, we may assume that X, Y, Z are all positive odd integers. We shall show that the following results.

The equations

$$\begin{aligned} X^2 &+ Y^2 = 2Z^2 , \\ X^2 &+ 2^mY^2 = Z^2 , \\ X^2 &+ Y^2 = 2Z^4 , \\ X^2 &+ 2^mY^2 = Z^4 , \\ X^2 &+ Y^4 = 2Z^2 , \\ X^4 &+ 2^mY^2 = Z^2 \end{aligned}$$

and

$$X^2 + 2^mY^4 = Z^2$$

have independently infinite solutions.

The equation

$$2^aX^4 + 2^bY^4 = 2^cZ^4$$

has only one trivial solution.

The equation

$$X^4 + Y^4 = 2Z^2$$

has only one trivial solution. (A.M.Legendre)

The equation

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$$X^4 \quad + \quad 2^m Y^2 \quad = \quad Z^4$$

has no solutions in nonzero integers.

The equations

$$\begin{array}{rcrcl} X^2 & + & Y^4 & = & 2Z^4 \quad , \\ X^4 & + & 2^m Y^4 & = & Z^2 \end{array}$$

and

$$X^2 \quad + \quad 2^m Y^4 \quad = \quad Z^4$$

have infinite solutions. In the latter half, we shall give one-to-one correspondences between solutions of these three equations, and in section 7, determine all solutions of these equations.

2. Pythagorian Triples

We remind first the following three theorems which are all well-known (see [1],[2],[3],[4] or [5]).

Theorem 1. *Let X, Y, Z be a solution of the equation*

$$X^2 + Y^2 = Z^2$$

with positive integers X, Y, Z such that $(X, Y) = 1$ and X odd. Then there exist unique integers u and v of opposite parity with $(u, v) = 1$ and $u > v > 0$ such that

$$\begin{array}{rcl} X & = & u^2 - v^2 \quad , \\ Y & = & 2uv \quad , \\ Z & = & u^2 + v^2 \quad . \end{array}$$

Theorem 2. *The equation*

$$X^4 + Y^4 = Z^2$$

has no solutions in nonzero integers X, Y, Z.

Theorem 3. *The equation*

$$X^4 + Y^2 = Z^4$$

has no solutions in nonzero integers X, Y, Z.

3. On the Diophantine Equation $2^a X^2 + 2^b Y^2 = 2^c Z^2$

Lemma 4. *Let m ba a non-negative integer. If a set of three odd integers X, Y, Z satisfies the equation*

$$X^2 + Y^2 = 2^m Z^2 ,$$

then $m = 1$.

Proof. Since the square of an odd integer is congruent to 1 modulo 4, we have

$$2^m Z^2 = X^2 + Y^2 \equiv 1 + 1 = 2 \ (mod\, 4) .$$

This implies $m = 1$ and completes the proof.

Lemma 5. *Let m be a non-negative integer. If a set of three odd integers X, Y, Z satisfies the equation*

$$X^2 + 2^m Y^2 = Z^2 ,$$

then $m \geqq 3$.

Proof. Since the square of an odd integer is congruent to 1 modulo 8, we have

$$2^m Y^2 = Z^2 - X^2 \equiv 1 - 1 = 0 \ (mod\, 8) .$$

This implies $m \geqq 3$ and completes the proof.

Theorem 6. (see L.J.Mordell [4] p.13) *Let X, Y, Z be a solution of the equation*

$$X^2 + Y^2 = 2Z^2$$

with positive odd integers X, Y, Z such that $(X, Y) = 1$. Then there exist non-negative integers c and d of opposite parity with $(c, d) = 1$ and $c > d \geqq 0$ such that

$$u = c^2 - d^2 , \ v = 2cd$$

and

$$\begin{array}{rcl} X & = & u + v \quad , \\ Y & = & |u - v| \quad , \\ Z & = & c^2 + d^2 \quad , \end{array}$$

or the corresponding formulas hold with X and Y interchanged.

Proof. From $X^2+Y^2=2Z^2$, we have $\left(\dfrac{X+Y}{2}\right)^2+\left(\dfrac{X-Y}{2}\right)^2=Z^2$ where $\dfrac{X+Y}{2}$, $\dfrac{X-Y}{2}$, Z are integers with pairwise relatively prime. We suppose that $X>Y$. Then, by Theorem 1, there exist positive integers c and d of opposite parity with $(c,d)=1$ and $c>d>0$ such that

$$\frac{X+Y}{2}=c^2-d^2\,,\ \frac{X-Y}{2}=2cd\,,\ Z=c^2+d^2$$

or

$$\frac{X+Y}{2}=2cd\,,\ \frac{X-Y}{2}=c^2-d^2\,,\ Z=c^2+d^2\,.$$

Then we have $X=c^2-d^2+2cd$ and $Y=|c^2-d^2-2cd|$. If $X<Y$, then we have $Y=c^2-d^2+2cd$ and $X=|c^2-d^2-2cd|$. When $X=Y$, we have $X=Y=Z=1$, and we set $c=1$ and $d=0$. Hence the proof is complete.

Theorem 7. *Let m be a non-negative integer. Let X,Y,Z be a solution of the equation*

$$X^2+2^mY^2=Z^2$$

with positive odd integers X,Y,Z such that $(X,Y)=1$. Then $m\geqq 3$, and there exist positive odd integers a and b with $(a,b)=1$ such that

$$\begin{array}{rcll} X &=& |a^2-2^{m-2}b^2| & , \\ Y &=& ab & , \\ Z &=& a^2+2^{m-2}b^2 & . \end{array}$$

Proof. By Lemma 5, we have $m\geqq 3$. Since $2^mY^2=Z^2-X^2=(Z+X)(Z-X)$ and $(Z+X,Z-X)=2$, there exist positive odd integers a amd b with $(a,b)=1$ such that $Y=ab$ and

$$Z+X=2a^2\,,\ Z-X=2^{m-1}b^2$$

or

$$Z+X=2^{m-1}b^2\,,\ Z-X=2a^2\,.$$

Then we have $Z=a^2+2^{m-2}b^2$ and $X=|a^2-2^{m-2}b^2|$. Conversely, $X^2+2^mY^2=\left(a^2-2^{m-2}b^2\right)^2+2^m(ab)^2=\left(a^2+2^{m-2}b^2\right)^2=Z^2$. Hence the proof is complete.

Theorem 8. *Let X, Y, Z be a solution of the equation*

$$X^2 + Y^2 = 2Z^4$$

with positive odd integers X, Y, Z such that $(X, Y) = 1$. Then there exist non-negative integers c and d of opposite parity with $(c, d) = 1$ and $c > d \geqq 0$ such that

$$\begin{array}{ll} u = c^2 - d^2 \quad , & v = 2cd \quad , \\ s = u^2 - v^2 \quad , & t = 2uv \end{array}$$

and

$$\begin{array}{rcl} X & = & |s + t| \quad , \\ Y & = & |s - t| \quad , \\ Z & = & c^2 + d^2 \quad , \end{array}$$

or the corresponding formulas hold with X and Y interchanged.

Proof. From $X^2 + Y^2 = 2Z^4$, we have $\left(\frac{X+Y}{2}\right)^2 + \left(\frac{X-Y}{2}\right)^2 = Z^4$ where $\frac{X+Y}{2}$, $\frac{X-Y}{2}$, Z are integers with pairwise relatively prime. If $X \neq Y$, then, by Theorem 1, there exist positive integers U and V of opposite parity with $(U, V) = 1$ and $U > V > 0$ such that

$$\frac{X+Y}{2} = U^2 - V^2\,,\ \left|\frac{X-Y}{2}\right| = 2UV\,,\ Z^2 = U^2 + V^2$$

or

$$\frac{X+Y}{2} = 2UV\,,\ \left|\frac{X-Y}{2}\right| = U^2 - V^2\,,\ Z^2 = U^2 + V^2\,.$$

Applying Theorem 1 again, there exist positive integers c and d of opposite parity with $(c, d) = 1$ and $c > d > 0$ such that $Z = c^2 + d^2$ and $U = c^2 - d^2$, $V = 2cd$ or $U = 2cd$, $V = c^2 - d^2$. We set $u = c^2 - d^2$, $v = 2cd$, $s = u^2 - v^2$ and $t = 2uv$. Then we have $X^2 + Y^2 = 2z^4 = 2\left(c^2 + d^2\right)^4 = 2\left(u^2 + v^2\right)^2 = 2\left(s^2 + t^2\right) = (s+t)^2 + (s-t)^2$, and $X^2Y^2 = \left(\left(\frac{X+Y}{2}\right)^2 - \left(\frac{X-Y}{2}\right)^2\right)^2 = \left((u^2 - v^2)^2 - (2uv)^2\right)^2 = \left(s^2 - t^2\right)^2 = (s+t)^2 \cdot (s-t)^2$. Thus $X^2 = (s+t)^2$, $Y^2 = (s-t)^2$ or $X^2 = (s-t)^2$, $Y^2 = (s+t)^2$. So we obtain that $X = |s+t|$, $Y = |s-t|$ or $X = |s-t|$, $Y = |s+t|$. If $X = Y$, then we set $c = 1$ and $d = 0$. Conversely, $(s+t)^2 + (s-t)^2 = 2\left(s^2 + t^2\right) = 2\left(u^2 + v^2\right)^2 = 2\left(c^2 + d^2\right)^4$. Hence the proof is complete.

Theorem 9. *Let m be a non-negative integer. Let X, Y, Z be a solution of the equation*

$$X^2 + 2^m Y^2 = Z^4$$

with positive odd integers X, Y, Z such that $(X, Y) = 1$. Then $m \geqq 5$, and there exist positive odd integers a and b with $(a, b) = 1$ such that

$$A = |a^2 - 2^{m-4}b^2|\,,\ B = ab$$

and

$$\begin{array}{rcl} X & = & |A^2 - 2^{m-2}B^2| \quad , \\ Y & = & AB \quad , \\ Z & = & a^2 + 2^{m-4}b^2 \quad . \end{array}$$

Proof. By Lemma 5, we have $m \geqq 3$. Since $2^m Y^2 = Z^4 - X^2 = (Z^2 + X)(Z^2 - X)$ and $\left(Z^2 + X, Z^2 - X\right) = 2$, there exist positive odd integers A and B with $(A, B) = 1$ such that $Y = AB$ and

$$Z^2 + X = 2A^2\,,\ Z^2 - X = 2^{m-1}B^2$$

or

$$Z^2 + X = 2^{m-1}B^2\,,\ Z^2 - X = 2A^2\,.$$

Thus, we have $Z^2 = A^2 + 2^{m-2}B^2$ and $X = |A^2 - 2^{m-2}B^2|$. Hence, by Theorem 7, we obtain that $m - 2 \geqq 3$, so $m \geqq 5$ and there exist positive odd integers a and b with $(a, b) = 1$ such that $A = |a^2 - 2^{m-4}b^2|$, $B = ab$ and $Z = a^2 + 2^{m-4}b^2$. Hence the proof is complete.

