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Type, cotype constants for $L_p(L_q)$, norms of the Rademacher matrices and interpolation

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Abstract. By applying the vector-valued interpolation arguments in [9] to the *Rademacher matrices* R_n, type t inequalities with 'type t constant' 1 are proved for $L_p(L_q)$ (L_q-valued L_p-space), where $t = \min\{p, q, p', q'\}$, $1/p + 1/p' = 1/q + 1/q' = 1$; or equivalently, it is shown that $\| R_n : l_t^n(L_p(L_q)) \to l_s^{2^n}(L_p(L_q)) \| = 2^{n/s}$, where $1 \leq s \leq t'$, $1/t + 1/t' = 1$. The constant t is optimal as far as the type constant is 1. By a duality argument analogous results are also obtained for cotype inequalities for $L_p(L_q)$. Some previous results by Milman [16], Cobos [5], and Cobos and Edmunds [6] are obtained as corollaries.

1. Introduction

In Kato and Miyazaki [9] (see also [17]), by applying vector-valued interpolation directly to the Littlewood matrices as operators

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between $L_p(L_q)$-valued $l_r^{2^n}$-spaces, the norms of these matrices are 'completely' determined (cf. [14]), which yields generalized Clarkson's inequalities (high-dimensional Clarkson-Boas-type inequalities) for $L_p(L_q)$ ([9]; cf. [3]): Those for L_p ([8]; cf. [3], [11]), the classical Clarkson's ones ([4]) and their Sobolev space versions by Milman [16] and Cobos [5] are immediate consequences.

In this paper, we apply the interpolation arguments in [9] to the *Rademacher matrices* to determine the norms of these matrices as operators of $l_t^n(L_p(L_q))$ to $l_s^{2^n}(L_p(L_q))$, where $t = \min\{p, q, p', q'\}$ and $1 \leqq s \leqq t'$ $(1/t + 1/t' = 1)$, which yields the type t inequalities for $L_p(L_q)$ with 'type t constant' 1. Here, t is optimal in the sense that if its 'type r constant' is 1, then $r \leqq t$. A similar treatment of type inequalities for the space B_p and interpolation argument for scalars are found in Maligranda and Persson [14] (see also [15]). By a duality argument analogous results are also obtained for cotype inequalities for $L_p(L_q)$. As corollaries, the previous results on type and cotype for Sovolev spaces by Milman [16] and Cobos [5] are obtained, and those on Besov and Triebel-Sobolev spaces by Cobos and Edmunds [6] are refined.

2. Preliminaries

Let $1 \leqq p, q \leqq \infty$. Let $L_p(L_q) := L_p(X, M, \mu ; L_q)$ be the L_q-valued L_p-space with the norm $\| f \|_{L_p(L_q)} :=$ the L_p-norm of $\| f(\cdot) \|_q$, where $L_q = L_q(Y, \mathcal{N}, \nu)$ is the usual L_q-space, and the measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are arbitrary but the former is assumed to be finite (cf., e.g., [7]).

For a Banach space E, let $l_r^n(E)$, $1 \leqq r \leqq \infty$, denote the space of E-valued sequences $\{x_j\}$ of length n with the norm $\| \{x_j\} \|_{l_r^n(E)} := (\Sigma_{j=1}^n \| x_j \|_E^r)^{1/r}$ (the usual modification is required if $r = \infty$).

2. 1. DEFINITION. We define *Rademacher matrices* $R_n = (r_{ij}^{(n)})$ ($2^n \times n$ matrices) recursively as follows:

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left(\begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & R_n \\ -1 & \end{array}\right) \qquad (n = 1, 2, \ldots).$$

Note that $r_{ij}^{(n)} = r_j((2i-1)/2^{n+1})$, where $r_j(t)$ are the Rademacher functions, that is, $r_j(t) = \mathrm{sgn}(\sin 2^j \pi t)$.

2. 2. DEFINITION. Let $1 \leqq p \leqq 2$. A Banach space E is called *of (Rademacher) type p* provided there exists a constant M such that

$$(2.1) \qquad \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\| dt \leqq M \left\{ \sum_{j=1}^n \| x_j \|^p \right\}^{1/p}$$

for all finite systems $\{x_j\}$ in E.

Let $2 \leqq q \leqq \infty$. A Banach space E is called *of (Rademacher) cotype q* provided there exists a constant M such that

(2. 2)
$$\left\{\sum_{j=1}^{n} \| x_j \|^q\right\}^{1/q} \leqq M \int_0^1 \left\| \sum_{j=1}^{n} r_j(t)x_j \right\| dt$$

for all finite systems $\{x_j\}$ in E.

By virtue of Khinchin-Kahane's inequality (see [1], [13]), (2. 1) and (2. 2) may be replaced by

(2. 3)
$$\left\{\int_0^1 \left\| \sum_{j=1}^{n} r_j(t)x_j \right\|^s dt\right\}^{1/s} \leqq M \left\{\sum_{j=1}^{n} \| x_j \|^p\right\}^{1/p}$$

and

(2. 4)
$$\left\{\sum_{j=1}^{n} \| x_j \|^q\right\}^{1/q} \leqq M \left\{\int_0^1 \left\| \sum_{j=1}^{n} r_j(t)x_j \right\|^s dt\right\}^{1/s}$$

with any $1 \leqq s < \infty$, respectively. Let $T_{p(s)}(E)$ resp. $C_{q(s)}(E)$ denote the smallest constant M satisfying (2. 3) resp. (2. 4) for all finite systems $\{x_j\}$ in E; and $T^{(n)}_{p(s)}(E)$ resp. $C^{(n)}_{q(s)}(E)$ the smallest constant M satisfying (2. 3) resp. (2. 4) for all *n elements* in E.

It is clear that if $1 \leqq s_1 \leqq s_2$, $1 \leqq T_{p(s_1)}(E) \leqq T_{p(s_2)}(E)$ and $C_{q(s_1)}(E) \geqq C_{q(s_2)}(E) \geqq 1$; further $T_{p(s)}(E) = \lim_{n\to\infty} T^{(n)}_{p(s)}(E)$ and $C_{q(s)}(E) = \lim_{n\to\infty} C^{(n)}_{q(s)}(E)$. Note that *all* Banach spaces are of type 1 and cotype ∞; and $T_{1(s)}(E) = C_{\infty(s)}(E) = 1$ for all $1 \leqq s < \infty$. Note also that

$$(2.5)\quad \left\{\int_0^1 \left\| \sum_{j=1}^n r_j(t)x_j \right\|^s dt\right\}^{1/s} = \left\{\frac{1}{2^n}\sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^n \theta_j x_j \right\|^s\right\}^{1/s}$$

$$= \left\{\frac{1}{2^n}\sum_{i=1}^{2^n} \left\| \sum_{j=1}^n r_{ij}^{(n)} x_j \right\|^s\right\}^{1/s}.$$

Owing to (2. 5) type and cotype properties are described by means of the operator norms of the Rademacher matrices (cf. [14]):

2. 3. PROPOSITION. Let E be a Banach space. (i) Let $1 < p \leqq 2$. Then, E is of type p if and only if there exist some s, $1 \leqq s < \infty$, and a constant M such that

$$\| R_n : l_p^n(E) \to l_s^{2^n}(E) \| \leqq M2^{n/s} \qquad (n = 1, 2, \ldots).$$

In this case, for any positive integer n

$$\| R_n : l_p^n(E) \to l_s^{2^n}(E) \| = T_{p(s)}^{(n)}(E)2^{n/s}.$$

(ii) Let $2 \leqq q < \infty$. Then, E is of cotype q if and only if there exist some s, $1 \leqq s < \infty$, and a constant M such that for the transposed matrix of R_n,

$$\| {}^tR_n : R_n(l_q^n(E)) \to l_q^n(E) \| \leqq M2^{n/s'} \qquad (n = 1, 2, \ldots),$$

where $R_n(l_q^n(E))$ $(\subset l_s^{2^n}(E))$ is the range of $R_n : l_q^n(E) \to l_s^{2^n}(E)$. In this case, for any positive integer n

$$\| {}^tR_n : R_n(l_q^n(E)) \to l_q^n(E) \| = C_{q(s)}^{(n)}(E)2^{n/s'}.$$

In particular, E is of cotype q if there exist some s, $1 \leqq s < \infty$, and an M such that

$$\| {}^tR_n : l_s^{2^n}(E) \to l_q^n(E) \| \leqq M 2^{n/s'} \qquad (n = 1, 2, \ldots).$$

In this case, $C_{q(s)}(E) \leqq M$.