Theorem 10. *Let X, Y, Z be a solution of the equation*

$$X^2 + Y^4 = 2Z^2$$

with positive odd integers X, Y, Z such that $(X, Y) = 1$. Then there exist integer b and positive odd integer a with $(a, b) = 1$ such that

$$\begin{array}{rclcrcl} u & = & (a+b)^2 + b^2 & , & v & = & 2ab\,, \\ s & = & u^2 - v^2 & , & t & = & 2uv \end{array}$$

and

$$\begin{array}{rcl} X & = & |s + t| \quad , \\ Y & = & |a^2 - 2b^2| \quad , \\ Z & = & u^2 + v^2 \quad . \end{array}$$

Proof. Let X, Y, Z be a solution of the equation $X^2 + Y^4 = 2Z^2$ with positive odd integers X, Y, Z such that $(X, Y) = 1$. When $X = Y$, we have $X = Y = Z = 1$, and we set $a = 1$ and $b = 0$. Thus we suppose that $X \neq Y$. Since $X^2 + Y^4 = 2Z^2$ and X, Y, Z are odd integers with pairwise relatively prime, we have

$$\left(\frac{X+Y^2}{2}\right)^2 + \left(\frac{X-Y^2}{2}\right)^2 = Z^2$$

where $\dfrac{X+Y^2}{2}$, $\dfrac{X-Y^2}{2}$ are integers of opposite parity, and $\dfrac{X+Y^2}{2}$, $\dfrac{X-Y^2}{2}$, Z are pairwise relatively prime.

In the case of (I) $X \equiv 1 \ (mod\ 4)$ and $X > Y^2$, by Theorem 1, there exist positive integers u and v of opposite parity with $(u, v) = 1$ and $u > v > 0$ such that $\dfrac{X+Y^2}{2} = u^2 - v^2$, $\dfrac{X-Y^2}{2} = 2uv$ and $Z = u^2 + v^2$. Hence we obtain that $X = u^2 - v^2 + 2uv$, $Y^2 = u^2 - v^2 - 2uv$ and $u - v > 0$.

In the case of (II) $X \equiv 1 \ (mod\ 4)$ and $X < Y^2$, by Theorem 1, there exist positive integer u and negative integer v of opposite parity with $(u, v) = 1$ and $u > -v > 0$ such that $\dfrac{X+Y^2}{2} = u^2 - v^2$, $\dfrac{Y^2-X}{2} = -2uv$ and $Z = u^2 + v^2$. Hence we obtain that $X = u^2 - v^2 + 2uv$, $Y^2 = u^2 - v^2 - 2uv$ and $u - v > 0$.

In the case of (III) $X \equiv -1 \ (mod\ 4)$ and $X > Y^2$, by Theorem 1, there exist positive integer u and negative integer v of opposite parity with $(u, v) = 1$ and $-v > u > 0$ such that $\dfrac{X-Y^2}{2} = v^2 - u^2$, $\dfrac{X+Y^2}{2} = -2uv$ and $Z = u^2 + v^2$. Hence we obtain that $X = -(u^2 - v^2 + 2uv)$, $Y^2 = u^2 - v^2 - 2uv$ and $u - v > 0$.

In the case of (IV) $X \equiv -1 \ (mod\ 4)$ and $X < Y^2$, by Theorem 1, there exist positive integer u and negative integer v of opposite parity with $(u, v) = 1$ and $u > -v > 0$ such that $\dfrac{Y^2-X}{2} = u^2 - v^2$, $\dfrac{X+Y^2}{2} = -2uv$ and $Z = u^2 + v^2$. Hence we obtain that $X = -(u^2 - v^2 + 2uv)$, $Y^2 = u^2 - v^2 - 2uv$ and $u - v > 0$.

In any case of (I), (II), (III) and (IV), from $Y^2 = u^2 - v^2 - 2uv$, we obtain that u is odd and v even,and $2v^2 = (u-v)^2 - Y^2 = (u-v+Y)(u-v-Y)$. Since $Y^2 = u^2 - v^2 - 2uv$, Y and u odd, v even and $(u, v) = 1$, we have $(u-v+Y, u-v-Y) = 2$. Then there exist nonzero integer b and positive odd integer a with $(a, b) = 1$ such that $v = 2ab$ and

$$u - v + Y = 2a^2 \, , \; u - v - Y = 4b^2$$

or

$$u - v + Y = 4b^2 \, , \; u - v - Y = 2a^2 \, .$$

Thus we have $u - v = a^2 + 2b^2$ and $Y = |a^2 - 2b^2|$. So we obtain $u = a^2 + 2b^2 + v = a^2 + 2b^2 + 2ab = (a+b)^2 + b^2$. Hence the proof is complete.

Theorem 11. *Let m be a non-negative integer. Let X, Y, Z be a solution of the equation*

$$X^4 + 2^m Y^2 = Z^2$$

with positive odd integers X, Y, Z such that $(X, Y) = 1$. Then $m = 3$ or $m \geqq 5$.

In the case of $m = 3$, there exist non-negative integers c and d of opposite parity with $(c, d) = 1$ and $c > d \geqq 0$ such that

$$u = c^2 - d^2\,,\ v = 2cd\,,\ B = c^2 + d^2$$

and

$$X = u + v\ ,\ A = |u - v|$$

or

$$X = |u - v|\,,\ A = u + v$$

and

$$\begin{array}{rcl} Y & = & AB\ , \\ Z & = & A^2 + 2B^2\ . \end{array}$$

In the case of $m \geqq 5$, there exist positive odd integers a and b with $(a, b) = 1$, such that

$$A = a^2 + 2^{m-4}b^2\,,\ B = ab$$

and

$$\begin{array}{rcl} X & = & |a^2 - 2^{m-4}b^2|\ , \\ Y & = & AB\ , \\ Z & = & A^2 + 2^{m-2}B^2\ . \end{array}$$

Proof. By Lemma 5, we have $m \geqq 3$. Since $2^m Y^2 = Z^2 - X^4 = (Z + X^2)(Z - X^2)$ and $(Z + X^2, Z - X^2) = 2$, there exist positive odd integers A and B with $(A, B) = 1$ such that $Y = AB$ and

$$\text{(I) } Z + X^2 = 2A^2\,,\ Z - X^2 = 2^{m-1}B^2$$

or

$$\text{(II) } Z + X^2 = 2^{m-1}B^2\,,\ Z - X^2 = 2A^2\,.$$

In the case of (I) $Z + X^2 = 2A^2$ and $Z - X^2 = 2^{m-1}B^2$, we have $Z = A^2 + 2^{m-2}B^2$ and $X^2 = A^2 - 2^{m-2}B^2$, or $X^2 + 2^{m-2}B^2 = A^2$. Hence, by Theorem 7, we obtain that $m - 2 \geqq 3$, so $m \geqq 5$ and there exist positive odd integers a and b with $(a, b) = 1$ such that $X = |a^2 - 2^{m-4}b^2|$, $B = ab$ and $A = a^2 + 2^{m-4}b^2$.

In the case of (II) $Z + X^2 = 2^{m-1}B^2$ and $Z - X^2 = 2A^2$, we have $Z = A^2 + 2^{m-2}B^2$ and $X^2 = 2^{m-2}B^2 - A^2$, or $X^2 + A^2 = 2^{m-2}B^2$. By Lemma 4, we have $m - 2 = 1$, so $m = 3$. By Theorem 6, there exist non-negative integers c

and d of opposite parity with $(c, d) = 1$ and $c > d \geqq 0$, such that $B = c^2 + d^2$ and $X = u + v$, $A = |u - v|$ or $X = |u - v|$, $A = u + v$ where $u = c^2 - d^2$ and $v = 2cd$. Hence the proof is complete.

Theorem 12. *Let m be a non-negative integer. Let X, Y, Z be a solution of the equation*

$$X^2 + 2^m Y^4 = Z^2$$

with positive odd integers X, Y, Z such that $(X, Y) = 1$. Then $m \geqq 3$, and there exist positive odd integers a and b with $(a, b) = 1$ such that

$$\begin{aligned} X &= |a^4 - 2^{m-2}b^4| \quad , \\ Y &= ab \quad , \\ Z &= a^4 + 2^{m-2}b^4 \quad . \end{aligned}$$

Proof. By Lemma 5, we have $m \geqq 3$. Since $2^m Y^4 = Z^2 - X^2 = (Z+X)(Z-X)$ and $(Z + X, Z - X) = 2$, there exist positive odd integers a and b with $(a, b) = 1$ such that $Y = ab$ and

$$Z + X = 2a^4\,,\ Z - X = 2^{m-1}b^4$$

or

$$Z + X = 2^{m-1}b^4\,,\ Z - X = 2a^4\,.$$

Then we have $Z = a^4 + 2^{m-2}b^4$ and $X = |a^4 - 2^{m-2}b^4|$. Hence the proof is complete.

Example 1. For example, applying Theorem 6, 7, 8, 9, 10, 11 and 12, when $m = 5$, $c = 9$, $d = 2$, $a = 7$ and $b = 3$, we have the following equations.

$$\begin{aligned}
113^2 &+ 41^2 = 2 \cdot 85^2 \\
23^2 &+ 2^5 \cdot 21^2 = 121^2 \\
10177^2 &+ 911^2 = 2 \cdot 85^4 \\
2567^2 &+ 2^5 \cdot 651^2 = 67^4 \\
19273^2 &+ 31^4 = 2 \cdot 13645^2 \\
113^4 &+ 2^3 \cdot 3485^2 = 16131^2 \\
41^4 &+ 2^3 \cdot 9605^2 = 27219^2 \\
31^4 &+ 2^5 \cdot 1407^2 = 8017^2 \\
1753^2 &+ 2^5 \cdot 21^4 = 3049^2
\end{aligned}$$

4. The equation $2^a X^4 + 2^b Y^4 = 2^c Z^4$ has only one trivial solution

Let a, b, c be non-negative integers. In this section, we shall determine the solutions of the Diophantine equation $2^a X^4 + 2^b Y^4 = 2^c Z^4$ in nonzero integers X, Y, Z ([7]). The following theorem was proved by A.M.Legendre.

Theorem 13. *Let X, Y, Z be a solution of the equation*

$$X^4 + Y^4 = 2Z^2$$

in non-negative integers. Then

$$X^2 = Y^2 = Z .$$

Proof. Let X, Y, Z be the solution of the equation $X^4 + Y^4 = 2Z^2$ in non-negative integers. Then, we obtain

$$\left(2Z^2\right)^2 = \left(X^4 + Y^4\right)^2 = \left(X^4 - Y^4\right)^2 + 4X^4Y^4$$

and so

$$(XY)^4 + \left(\frac{X^4 - Y^4}{2}\right)^2 = Z^4 .$$

This equation implies that $\dfrac{X^4 - Y^4}{2}$ is an integer. By Theorem 3, we have that $XY = 0$ and $|\dfrac{X^4 - Y^4}{2}| = Z^2$ or that $\dfrac{X^4 - Y^4}{2} = 0$ and $XY = Z$. When $XY = 0$, since $X^4 + Y^4 = 2Z^2$, we obtain $X = Y = Z = 0$. When $\dfrac{X^4 - Y^4}{2} = 0$ and $XY = Z$, we obtain $X^2 = Y^2 = Z$. Hence the proof is complete.

Corollary 14. *Let X, Y, Z be a solution of the equation*

$$X^4 + Y^4 = 2Z^4$$

in non-negative integers. Then

$$X = Y = Z .$$

To prove Theorem 16,we shall recall the following Lemma 15. This lemma is slightly stronger than, and implies Fermat's last theorem for $n = 4$ (see [6]).

Lemma 15. *Let m be a non-negative integer. Then the equation*

$$X^4 + 2^m Y^4 = Z^4$$

has no solutions in odd integers X, Y, Z.

Theorem 16. *Let a, b, c be non-negative integers. If X, Y, Z is a solution of the equation*

$$2^a X^4 + 2^b Y^4 = 2^c Z^4$$

in positive odd integers, then

$$X = Y = Z \text{ and } a + 1 = b + 1 = c.$$

Proof. Let a, b and c be non-negative integers. Let X, Y, Z be the solution of the equation

$$2^a X^4 + 2^b Y^4 = 2^c Z^4$$

in positive odd integersX, Y, Z.

We shall first show that $a = b$. If $a \neq b$, then, without loss of generality, we may assume that $a < b$. Set $b = a + m$. Consequently we obtain that $c = a$ and

$$X^4 + 2^m Y^4 = Z^4,$$

where X, Y and Z are positive odd integers, and m is a positive integer. By Lemma 15, this equation is imposible. Thus $a = b$.

It follows from $a = b$ that $c = a + 1$ and

$$X^4 + Y^4 = 2Z^4$$

with positive odd integers X, Y, Z. Hence, according to Corollary 14, we have $X = Y = Z$. This completes the proof.

5. The equation $X^4 + 2^m Y^2 = Z^4$ has no solutions in nonzero integers

Let m be a non-negative integer. In this section, we shall prove that the equation $X^4 + 2^m Y^2 = Z^4$ has no solutions in nonzero integers X, Y, Z.

Lemma 17. *Let m be a non-negative integer. If a set of three odd integers X, Y, Z satisfies the equation*

$$X^4 + 2^m Y^4 = Z^2,$$

then $m \geqq 3$ and $m \equiv -1 \pmod 4$.

Proof. Since the square of an odd integer is congruent to 1 modulo 8, we have $2^m Y^4 = Z^2 - X^4 \equiv 1 - 1 = 0 \pmod 8$. This implies $m \geqq 3$.

We suppose that there is a set of four integers X, Y, Z, m satisfying $X^4 + 2^m Y^4 = Z^2$ with X, Y, Z odd, $m > 3$ and $m \not\equiv -1 \pmod 4$, and we assume that the set of positive integers x, y, z, m satisfying $x^4 + 2^m y^4 = z^2$ with x, y, z odd, $m > 3$ and $m \not\equiv -1 \pmod 4$, is such that m is least. Canceling the greatest common divisor of x^4 and y^4, we may assume that x, y, z are pairwise relatively prime. We have $2^m y^4 = z^2 - x^4 = (z + x^2)(z - x^2)$, and since z, x are both odd integers and relatively prime, we have $(z + x^2, z - x^2) = 2$. Hence there exist positive odd integers a and b with $(a, b) = 1$ such that

$$\text{(I) } z + x^2 = 2a^4, \ z - x^2 = 2^{m-1} b^4$$

or

$$\text{(II) } z + x^2 = 2^{m-1} b^4, \ z - x^2 = 2a^4.$$

In the case of (I) $z + x^2 = 2a^4$ and $z - x^2 = 2^{m-1} b^4$, we obtain $x^2 = a^4 - 2^{m-2} b^4$, $2^{m-2} b^4 = a^4 - x^2 = (a^2 + x)(a^2 - x)$, $m - 2 \geqq 3$, and so $m \geqq 5$. Also note that a and x both odd integers and relatively prime and $(a^2 + x, a^2 - x) = 2$. Hence there exist positive odd integers A and B with $(A, B) = 1$ such that

$$a^2 + x = 2A^4, \ a^2 - x = 2^{m-3} B^4$$

or

$$a^2 + x = 2^{m-3} B^4, \ a^2 - x = 2A^4.$$

Thus, we obtain $a^2 = A^4 + 2^{m-4} B^4$, where a, A, B are odd integers. Hence, by Lemma 5, we obtain that $m - 4 \geqq 3$, and since $m \not\equiv -1 \pmod 4$, we have $m > 7$. Further $m - 4 < m$ and $m - 4 \equiv m \not\equiv -1 \pmod 4$. This contradicts the choice of m.

In the case of (II) $z + x^2 = 2^{m-1}b^4$ and $z - x^2 = 2a^4$, we obtain $x^2 = 2^{m-2}b^4 - a^4$. Since $2^{m-2}b^4 = x^2 + a^4 \equiv 1 + 1 = 2 \pmod 4$, we have $m - 2 = 1$, so $m = 3$. This contradicts the choice of m. Hence the lemma is proved.

Lemma 18. *Let m be a non-negative integer. If a set of three odd integers X, Y, Z satisfies the equation*

$$X^2 + 2^m Y^4 = Z^4,$$

then $m \geqq 5$ and $m \equiv 1 \pmod 4$.

Proof. Let m be a non-negative integer. Let X, Y, Z be a solution of the equation $X^2 + 2^m Y^4 = Z^4$ in odd integers X, Y, Z. Hence, by Lemma 5, we have $m \geqq 3$ and

$$(X^2)^2 = (Z^4 - 2^m Y^4)^2 = (Z^4 + 2^m Y^4)^2 - 2^{m+2} Y^4 Z^4,$$

and so

$$X^4 + 2^{m+2}(YZ)^4 = (Z^4 + 2^m Y^4)^2,$$

where $X, YZ, Z^4 + 2^m Y^4$ are odd integers. By Lemma 17, we have $m + 2 \equiv -1 \pmod 4$, so $m \equiv 1 \pmod 4$. Also we note $m \geqq 5$. This completes the proof.

Theorem 19. *Let m be a non-negative integer. Then the equation*

$$X^4 + 2^m Y^2 = Z^4$$

has no solutions in odd integers X, Y, Z.

Proof. Let m be a non-negative integer. Suppose that there is a solution X, Y, Z of the equation $X^4 + 2^m Y^2 = Z^4$ in odd integers X, Y, Z. Hence, by Lemma 5, we have $m \geqq 3$ and

$$(2^m Y^2)^2 = (Z^4 - X^4)^2 = (Z^4 + X^4)^2 - 4X^4 Z^4.$$

Since X, Z are both odd integers, so is $\dfrac{X^4 + Z^4}{2}$, and we obtain

$$(XZ)^4 + 2^{2m-2} Y^4 = \left(\frac{X^4 + Z^4}{2}\right)^2$$

where $XZ, Y, \dfrac{X^4 + Z^4}{2}$ are odd integers and $2m - 2 \not\equiv -1 \pmod 4$. By Lemma 17, the last equation is impossible. Hence the theorem is proved.

Remark. It is shown that let m be a non-negative integer, then the equation $X^4 + 2^m Y^2 = Z^4$ has no solutions in nonzero integers X, Y, Z (see [7]).

6. On the Diophantine Equations $X^4 + 2^m Y^4 = Z^2$ and $X^2 + 2^m Y^4 = Z^4$

In this section, we shall give ono-to-one correspondences between solutions of the equation $x^2 + y^4 = 2z^4$ and of $x^4 + 2^3 \cdot y^4 = z^2$, between solutions of the equation $x^4 + 2^{4a-1} \cdot y^4 = z^2$ and of $x^2 + 2^{4a+1} \cdot y^4 = z^4$, and between solutions of the equation $x^2 + 2^{4a+1} \cdot y^4 = z^4$ and of $x^4 + 2^{4a+3} \cdot y^4 = z^2$.

Theorem 20. *Let x, y, z be a solution of the equation*

$$x^2 + y^4 = 2z^4 \cdots\cdots\cdots\cdots\cdots\cdots \circledast$$

in positive odd integers x, y, z which are pairwise relatively prime.
Set

$$\begin{aligned} U &= yz \quad , \\ V &= y^4 + 2z^4 \quad . \end{aligned}$$

Then x, U, V is a solution of the equation

$$x^4 + 2^3 \cdot U^4 = V^2$$

in positive odd integers x, U, V which are pairwise relatively prime.

Conversely, let x, U, V be a solution of the equation

$$x^4 + 2^3 \cdot U^4 = V^2 \cdots\cdots\cdots\cdots\cdots\cdots (*\text{-}3)$$

in positive odd integers x, U, V which are pairwise relatively prime. Then there exist unique positive odd integers y, z with $(y, z) = 1$ such that

$$\begin{aligned} y^4 &= \frac{V - x^2}{2} \quad , \\ z^4 &= \frac{V + x^2}{4} \end{aligned}$$

and

$$x^2 + y^4 = 2z^4 .$$

Furthermore, above two correspondences $\circledast \longrightarrow (*\text{-}3)$ and $(*\text{-}3) \longrightarrow \circledast$ are mutual inverses.

Proof. Let x, y, z be a solution of the equation $\circledast$ in positive odd integers x, y, z which are pairwise relatively prime. We set $U = yz$ and $V = y^4 + 2z^4$. Then we have $x^4 + 2^3 \cdot U^4 = \left(2z^4 - y^4\right)^2 + 2^3 \cdot (yz)^4 = \left(2z^4 + y^4\right)^2 = V^2$. Since $(x, y) = (x, z) = 1$, we have $(x, yz) = 1$. Also we note that x, U, V are positive odd integers and pairwise relatively prime.

Conversely, let x, U, V be a solution of the equation $(*{-}3)$ in positive odd integers x, U, V which are pairwise relatively prime. Since $2^3 \cdot U^4 = V^2 - x^4 = (V + x^2)(V - x^2)$ and $(V + x^2, V - x^2) = 2$, there exist positive odd integers y, z with $(y, z) = 1$ such that

$$\text{(I) } V + x^2 = 2y^4\,,\ V - x^2 = 4z^4$$

or

$$\text{(II) } V + x^2 = 4z^4\,,\ V - x^2 = 2y^4\,.$$

Suppose that (I) $V + x^2 = 2y^4$ and $V - x^2 = 4z^4$, we have $x^2 = y^4 - 2z^4$ or $x^2 + 2z^4 = y^4$ with positive odd integers x, y, z. But by Lemma 5, the last equation is impossible. Thus we obtain (II) $y^4 = \dfrac{V - x^2}{2}$, $z^4 = \dfrac{V + x^2}{4}$, and $x^2 = 2z^4 - y^4$. Hence $x^2 + y^4 = 2z^4$. Also we note that x, y, z are positive odd integers and pairwise relatively prime.