Indeed, (i) is trivial. To see (ii) note that (2. 4) is rewritten as

$$\| (x_j) \|_{l_q^n(E)} \leqq M 2^{-n/s} \| R_n(x_j) \|_{l_s^{2^n}(E)} \qquad \text{for all } (x_j) \in l_q^n(E),$$

which is equivalent to

$$\| {}^tR_n R_n(x_j) \|_{l_q^n(E)} \leqq M 2^{n/s'} \| R_n(x_j) \|_{l_s^{2^n}(E)} \qquad \text{for all } (x_j) \in l_q^n(E)$$

since ${}^tR_n R_n = 2^n E_n$ (E_n is the $n \times n$ unit matrix).

3. Type, cotype constants for $L_p(L_q)$ and the norms of the Rademacher matrices

The following lemma is immediate to see by induction.

3. 1. LEMMA. Let H be a Hilbert space. Then, for an arbitrary positive integer n and for all $x_1, x_2, \ldots, x_n$ in H,

$$\left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{n} r_{ij}^{(n)} x_j \right\|_H^2 \right\}^{1/2} = 2^{n/2} \left\{ \sum_{j=1}^{n} \| x_j \|_H^2 \right\}^{1/2}.$$

Hence

$$\| R_n : l_2^n(H) \to l_2^{2^n}(H) \|$$

$$= \| {}^tR_n : l_2^{2^n}(H) \to l_2^n(H) \| = 2^{n/2}.$$

3. 2. THEOREM. Let $1 < p, q < \infty$ and let $t = \min\{p, q, p', q'\}$. Then, for any s with $1 \leqq s \leqq t'$

$$(3.1) \quad \| R_n : l_t^n(L_p(L_q)) \to l_s^{2^n}(L_p(L_q)) \| = 2^{n/s} \quad (n = 1, 2, \dots):$$

In other words, $L_p(L_q)$ is of type t and $T_{t(s)}(L_p(L_q)) = 1$ for all $1 \leqq s \leqq t'$.

PROOF. It is enough to show (3. 1) for $s = t'$. Let us show

$$(3.2) \qquad \| R_n : l_t^n(L_p(L_q)) \to l_{t'}^{2^n}(L_p(L_q)) \| \leqq 2^{n/t'}.$$

(i) Let $1 < p \leqq q \leqq 2$ $(t = p)$. If $p = q = 2$, Lemma 3. 1 gives the conclusion. So, we assume this is not the case. Put $\theta = 2/p'$ $(0 < \theta < 1)$ and $1/q_0 = (1/q - 1/p')/(1/p - 1/p')$. Then, since $(1 - \theta)/1 + \theta/2 = 1/p$, $(1 - \theta)/\infty + \theta/2 = 1/p'$ and $(1 - \theta)/q_0 + \theta/2 = 1/q$, we have

$$(L_1(L_{q_0}), L_2(L_2))_{[\theta]} = L_p(L_q) \quad \text{with equal norms}$$

by Theorems 5. 1. 1 and 5. 1. 2 of [2]. Further, using Theorem 5. 1. 2 (with 5. 1. 1 and 4. 2. 1) of [2] duplicately, we have

$$(l_1^n(L_1(L_{q_0})), l_2^n(L_2(L_2)))_{[\theta]} = l_p^n(L_p(L_q)) \quad \text{with equal norms,}$$

$$(l_\infty^{2^n}(L_1(L_{q_0})),\ l_2^{2^n}(L_2(L_2))_{[\theta]} = l_{p'}^{2^n}(L_p(L_q)) \quad \text{with equal norms.}$$

By easy calculation we have

$$(3.3) \qquad M_1 = \| R_n : l_1^n(L_1(L_{q_0})) \to l_\infty^{2^n}(L_1(L_{q_0})) \| = 1$$

and by Lemma 3.1

$$(3.4) \qquad M_2 = \| R_n : l_2^n(L_2(L_2)) \to l_2^{2^n}(L_2(L_2)) \| = 2^{n/2}.$$

Therefore, we obtain

$$\| R_n : l_p^n(L_p(L_q)) \to l_{p'}^{2^n}(L_p(L_q)) \| \leqq M_1^{1-\theta} M_2^{\theta} = 2^{n/p'}$$

by Theorem 4.1.2 of [2] with (3.3) and (3.4).