Furthermore, we can prove that if $x^2 + y^4 = 2z^4$, $U = yz$ and $V = y^4 + 2z^4$, then $\dfrac{V - x^2}{2} = \dfrac{y^4 + 2z^4 - x^2}{2} = \dfrac{y^4 + y^4}{2} = y^4$ and $\dfrac{V + x^2}{4} = \dfrac{y^4 + 2z^4 + x^2}{4} = \dfrac{2z^4 + 2z^4}{4} = z^4$, and that if $x^4 + 2^3 \cdot U^4 = V^2$, $y^4 = \dfrac{V - x^2}{2}$ and $z^4 = \dfrac{V + x^2}{4}$, then $(yz)^4 = \dfrac{V - x^2}{2} \cdot \dfrac{V + x^2}{4} = \dfrac{V^2 - x^4}{8} = \dfrac{2^3 \cdot U^4}{8} = U^4$ and $y^4 + 2z^4 = \dfrac{V - x^2}{2} + 2 \cdot \dfrac{V + x^2}{4} = V$. Hence this shows that above two correspondences are mutual inverses. And the proof is complete.

Theorem 21. *Let a be a positive integer. Let x, y, z be a solution of the equation*

$$x^4 + 2^{4a-1} \cdot y^4 = z^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (*{-}4a - 1)$$

in positive odd integers x, y, z which are pairwise relatively prime.
Set

$$\begin{aligned} U &= |x^4 - 2^{4a-1} \cdot y^4|\,, \\ V &= xy\,. \end{aligned}$$

Then U, V, z is a solution of the equation

$$U^2 + 2^{4a+1} \cdot V^4 = z^4$$

in positive odd integers U, V, z which are pairwise relatively prime.

Conversely, let U, V, z be a solution of the equation

$$U^2 + 2^{4a+1} \cdot V^4 = z^4 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (*{-}4a + 1)$$

in positive odd integers U, V, z which are pairwiae relatively prime. Then there exist unique positive odd integers x, y with $(x, y) = 1$ such that

$$x^4 = \frac{z^2+U}{2}, \; y^4 = \frac{z^2-U}{2^{4a}} \quad \text{if} \quad U \equiv 1 \ (mod\ 4)$$

or

$$x^4 = \frac{z^2-U}{2}, \; y^4 = \frac{z^2+U}{2^{4a}} \quad \text{if} \quad U \equiv -1 \ (mod\ 4)$$

and

$$x^4 + 2^{4a-1} \cdot y^4 = z^2 .$$

Furthermore, above two correspondences $(*\text{-}4a-1) \to (*\text{-}4a+1)$ and $(*\text{-}4a+1) \to (*\text{-}4a-1)$ are mutual inverses.

Proof. Let x, y, z be a solution of the equation $(*\text{-}4a-1)$ in positive odd integers x, y, z which are pairwise relatively prime. We set $U = |x^4 - 2^{4a-1} \cdot y^4|$ and $V = xy$. Then we have $U^2 + 2^{4a+1} \cdot V^4 = \left(x^4 - 2^{4a-1} \cdot y^4\right)^2 + 2^{4a+1} \cdot (xy)^4 = \left(x^4 + 2^{4a-1} \cdot y^4\right)^2 = z^4$. And U, V, z are positive odd integers and pairwise relatively prime.

Conversely, let U, V, z be a solution of the equation $(*\text{-}4a+1)$ in positive odd integers U, V, z which are pairwise relatively prime. Since $2^{4a+1} \cdot V^4 = z^4 - U^2 = (z^2+U)(z^2-U)$ and $(z^2+U, z^2-U) = 2$, there exist positive odd integers x, y with $(x, y) = 1$ such that

$$z^2 + U = 2x^4, \; z^2 - U = 2^{4a} \cdot y^4 \quad \text{if} \quad U \equiv 1 \ (mod\ 4)$$

or

$$z^2 + U = 2^{4a} \cdot y^4, \; z^2 - U = 2x^4 \quad \text{if} \quad U \equiv -1 \ (mod\ 4).$$

Thus, we obtain $z^2 = x^4 + 2^{4a-1} \cdot y^4$.

Furthermore, note that $x^4 > 2^{4a-1} \cdot y^4$ if and only if $U \equiv 1 \ (mod\ 4)$, and it is easily proved that above two correspondences are mutual inverses. This completes the proof.

Theorem 22. *Let a be a positive integer. Let x, y, z be a solution of the equation*

$$x^2 + 2^{4a+1} \cdot y^4 = z^4 \quad \cdots\cdots\cdots\cdots\cdots \quad (*\text{-}4a+1)$$

in positive odd integers x, y, z which are pairwise relatively prime.
Set

$$\begin{array}{rcl} U & = & yz \quad , \\ V & = & z^4 + 2^{4a+1} \cdot y^4 \quad . \end{array}$$

Then x, U, V is a solution of the equation

$$x^4 + 2^{4a+3} \cdot U^4 = V^2$$

in positive odd integers x, U, V which are pairwise relatively prime.

Conversely, let x, U, V be a solution of the equation

$$x^4 + 2^{4a+3} \cdot U^4 = V^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (*\text{-}4a+3)$$

in positive odd integers x, U, V which are pairwise relatively prime. Then there exist unique positive odd integers y, z with $(y, z) = 1$ such that

$$\begin{aligned} y^4 &= \frac{V - x^2}{2^{4a+2}}, \\ z^4 &= \frac{V + x^2}{2} \end{aligned}$$

and

$$x^2 + 2^{4a+1} \cdot y^4 = z^4 .$$

Furthermore, above two correspondences $(*\text{-}4a+1) \to (*\text{-}4a+3)$ and $(*\text{-}4a+3) \to (*\text{-}4a+1)$ are mutual inverses.

Proof. Let x, y, z be a solution of the equation $(*\text{-}4a+1)$ in positive odd integers x, y, z which are pairwise relatively prime. We set $U = yz$ and $V = z^4 + 2^{4a+1} \cdot y^4$. Then we have $x^4 + 2^{4a+3} \cdot U^4 = \left(z^4 - 2^{4a+1} \cdot y^4\right)^2 + 2^{4a+3} \cdot (yz)^4 = \left(z^4 + 2^{4a+1} \cdot y^4\right)^2 = V^2$. And x, U, V are positive odd integers and pairwise relatively prime.

Conversely, let x, U, V be a solution of the equation $(*\text{-}4a+3)$ in positive odd integers x, U, V which are pairwise relatively prime. Since $2^{4a+3} \cdot U^4 = V^2 - x^4 = (V + x^2)(V - x^2)$ and $(V + x^2, V - x^2) = 2$, there exist odd integers y, z with $(y, z) = 1$ such that

$$\text{(I) } V + x^2 = 2z^4 , \quad V - x^2 = 2^{4a+2} \cdot y^4$$

or

$$\text{(II) } V + x^2 = 2^{4a+2} \cdot y^4 , \quad V - x^2 = 2z^4 .$$

Suppose that (II) $V + x^2 = 2^{4a+2} \cdot y^4$ and $V - x^2 = 2z^4$, we have $x^2 = 2^{4a+1} \cdot y^4 - z^4$, or $x^2 + z^4 = 2^{4a+1} \cdot y^4$ with positive odd integers x, y, z. But by Lemma 4, the last equation is impossible, since $a > 0$. Thus we obtain (I) $y^4 = \dfrac{V - x^2}{2^{4a+2}}$ and $z^4 = \dfrac{V + x^2}{2}$, and $x^2 = z^4 - 2^{4a+1} \cdot y^4$. Hence $x^2 + 2^{4a+1} \cdot y^4 = z^4$.

Furthermore, it is easily shown that above two correspondences are mutual inverses. This completes the proof.

Example 2. Since $1^2 + 1^4 = 2 \cdot 1^4$, applying Theorem 20, Theorem 21 and Theorem 22, we obtain the following correspondences.

$$\begin{array}{rcrcrl}
1^2 & + & 1^4 & = & 2 \cdot 1^4 & \cdots\cdots\cdots ① \\
 & & & \updownarrow & & \\
1^4 & + & 2^3 \cdot 1^4 & = & 3^2 & \cdots\cdots\cdots (1\text{–}3) \\
 & & & \updownarrow & & \\
7^2 & + & 2^5 \cdot 1^4 & = & 3^4 & \cdots\cdots\cdots (1\text{–}5) \\
 & & & \updownarrow & & \\
7^4 & + & 2^7 \cdot 3^4 & = & 113^2 & \cdots\cdots\cdots (1\text{–}7) \\
 & & & \updownarrow & & \\
7967^2 & + & 2^9 \cdot 21^4 & = & 113^4 & \cdots\cdots\cdots (1\text{–}9) \\
 & & & \updownarrow & & \\
7967^4 & + & 2^{11} \cdot 2373^4 & = & 262621633^2 & \cdots\cdots\cdots (1\text{–}11) \\
 & & & \updownarrow & & \\
60912456065182847^2 & + & 2^{13} \cdot 18905691^4 & = & 262621633^4 & \cdots\cdots\cdots (1\text{–}13)
\end{array}$$

Similarly, from $239^2 + 1^4 = 2 \cdot 13^4$, we obtain that

$$\begin{array}{rcrcrl}
239^2 & + & 1^4 & = & 2 \cdot 13^4 & \cdots\cdots ② \\
 & & & \updownarrow & & \\
239^4 & + & 2^3 \cdot 13^4 & = & 57123^2 & \cdots\cdots (2\text{–}3) \\
 & & & \updownarrow & & \\
3262580153^2 & + & 2^5 \cdot 3107^4 & = & 57123^4 & \cdots\cdots (2\text{–}5) \\
 & & & \updownarrow & & \\
3262580153^4 & + & 2^7 \cdot 177481161^4 & = & 1065039335715621873^2 & \cdots\cdots (2\text{–}7).
\end{array}$$

Example 3. Similarly, we obtain that

$$\begin{array}{rrcrl}
2750257^2 + & 1343^4 & = & 2 \cdot 1525^4 & \cdots ③ \\
 & & \updownarrow & & \\
2750257^4 + & 2^3 \cdot 2048075^4 & = & 14070212996451^2 & \cdots (3\text{–}3) \\
 & & \updownarrow & & \\
835453168961784283676545 99^2 + & 2^5 \cdot 5632732605275^4 & = & 14070212996451^4 & \cdots (3\text{–}5) \\
 & & \updownarrow & &
\end{array}$$

$$835453168961784283676545 99^4 + 2^7 \cdot 7925374750827360555887902 5^4$$
$$= 7140512958133781039961302579465950399610603419085080 1^2 \cdots (3\text{–}7)$$

We have

$x^2 + y^4 = 2z^4$		①	②	③	······
$x^4 + 2^3 \cdot y^4 = z^2$	:	(1–3)	(2–3)	(3–3)	⋮
$x^2 + 2^5 \cdot y^4 = z^4$	:	(1–5)	(2–5)	(3–5)	⋮
$x^4 + 2^7 \cdot y^4 = z^2$	:	(1–7)	(2–7)	(3–7)	⋮
$x^2 + 2^9 \cdot y^4 = z^4$	:	(1–9)	⋮	⋮	⋮
$x^4 + 2^{11} \cdot y^4 = z^2$	:	(1–11)	⋮	⋮	⋮
$x^2 + 2^{13} \cdot y^4 = z^4$	:	(1–13)	⋮	⋮	⋮
⋮		⋮	⋮	⋮	⋮
⋮		⋮	⋮	⋮	⋮

7. All Solutions of the Diophantine Equation $X^2 + Y^4 = 2Z^4$

Let m be a non-negative integer and let x, y, z be positive odd integers. We shall determine the solutions of the Diophantine equation $x^2 + y^4 = 2z^4$. Finally, in this section, we shall show that the equation $1^2 + 1^4 = 2 \cdot 1^4$ induces all solutions of the equation $x^2 + y^4 = 2z^4$, of the equation $x^4 + 2^m \cdot y^4 = z^2$ and of the equation $x^2 + 2^m \cdot y^4 = z^4$(see L.J.Mordell [4]), and that above three Diophantine equations $x^2 + y^4 = 2z^4, x^4 + 2^m \cdot y^4 = z^2, x^2 + 2^m \cdot y^4 = z^4$ have infinite solutions.

Lemma 23. *Let x, y, z be a solution of the equation*

$$x^2 + y^4 = 2z^4$$

with positive odd integers x, y, z which are pairwise relatively prime. And a set of four integers c, d, a, b satisfies the conditions :

$$\begin{gathered} c^2 - d^2 = (a+b)^2 + b^2\,, \\ c > 0,\ a > 0,\ (c, d) = 1,\ (a, b) = 1,\ cd = ab, \\ c \textit{ and } a \textit{ odd}, d \textit{ and } b \textit{ even}, \\ y = |a^2 - 2b^2|\,,\ z = c^2 + d^2\,. \end{gathered}$$

Set $u = c^2 - d^2,\ v = 2cd,\ s = u^2 - v^2,\ t = 2uv$. Then we have

$$\begin{array}{llll} d = 0 & \textit{if and only if} & z = 1 & , \\ d > 0 & \textit{if and only if} & x \equiv 1 \pmod 4 \quad \textit{and} \quad x > y^2 & , \\ d < 0 & \textit{if and only if} & x \equiv -1 \pmod 4 \quad \textit{or} \quad x < y^2 & , \end{array}$$

$$s^2+t^2=z^4\,,\ 4st=x^2-y^4\,,\ x=|s+t|$$

and

$$2a^2=u-v+y\,,\ 4b^2=u-v-y\ \ \textit{if and only if}\ \ y\equiv 1\ (\mathit{mod}\,4)\ ,$$
$$2a^2=u-v-y\,,\ 4b^2=u-v+y\ \ \textit{if and only if}\ \ y\equiv -1\ (\mathit{mod}\,4)\,.$$

Proof. First, we notice that $\left(c^2+d^2\right)^2 = u^2+v^2$, $\left(u^2+v^2\right)^2 = s^2+t^2$ and $y^2 = \left(a^2-2b^2\right)^2 = \left(a^2+2b^2\right)^2 - 8a^2b^2 = \left(c^2-d^2-2ab\right)^2 - 8(ab)^2 = \left(c^2-d^2-2cd\right)^2 - 8(cd)^2 = (u-v)^2-2v^2 = u^2-v^2-2uv = s-t > 0$. So we obtain $z^4 = \left(c^2+d^2\right)^4 = \left(u^2+v^2\right)^2 = s^2+t^2$. Since $x^2 = 2z^4-y^4 = 2(s^2+t^2)-(s-t)^2 = (s+t)^2$, we have $x=|s+t|$. Also, $x^2-y^4 = (s+t)^2-(s-t)^2 = 4st$. It is easily proved that $z=1$ if and only if $x=y=z=1$, and if and only if $c=a=1$, $d=b=0$, and if and only if $d=0$.

Next, we shall show that $d>0$ if and only if $x\equiv 1\ (\mathit{mod}\ 4)$ and $x>y^2$. We note that $u=c^2-d^2=(a+b)^2+b^2>0$. If $d>0$, then $v=2cd>0$ and $t=2uv>0$. Since $s-t>0$ and $t>0$, so $s>0$. Since $x^2-y^4=4st>0$, we have $x>y^2$. Further, $x=|s+t|=s+t=u^2-v^2+2uv\equiv 1\ (\mathit{mod}\ 4)$. Conversely, suppose that $x\equiv 1\ (\mathit{mod}\,4)$ and $x>y^2$. Since $x=|s+t|$ and $s+t\equiv 1\ (\mathit{mod}\,4)$, we have $x=s+t$, so $s+t>0$. And $4st=x^2-y^4>0$, so we have $s>0$ and $t>0$. Since $u=c^2-d^2>0$ and $t=2uv>0$, we have $v>0$. Thus $d>0$. Also we note that $d<0$ if and only if $x\equiv -1\ (\mathit{mod}\,4)$ or $x<y^2$.

Finally, we have $2a^2+4b^2 = 2\left(a^2+2b^2\right) = 2\left(c^2-d^2-2ab\right) = 2(u-v) = (u-v+y)+(u-v-y)$ and $\left(2a^2\right)\left(4b^2\right) = \left(a^2+2b^2\right)^2 - \left(a^2-2b^2\right)^2 = \left(c^2-d^2-2ab\right)^2 - y^2 = (u-v)^2-y^2 = (u-v+y)(u-v-y)$. We notice that $2a^2\equiv 2\ (\mathit{mod}\,4)$, $4b^2\equiv 0\ (\mathit{mod}\ 4)$ and $u-v=c^2-d^2-2cd\equiv 1\ (\mathit{mod}\ 4)$. And since $y\equiv 1\ (\mathit{mod}\ 4)$ if and only if $u-v+y\equiv 2\ (\mathit{mod}\ 4)$, $u-v-y\equiv 0\ (\mathit{mod}\ 4)$, we have $2a^2 = u-v+y$, $4b^2 = u-v-y$ if and only if $y\equiv 1\ (\mathit{mod}\ 4)$. Similarly, we have $2a^2=u-v-y$, $4b^2=u-v+y$ if and only if $y\equiv -1\ (\mathit{mod}\,4)$. Hence the proof is complete.

Lemma 24. *Let x,y,z be a solution of the equation*

$$x^2+y^4=2z^4$$

with positive odd integers x,y,z which are pairwise relatively prime. And a set of four integers c,d,a,b satisfies the conditions :

$$c^2-d^2=(a+b)^2+b^2\,,$$
$$c>0\,,\ a>0\,,\ (c,d)=1\,,\ (a,b)=1\,,\ cd=ab\,,$$
$$c\ \textit{and}\ a\ \textit{odd},\ d\ \textit{and}\ b\ \textit{even},$$
$$y=|a^2-2b^2|\,,\ z=c^2+d^2\,.$$

And a set of four integers c', d', a', b' satisfies the conditions :

$$c'^2 - d'^2 = (a' + b')^2 + b'^2 ,$$
$$c' > 0 ,\ a' > 0 ,\ (c', d') = 1 ,\ (a', b') = 1 ,\ c'd' = a'b' ,$$
$$c' \text{ and } a' \text{ odd, } d' \text{ and } b' \text{ even,}$$
$$y = |a'^2 - 2b'^2| ,\ z = c'^2 + d'^2 .$$

Then we have that $c = c'$, $d = d'$, $a = a'$ and $b = b'$.

Proof. We set $u = c^2 - d^2$, $v = 2cd$, $s = u^2 - v^2$, $t = 2uv$ and $u' = c'^2 - d'^2$, $v' = 2c'd'$, $s' = u'^2 - v'^2$, $t' = 2u'v'$. Then by Lemma 23, $s^2 + t^2 = z^4 = s'^2 + t'^2$ and $s^2 \cdot t^2 = \left(\frac{x^2 - y^4}{4}\right)^2 = s'^2 \cdot t'^2$. Since s and s' are odd, t and t' even, so we have $s^2 = s'^2$ and $t^2 = t'^2$. Since $\left(u^2 + v^2\right)^2 = \left(u^2 - v^2\right)^2 + (2uv)^2 = s^2 + t^2 = s'^2 + t'^2 = \left(u'^2 + v'^2\right)^2$ and $u^2 \cdot v^2 = \left(\frac{t}{2}\right)^2 = \left(\frac{t'}{2}\right)^2 = u'^2 \cdot v'^2$, u and u' odd, v and v' even, we have $u^2 = u'^2$ and $v^2 = v'^2$. Similarly, since $c^2 + d^2 = z = c'^2 + d'^2$, $c^2 \cdot d^2 = \left(\frac{v}{2}\right)^2 = \left(\frac{v'}{2}\right)^2 = c'^2 \cdot d'^2$, c and c' odd, d and d' even, we have $c^2 = c'^2$ and $d^2 = d'^2$. Further, c and c' are positive, $dd' \geqq 0$, so we have $c = c'$ and $d = d'$. Hence $u = u'$ and $v = v'$. Also by Lemma 23, if $y \equiv 1\ (mod\ 4)$, then

$$2a^2 = u - v + y = u' - v' + y = 2a'^2$$

and

$$2b^2 = u - v - y = u' - v' - y = 2b'^2 .$$

Similarly, when $y \equiv -1\ (mod\ 4)$, we have $a^2 = a'^2$ and $b^2 = b'^2$. Further, a and a' are positive, $bb' \geqq 0$, so we have $a = a'$ and $b = b'$. Hence the proof is complete.

Theorem 25. *Let x, y, z be a solution of the equation*

$$x^2 + y^4 = 2z^4 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (*)$$

with positive odd integers x, y, z which are pairwise relatively prime. Then there exist unique integers c, d, a, b such that

$$c > 0,\ a > 0,\ (c, d) = 1,\ (a, b) = 1,\ cd = ab ,$$
$$c \text{ and } a \text{ odd, } d \text{ and } b \text{ even,}$$
$$y = |a^2 - 2b^2| ,\ z = c^2 + d^2$$

and

$$c^2 - d^2 = (a + b)^2 + b^2 .$$

Conversely, let c, d, a, b be a solution of the equation

$$c^2 - d^2 = (a+b)^2 + b^2 \cdots\cdots\cdots\cdots\cdots\cdots \boxed{*}$$

with integers c, d, a, b *such that*

$$c > 0,\ a > 0,\ (c,d) = 1,\ (a,b) = 1,\ cd = ab\,,$$
$$c \text{ and } a \text{ odd, } d \text{ and } b \text{ even.}$$

Then there exist unique positive odd integers x, y, z *with pairwise relatively prime such that*

$$y = |a^2 - 2b^2|\,,\ z = c^2 + d^2$$

and

$$x^2 + y^4 = 2z^4\,.$$

Furthermore, above two correspondences $\circledast \to \boxed{*}$ *and* $\boxed{*} \to \circledast$ *are mutual inverses.*

Proof. Let x, y, z be a solution of the equation $\circledast$ with positive odd integers x, y, z which are pairwise relatively prime. When $x = y$, we have $x = y = z = 1$, and we set $c = a = 1$ and $d = b = 0$. Thus we suppose that $x \neq y$. Since $x^2 + y^4 = 2z^4$ and x, y, z are odd integers with pairwise relatively prime, we have $\left(\dfrac{x+y^2}{2}\right)^2 + \left(\dfrac{x-y^2}{2}\right)^2 = z^4$ where $\dfrac{x+y^2}{2}, \dfrac{x-y^2}{2}$ are integers of opposite parity, and $\dfrac{x+y^2}{2}, \dfrac{x-y^2}{2}, z$ are pairwise relatively prime.

In the case of (I) $x \equiv 1 \ (mod\ 4)$ and $x > y^2$, by Theorem 1, there exist positive integers U and V of opposite parity with $(U, V) = 1$ and $U > V > 0$ such that $\dfrac{x+y^2}{2} = U^2 - V^2$, $\dfrac{x-y^2}{2} = 2UV$ and $z^2 = U^2 + V^2$. Since $y^2 = U^2 - V^2 - 2UV$, U is odd and V even. Applying Theorem 1 again, there exist positive integers c and d of opposite parity with $(c, d) = 1$ and $c > d > 0$ such that $U = c^2 - d^2$, $V = 2cd$, and $z = c^2 + d^2$. We set $u = c^2 - d^2 = U$ and $v = 2cd = V$. Since $u = U > V = v > 0$, we have $u - v > 0$. Hence we obtain $y^2 = u^2 - v^2 - 2uv$ where u is odd, v even, $u - v > 0$ and u, v, y are pairwise relatively prime.

In the case of (II) $x \equiv 1 \ (mod\ 4)$ and $x < y^2$, by Theorem 1, there exist positive integers U and V of opposite parity with $(U, V) = 1$ and $U > V > 0$ such that $\dfrac{x+y^2}{2} = U^2 - V^2$, $\dfrac{y^2-x}{2} = 2UV$ and $z^2 = U^2 + V^2$. Since $y^2 = U^2 - V^2 + 2UV$, U is odd and V even. Applying Theorem 1 again, there exist positive integers c and negative integer d of opposite parity with $(c, d) = 1$ and $c > -d > 0$ such that $U = c^2 - d^2$, $V = -2cd$ and $z = c^2 + d^2$. We set $u = c^2 - d^2 = U$ and $v = 2cd = -V$. Since $u = U > V = -v > 0$, we have $u - v > 0$. Hence we obtain $y^2 = u^2 - v^2 - 2uv$ where u is odd, v even, $u - v > 0$ and u, v, y are pairwise relatively prime.

In the case of (III) $x \equiv -1 \ (mod\ 4)$ and $x > y^2$, by Theorem 1, there exist positive integers U and V of opposite parity with $(U, V) = 1$ and $U > V > 0$ such that $\dfrac{x - y^2}{2} = U^2 - V^2$, $\dfrac{x + y^2}{2} = 2UV$ and $z^2 = U^2 + V^2$. Since $y^2 = V^2 - U^2 + 2UV$, V is odd and U even. Applying Theorem 1 again, there exist positive integer c and negative integer d of opposite parity with $(c, d) = 1$ and $c > -d > 0$ such that $V = c^2 - d^2$, $U = -2cd$ and $z = c^2 + d^2$. We set $u = c^2 - d^2 = V$ and $v = 2cd = -U$. Since $-v = U > V = u > 0$, we have $u - v > 0$. Hence we obtain $y^2 = u^2 - v^2 - 2uv$ where u is odd, v even, $u - v > 0$ and u, v, y are pairwise relatively prime.

In the case of (IV) $x \equiv -1 \ (mod\ 4)$ and $x < y^2$, by Theorem 1, there exist positive integers U and V of opposite parity with $(U, V) = 1$ and $U > V > 0$ such that $\dfrac{y^2 - x}{2} = U^2 - V^2$, $\dfrac{x + y^2}{2} = 2UV$ and $z^2 = U^2 + V^2$. Since $y^2 = U^2 - V^2 + 2UV$, U is odd and V even. Applying Theorem 1 again, there exist positive integer c and negative integer d of opposite parity with $(c, d) = 1$ and $c > -d > 0$ such that $U = c^2 - d^2$, $V = -2cd$ and $z = c^2 + d^2$. We set $u = c^2 - d^2 = U$ and $v = 2cd = -V$. Since $u = U > V = -v > 0$, we have $u - v > 0$. Hence we obtain $y^2 = u^2 - v^2 - 2uv$ where u is odd, v even, $u - v > 0$ and u, v, y are pairwise relatively prime.

In any case of (I), (II), (III) and (IV), we have $2v^2 = (u - v)^2 - y^2 = (u - v + y)(u - v - y)$ and $(u - v + y, u - v - y) = 2$. Hence there exist even integer b and positive odd integer a with $(a, b) = 1$ such that $v = 2ab$ and

$$u - v + y = 2a^2\,,\ u - v - y = 4b^2$$

or

$$u - v + y = 4b^2\,,\ u - v - y = 2a^2\,.$$

Thus we have $u - v = a^2 + 2b^2$ and $y = |a^2 - 2b^2|$. So we obtain $u = c^2 - d^2 = a^2 + 2b^2 + v = a^2 + 2b^2 + 2ab = (a + b)^2 + b^2$. Moreover, $2cd = v = 2ab$, and from $c^2 - d^2 = (a + b)^2 + b^2 \equiv 1 \ (mod\ 4)$, it is shown that c is odd and d even. Furthermore, by Lemma 24, these integers c, d, a and b are uniquely determined.

Conversely, let c, d, a, b be a solution of the equation $\boxed{*}$ with integers c, d, a, b such that $c > 0$, $a > 0$, $(c, d) = 1$, $(a, b) = 1$, $cd = ab$, c and a odd, d and b even. And we set $u = c^2 - d^2$, $v = 2cd$, $s = u^2 - v^2$, $t = 2uv$, $y = |a^2 - 2b^2|$ and $z = c^2 + d^2$. First we show that $y^2 = \left(a^2 - 2b^2\right)^2 = \left(a^2 + 2b^2\right)^2 - 8a^2b^2 = \left(c^2 - d^2 - 2ab\right)^2 - 8(ab)^2 = \left(c^2 - d^2 - 2cd\right)^2 - 8(cd)^2 = (u - v)^2 - 2v^2 = u^2 - v^2 - 2uv = s - t$. Hence, set $x = |s + t|$, we obtain $x^2 + y^4 = (s + t)^2 + (s - t)^2 = 2\left(s^2 + t^2\right) = 2\Big(\left(u^2 - v^2\right)^2 + (2uv)^2\Big) = 2\left(u^2 + v^2\right)^2 = 2\Big(\left(c^2 - d^2\right)^2 + (2cd)^2\Big)^2 = 2\Big(\left(c^2 + d^2\right)^2\Big)^2 = 2\left(c^2 + d^2\right)^4 = 2z^4$. Since $(c, d) = (a, b) = 1$, c and a odd, d and

b even, it is proved that x, y, z are positive odd integers and pairwise relatively prime.

Furthermore, it is easily shown that above two correspondences are mutual inverses. Hence the proof of Theorem 25 is complete.

Example 4. Applying Theorem 25, we obtain the following correspondences.

$$
\begin{array}{rcrcl}
1^2 & + & 1^4 & = & 2\cdot 1^4 \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{①} \\
 & & & \updownarrow & \qquad\qquad (1^2+0^2=1) \\
1^2 & - & 0^2 & = & (1+0)^2+0^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \boxed{1} \\
 & & & & \\
239^2 & + & 1^4 & = & 2\cdot 13^4 \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{②} \\
 & & & \updownarrow & \qquad\qquad (3^2+2^2=13) \\
3^2 & - & 2^2 & = & (3-2)^2+2^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \boxed{2} \\
 & & & & \\
2750257^2 & + & 1343^4 & = & 2\cdot 1525^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{③} \\
 & & & \updownarrow & \qquad\qquad (39^2+2^2=1525) \\
39^2 & - & 2^2 & = & (3+26)^2+26^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \boxed{3}
\end{array}
$$

Figure 1. We have

$$
\begin{array}{lccccc}
\begin{array}{c} c^2-d^2=(a+b)^2+b^2 \\ cd=ab \end{array} & & \boxed{1} & \boxed{2} & \boxed{3} & \cdots\cdots \\
x^2+y^4=2z^4 & & \text{①} & \text{②} & \text{③} & \cdots\cdots \\
x^4+2^3\cdot y^4=z^2 & : & (1\text{–}3) & (2\text{–}3) & (3\text{–}3) & \vdots \\
x^2+2^5\cdot y^4=z^4 & : & (1\text{–}5) & (2\text{–}5) & (3\text{–}5) & \vdots \\
x^4+2^7\cdot y^4=z^2 & : & (1\text{–}7) & (2\text{–}7) & (3\text{–}7) & \vdots \\
x^2+2^9\cdot y^4=z^4 & : & (1\text{–}9) & \vdots & \vdots & \vdots \\
x^4+2^{11}\cdot y^4=z^2 & : & (1\text{–}11) & \vdots & \vdots & \vdots \\
x^2+2^{13}\cdot y^4=z^4 & : & (1\text{–}13) & \vdots & \vdots & \vdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Theorem 26. *Let x, y, z be a solution of the equation*

$$x^2 + y^4 = 2z^4 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (*)$$

with positive odd integers x, y, z which are pairwise relatively prime. Set

$$d_a = y\,,\ c_b = z\,,$$
$$c_{a,1} = \left(-x - yz, y^2 + 2z^2\right)\,,\ c_{a,2} = \left(+x - yz, y^2 + 2z^2\right)\,,$$
$$d_{b,1} = \frac{+x - yz}{c_{a,2}}\,,\ d_{b,2} = \frac{-x - yz}{c_{a,1}}\,,$$
$$\text{(I)}\ C_1 = c_{a,1} \cdot c_b\,,\ D_1 = d_a \cdot d_{b,1}\,,\ A_1 = c_{a,1} \cdot d_a\,,\ B_1 = c_b \cdot d_{b,1}\,,$$
$$\text{(II)}\ C_2 = c_{a,2} \cdot c_b\,,\ D_2 = d_a \cdot d_{b,2}\,,\ A_2 = c_{a,2} \cdot d_a\,,\ B_2 = c_b \cdot d_{b,2}\,.$$

Then

$$C_1,\ D_1,\ A_1,\ B_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \boxed{*\text{–I}}$$

and

$$C_2,\ D_2,\ A_2,\ B_2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \boxed{*\text{–II}}$$

are solutions of the equation

$$C^2 - D^2 = (A + B)^2 + B^2$$

with integers C, D, A, B such that

$$C > 0,\ A > 0,\ (C, D) = 1,\ (A, B) = 1,\ CD = AB\,,$$
$$C \text{ and } A \text{ odd, } D \text{ and } B \text{ even.}$$

And $z < {C_1}^2 + {D_1}^2$ if $z \neq 1$, $z < {C_2}^2 + {D_2}^2$.

Conversely, let C, D, A, B be a solution of the equation

$$C^2 - D^2 = (A + B)^2 + B^2 \quad \cdots\cdots\cdots\cdots\cdots \quad \boxed{*\text{–}*}$$

with integers C, D, A, B such that

$$C > 0,\ A > 0,\ (C, D) = 1,\ (A, B) = 1,\ CD = AB\,,$$
$$C \text{ and } A \text{ odd, } D \text{ and } B \text{ even.}$$

Set

$$c_a = (C, A),\ c_b = (C, B),\ d_a = (D, A),\ d_b = \frac{D}{d_a}\,,$$

$$\begin{aligned} x &= \frac{\left|{d_a}^2 \cdot d_b + 2{c_b}^2 \cdot d_b + c_a \cdot c_b \cdot d_a\right|}{c_a}\,, \\ y &= d_a\,, \\ z &= c_b\,. \end{aligned}$$

Then x, y, z is a solution of the equation

$$x^2 + y^4 = 2z^4$$

with positive odd integers x, y, z which are pairwise relatively prime. And $z < C^2 + D^2$ if $C^2 + D^2 \neq 1$.

Furthermore, $AD + 2BC + AC \neq 0$. And if $AD + 2BC + AC > 0$, then two correspondences $\boxed{*-*} \to \circledast$ and $\circledast \to \boxed{*-\mathrm{I}}$ are mutual inverses. And if $AD + 2BC + AC < 0$, then two correspondences $\boxed{*-*} \to \circledast$ and $\circledast \to \boxed{*-\mathrm{II}}$ are mutual inverses.

Proof. Let x, y, z be a solution of the equation $\circledast$ with positive odd integers x, y, z which are pairwise relatively prime. Set

$$\begin{gathered} d_a = y\,,\ c_b = z\,, \\ c_{a,1} = \left(-x - yz, y^2 + 2z^2\right)\,,\ c_{a,2} = \left(+x - yz, y^2 + 2z^2\right)\,, \\ d_{b,1} = \frac{+x - yz}{c_{a,2}}\,,\ d_{b,2} = \frac{-x - yz}{c_{a,1}}\,, \\ \text{(I)}\ C_1 = c_{a,1} \cdot c_b\,,\ D_1 = d_a \cdot d_{b,1}\,,\ A_1 = c_{a,1} \cdot d_a\,,\ B_1 = c_b \cdot d_{b,1}\,, \\ \text{(II)}\ C_2 = c_{a,2} \cdot c_b\,,\ D_2 = d_a \cdot d_{b,2}\,,\ A_2 = c_{a,2} \cdot d_a\,,\ B_2 = c_b \cdot d_{b,2}\,. \end{gathered}$$

Then it is easily proved that $C_1 > 0$, $A_1 >$, $C_1 \cdot D_1 = A_1 \cdot B_1$, C_1 and A_1 odd, D_1 and B_1 even. And since $(-x - yz, y) = 1$, $(-x - yz, x - yz) = 2$, $(z, y) = 1$ and $(z, x - yz) = 1$, we have $(c_{a,1}, d_a) = (c_{a,1}, d_{b,1}) = (c_b, d_a) = (c_b, d_{b,1}) = 1$, so $(C_1, D_1) = 1$. Similarly, we have $(A_1, B_1) = 1$. Similarly, we obtain that $C_2 > 0$, $A_2 > 0$, $C_2 \cdot D_2 = A_2 \cdot B_2$, C_2 and A_2 odd, D_2 and B_2 even and $(C_2, D_2) = 1$, $(A_2, B_2) = 1$. Further, we note that $(-x - yz)(x - yz) = -x^2 + y^2z^2 = y^4 - 2z^4 + y^2z^2 = \left(y^2 + 2z^2\right)\left(y^2 - z^2\right)$, $(-x - yz, x - yz) = 2$ and $y^2 + 2z^2$ is odd. Then we have $y^2 + 2z^2 = c_{a,1} \cdot c_{a,2}$. Since $y^2 + 2z^2 = c_{a,1} \cdot c_{a,2}$ and $x - yz = c_{a,2} \cdot d_{b,1}$, we have $c_{a,1} \cdot (x - yz) = d_{b,1} \cdot \left(y^2 + 2z^2\right)$, so $c_{a,1}{}^2 \cdot x = c_{a,1} \cdot d_{b,1} \cdot \left(y^2 + 2z^2\right) + c_{a,1}{}^2 \cdot yz = c_{a,1} \cdot d_{b,1} \cdot d_a{}^2 + 2c_{a,1} \cdot d_{b,1} \cdot c_b{}^2 + c_{a,1}{}^2 \cdot d_a \cdot c_b = A_1D_1 + 2B_1C_1 + A_1C_1$. From $\left(c_{a,1}{}^2 \cdot x\right)^2 + (c_{a,1} \cdot y)^4 = 2(c_{a,1} \cdot z)^4$, we obtain $(A_1D_1 + 2B_1C_1 + A_1C_1)^2 + A_1{}^4 = 2C_1{}^4$. Then $(A_1D_1)^2 + 4(B_1C_1)^2 + (A_1C_1)^2 + 4A_1B_1C_1D_1 + 4A_1B_1C_1{}^2 + 2A_1{}^2C_1D_1 + A_1{}^4 - 2C_1{}^4 = A_1{}^4 + 2A_1{}^3B_1 + 2A_1{}^2B_1{}^2 + 2C_1{}^2A_1{}^2 + 4C_1{}^2A_1B_1 + 4C_1{}^2B_1{}^2 - A_1{}^2C_1{}^2 + A_1{}^2D_1{}^2 - 2C_1{}^4 + 2C_1{}^2D_1{}^2 =$

$\left({A_1}^2+2{C_1}^2\right)\left({A_1}^2+2A_1B_1+2{B_1}^2\right)-\left({A_1}^2+2{C_1}^2\right)\left({C_1}^2-{D_1}^2\right)=0$. So we have $(A_1+B_1)^2+{B_1}^2={C_1}^2-{D_1}^2$. Similarly, since $y^2+2z^2=c_{a,1}\cdot c_{a,2}$ and $-x-yz=c_{a,1}\cdot d_{b,2}$, we have $(A_2+B_2)^2+{B_2}^2={C_2}^2-{D_2}^2$. Further, if $z\neq 1$, then $x-yz\neq 0$, so $d_{b,1}\neq 0$, and $D_1\neq 0$. Hence, if $z\neq 1$, then $z=c_b<(c_{a,1}\cdot c_b)^2+{D_1}^2={C_1}^2+{D_1}^2$. And since $-x-yz\neq 0$, we have $D_2\neq 0$ and $Z=c_b<(c_{a,2}\cdot c_b)^2+{D_2}^2={C_2}^2+{D_2}^2$.

Conversely, let C,D,A,B be a solution of the equation $\boxed{*\!-\!*}$ with integers C,D,A,B such that $C>0$, $A>0$, $(C,D)=1$, $(A,B)=1$, $CD=AB$, C and A odd, D and B even. We set

$$c_a=(C,A)\,,\ c_b=(C,B)\,,\ d_a=(D,A)\,,\ d_b=\frac{D}{d_a}\,.$$

Then it is easily proved that $C=c_a\cdot c_b$, $D=d_a\cdot d_b$, $A=c_a\cdot d_a$ and $B=c_b\cdot d_b$. From the equation $\boxed{*\!-\!*}$, we have $\left(A^2+2C^2\right)\left(C^2-D^2\right)=(A^2+2C^2)(A^2+2AB+2B^2)$, so $A^4+2A^3B+2A^2B^2+2C^2A^2+4C^2AB+4C^2B^2-A^2C^2+A^2D^2-2C^4+2C^2D^2=A^2D^2+4B^2C^2+A^2C^2+\left(2A^2B^2+2C^2D^2\right)+4ABC^2+2A^3B+A^4-2C^4=(AD)^2+(2BC)^2+(AC)^2+4ABCD+4ABC^2+2A^2CD+A^4-2C^4=(AD+2BC+AC)^2+A^4-2C^4=0$. Thus we have

$$(AD+2BC+AC)^2+A^4=2C^4\,.$$

This equation implies $AD+2BC+AC\neq 0$ and $(c_a\cdot d_a\cdot d_a\cdot d_b+2c_b\cdot d_b\cdot c_a\cdot c_b+c_a\cdot d_a\cdot c_a\cdot c_b)^2+(c_a\cdot d_a)^4=2(c_a\cdot c_b)^4$. We set $x=\dfrac{|{d_a}^2\cdot d_b+2{c_b}^2\cdot d_b+c_a\cdot c_b\cdot d_a|}{c_a}$, $y=d_a$ and $z=c_b$, then we obtain $x^2+y^4=2z^4$ where x,y,z are positive odd integers. Since $(C,D)=1$, so $(c_b,d_a)=1$. Hence x,y,z are pairwise relatively prime. We note that $C^2+D^2=1$ if and only if $D=0$. Thus if $C^2+D^2\neq 1$, then $z=c_b<(c_a\cdot c_b)^2+D^2=C^2+D^2$.

Furthermore, since $(x-yz)(-x-yz)=\left(y^2-z^2\right)\left(y^2+2z^2\right)$, $(x-yz,-x-yz)=2$ and y^2+2z^2 odd, we have $y^2+2z^2=\left(x-yz,y^2+2z^2\right)\cdot\left(-x-yz,y^2+2z^2\right)$. If $AD+2BC+AC>0$, then ${d_a}^2\cdot d_b+2{c_b}^2\cdot d_b+c_a\cdot c_b\cdot d_a=\dfrac{AD+2BC+AC}{c_a}>$

0. Hence $x = \dfrac{{d_a}^2 \cdot d_b + 2{c_b}^2 \cdot d_b + c_a \cdot c_b \cdot d_a}{c_a} = \dfrac{\left({d_a}^2 + 2{c_b}^2\right) d_b}{c_a} + c_b \cdot d_a$, so $\dfrac{d_b}{c_a} = \dfrac{x - c_b \cdot d_a}{{d_a}^2 + 2{c_b}^2} = \dfrac{x - yz}{y^2 + 2z^2}$. Since $(c_a, d_b) = 1$, the last equation show that $x - yz = d_b \cdot \left(x - yz, y^2 + 2z^2\right)$ and $y^2 + 2z^2 = c_a \cdot \left(x - yz, y^2 + 2z^2\right)$. Hence we obtain that $c_a = \left(-x - yz, y^2 + 2z^2\right)$ and $d_b = \dfrac{x - yz}{\left(x - yz, y^2 + 2z^2\right)}$. Thus we note that if $AD + 2BC + AC > 0$, then two correspondences $\boxed{*\text{–}*} \to \circledast$ and $\circledast \to \boxed{*\text{–I}}$ are mutual inverses.

Similarly, if $AD + 2BC + AC < 0$, then we obtain that $c_a = \left(x - yz, y^2 + 2z^2\right)$ and $d_b = \dfrac{-x - yz}{\left(-x - yz, y^2 + 2z^2\right)}$. And we note that if $AD + 2BC + AC < 0$, then two correspondences $\boxed{*\text{–}*} \to \circledast$ and $\circledast \to \boxed{*\text{–II}}$ are mutual inverses. Hence the proof of Theorem 26 is complete.

Example 5. Applying Theorem 26 to the equation ①, we obtain that

$$
\begin{array}{lrcrcll}
 & 1^2 & + & 1^4 & = & 2 \cdot 1^4 & \cdots\cdots\cdots\cdots\cdots ① \\
 & & & & \updownarrow & & \\
 & 1^2 & - & 0^2 & = & (1+0)^2 + 0^2 & \cdots\cdots \boxed{1\text{–I}} = \boxed{1} \\
\text{and} & & & & & & \\
 & 3^2 & - & 2^2 & = & (3-2)^2 + 2^2 & \cdots\cdots \boxed{1\text{–II}} = \boxed{2}
\end{array}
$$

Similarly, we obtain that

$$
\begin{array}{lrcrcll}
 & 239^2 & + & 1^4 & = & 2 \cdot 13^4 & \cdots\cdots\cdots\cdots\cdots ② \\
 & & & & \updownarrow & & \\
 & 39^2 & - & 2^2 & = & (3+26)^2 + 26^2 & \cdots\cdots \boxed{2\text{–I}} = \boxed{3} \\
\text{and} & & & & & & \\
 & 1469^2 & - & 84^2 & = & (113-1092)^2 + 1092^2 & \cdots\cdots \boxed{2\text{–II}} = \boxed{4}
\end{array}
$$

Example 6. For example, applying Theorem 26 to the equation ③,

$$2750257^2 + 1343^4 = 2 \cdot 1525^4 \cdots\cdots\cdots\cdots\cdots\cdots \text{③}$$

we have $x = 2750257$, $y = 1343$, $z = 1525$. Set $d_a = y = 1343$, $c_b = z = 1525$,

$$\begin{aligned}
c_{a,1} &= (-x - yz, y^2 + 2z^2) &&= (-4798332, 6454899) &&= 75123\,,\\
c_{a,2} &= (+x - yz, y^2 + 2z^2) &&= (\quad 702182, 6454899) &&= 113\,,
\end{aligned}$$

$$\begin{aligned}
d_{b,1} &= \frac{+x - yz}{c_{a,2}} = \frac{702182}{113} = 6214\,,\\
d_{b,2} &= \frac{-x - yz}{c_{a,1}} = \frac{-4798332}{57123} = -84\,,
\end{aligned}$$

$$\begin{array}{lclcrlclcr}
C_1 &=& 57123 \cdot 1525 &=& 87112575\,, & D_1 &=& 1343 \cdot 6214 &=& 8345402\,,\\
A_1 &=& 57123 \cdot 1343 &=& 76716189\,, & B_1 &=& 1525 \cdot 6213 &=& 9476350\,,\\
C_2 &=& 123 \cdot 1525 &=& 172325\,, & D_2 &=& 1343 \cdot (-84) &=& -112812\,,\\
A_2 &=& 123 \cdot 1343 &=& 151759\,, & B_2 &=& 1525 \cdot (-84) &=& -128100\,.
\end{array}$$

Hence, we obtain that

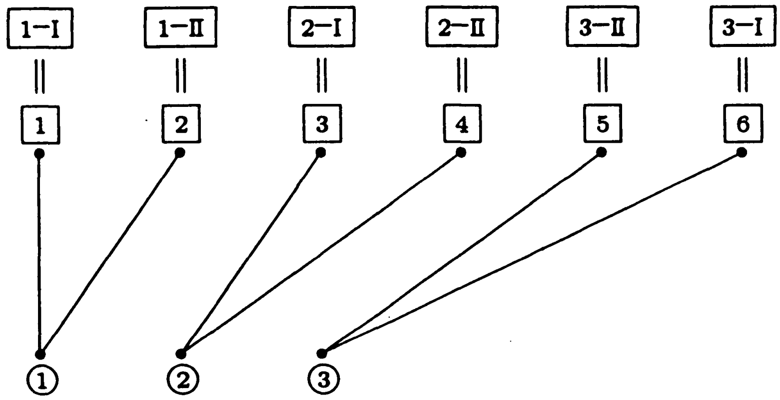
$$\begin{array}{rcrcl}
2750257^2 &+& 1343^4 &=& 2 \cdot 1525^4 \qquad\qquad \cdots\cdots \text{③}\\
&&&\updownarrow&\\
87112575^2 &-& 8345402^2 &=& (76716189 + 9476350)^2 + 9476350^2 \quad \cdot \boxed{3\text{–I}} = \boxed{6}
\end{array}$$

and

$$172325^2 - 112812^2 = (151759 - 128100)^2 + 128100^2 \quad \cdots \boxed{3\text{–II}} = \boxed{5}$$

Figure 2. We have

$$c^2 - d^2 = (a+b)^2 + b^2$$
$$cd = ab$$



$$x^2 + y^4 = 2z^4$$

Applying Theorem 25 to the equation $\boxed{4}$, $\boxed{5}$ and $\boxed{6}$, we obtain the following correspondences.

$$1469^2 - 84^2 = (113 - 1092)^2 + 1092^2 \qquad \cdots\cdots \boxed{4}$$
$$\updownarrow$$
$$3503833734241^2 + 232159^4 = 2 \cdot 2165017^4 \qquad \cdots\cdots ④$$

$$172325^2 - 112812^2 = (151759 - 128100)^2 + 128100^2 \qquad \cdots\cdots \boxed{5}$$
$$\updownarrow$$
$$2543305831910011724639^2 + 9788425919^4 = 2 \cdot 42422452969^4 \qquad \cdots\cdots ⑤$$

$$87112575^2 - 8345402^2 = (76716189 + 9476350)^2 + 9476350^2 \qquad \cdots\cdots \boxed{6}$$
$$\updownarrow$$
$$76285433470805578504147559981041^2 + 5705771236038721^4$$
$$= 2 \cdot 7658246457672229^4 \qquad \cdots\cdots ⑥$$

Applying Theorem 26 to the equation ④ and ⑤, we obtain that

$$3503833734241^2 + 2372159^4 = 2 \cdot 2165017^4 \qquad \cdots\cdots ④$$

$$\updownarrow$$

$$123672266091^2 - 14740596026^2$$
$$= (135504838557 - 13453415638)^2 + 13453415638^2 \qquad \cdots \boxed{4\text{–I}}$$
$$z = c^2 + d^2 = 15512114571284835412957$$

and

$$568580300012761^2 - 358778440454952^2$$
$$= (622980270315647 - 327449139293976)^2 \qquad \cdots \boxed{4\text{–II}}$$
$$+ 327449139293976^2$$
$$z = c^2 + d^2 = 452005526897888844293504165425$$

$$2543305831910011724639^2 + 9788425919^2 = 2 \cdot 42422452969^2 \qquad \cdots\cdots ⑤$$

$$\updownarrow$$

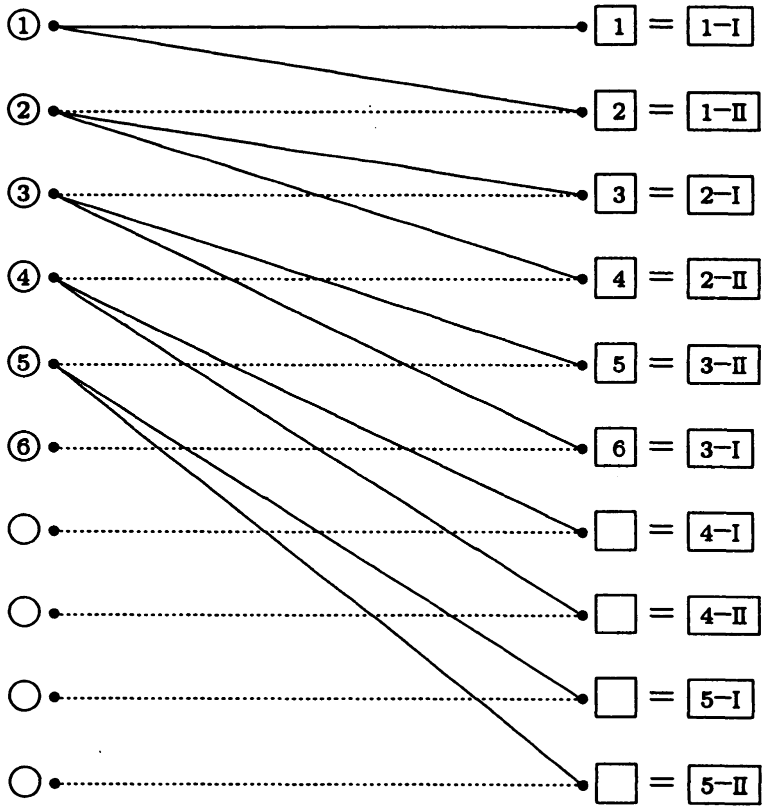
$$1114105387458447837 7^2 - 1480455646408040232^2$$
$$= (2570652399347305727 + 6416206298351572632)^2 \qquad \cdots \boxed{5\text{–I}}$$
$$+ 6416206298351572632^2$$
$$z = c^2 + d^2 = 126314830357375266295717376544111167953$$

and

$$5968929491057551114130 19^2 - 1102711716565404122454 50^2$$
$$= (1377252375803116234134 69 - 4779086680684630576229 50)^2 \qquad \cdots \boxed{5\text{–II}}$$
$$+ 477908668068463057622950^2$$
$$z = c^2 + d^2$$
$$= 36844092399067176322276741441513674938618483968 61$$

Figure 3. We have

$x^2 + y^4 = 2z^4$ $\qquad\qquad\qquad\qquad$ $c^2 - d^2 = (a+b)^2 + b^2\,,\; cd = ab$



By Theorem 25 and Theorem 26, above figure shows the following Note 27.

Note 27. *Let the equation* ⊛ *be*

$$x^2 + y^4 = 2z^4 \cdots\cdots\cdots\cdots\cdots\cdots \circledast$$

with positive odd integers x, y, z *which are pairwise relatively prime. Then the first six solutions of the equation* ⊛, *that is, with smallest values of* z, *are*

$$1^2 + 1^4 = 2 \cdot 1^4 \cdots\cdots ①,$$

$$239^2 + 1^4 = 2 \cdot 13^4 \cdots\cdots ②,$$

$$2750257^2 + 1343^4 = 2 \cdot 1525^4 \cdots\cdots ③,$$

$$3503833734241^2 + 2372159^4 = 2 \cdot 2165017^4 \cdots\cdots ④,$$

$$2543305831910011724639^2 + 9788425919^4 = 2 \cdot 42422452969^4 \cdots\cdots ⑤,$$

$$76285433470805578504147559981041^2 + 5705771236038721^4$$
$$= 2 \cdot 7658246457672229^4 \cdots\cdots ⑥.$$

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