(ii) Let $1 < q < p \leqq 2$ $(t = q)$. Put $\theta = 2/q'$ $(0 < \theta < 1)$ and $1/p_0 = (1/p - 1/q')/(1/q - 1/q')$. Then, $(1-\theta)/1 + \theta/2 = 1/q$, $(1-\theta)/\infty + \theta/2 = 1/q'$ and $(1-\theta)/p_0 + \theta/2 = 1/p$. Since

$$(L_{p_0}(L_1),\ L_2(L_2))_{[\theta]} = L_p(L_q) \quad \text{with equal norms,}$$

we have by

$$M_3 = \| R_n : l_1^n(L_{p_0}(L_1)),\ l_\infty^{2^n}(L_{p_0}(L_1)) \| = 1$$

and (3.4),

$$(3.5) \qquad \| R_n : l_q^n(L_p(L_q)) \to l_{q'}^{2^n}(L_p(L_q)) \| \leqq M_3^{1-\theta} M_2^{\theta} = 2^{n/q'},$$

or (3.2) with $t = q$.

(iii) In the case where $1 < q \leqq 2 < p$ and $q < p'$ $(t = q)$, we have (3.5) in the same way as the previous case (ii).

(iv) Let $1 < q \leqq 2 < p$ and $p' < q$ $(t = p')$. Let $\theta = 2/p$ and $1/q_0 = (1/q - 1/p)/(1/p' - 1/p)$. Then, $(1 - \theta)/\infty + \theta/2 = 1/p$, $(1 - \theta)/1 + \theta/2 = 1/p'$ and $(1 - \theta)/q_0 + \theta/2 = 1/q$. Therefore, we have by Theorem 5.1.2 of [2]

$$(L_\infty^0(L_{q_0}),\ L_2(L_2))_{[\theta]} = L_p(L_q) \quad \text{with equal norms,}$$

where $L_\infty^0(L_{q_0})$ stands for the completion in $L_\infty(L_{q_0})$ of the simple functions (with support of finite measure). Consequently, by

$$M_4 = \| R_n : l_1^n(L_\infty^0(L_{q_0})),\ l_\infty^{2^n}(L_\infty^0(L_{q_0})) \| = 1$$

and (3.4), we obtain

$$\| R_n : l_{p'}^n(L_p(L_q)) \to l_p^{2^n}(L_p(L_q)) \| \leqq M_4^{1-\theta} M_2^{\theta} = 2^{n/p}.$$

(v) Let $2 < p,\ q < \infty$ $(t = \min(p', q'))$. Then, we have

$$\| R_n : l_t^n(L_p(L_q)) \to l_{t'}^{2^n}(L_p(L_q)) \|$$

$$= \| {}^tR_n : l_t^{2^n}(L_{p'}(L_{q'})) \to l_{t'}^n(L_{p'}(L_{q'})) \|$$

$$\leqq 2^{n/t'},$$

where the inequality on ${}^tR_n$ is obtained analogously to (i) and (ii) with Lemma 3.1. (Note here that $L_{q'}$ has the Radon-Nikodym property and the measure space $(X, \mathcal{M}, \mu)$ is finite; cf. [7], esp., p.98).

(vi) The proof of the case $1 < p \leqq 2 < q$ ($t = \min(p, q')$) goes in the same way as (v) by using the analogous results on ${}^{t}R_n$ to (iii) and (iv).

Equality is attained in (3.2) with $(f, 0, \ldots, 0) \in l_t^n(L_p(L_q))$ ($f \neq 0$). This completes the proof.

3.3. COROLLARY. Let $1 < p, q < \infty$ and let $t = \min\{p, q, p', q'\}$. Then, for any s with $t \leqq s < \infty$,

$$\| {}^{t}R_n : l_s^{2^n}(L_p(L_q)) \to l_{t'}^n(L_p(L_q)) \| = 2^{n/s'} \quad (n = 1, 2, \ldots),$$

and hence $L_p(L_q)$ is of cotype t' and $C_{t'(s)}(L_p(L_q)) = 1$ for all $t \leqq s < \infty$.

This is a direct consequence of the above theorem and Proposition 2.3 (use duality).

3.4. REMARKS. (i) The constant $t = \min\{p, q, p', q'\}$ in Theorem 3.2 is optimal under the condition that 'the type constant' is 1, that is, if $T_{r(s)}(L_p(L_q)) = 1$ with some s, then $r \leqq t$: Note here that $L_p(L_q)$ is of type $m = \min\{p, q, 2\}$ (cf. [12], p. 348; [1], [13]); and m is optimal as far as only 'type' is under consideration, where L_p and L_q are assumed to be of infinite dimension. Note also that $t = m$ if $p \leqq q'$, and $t < m$ if $p > q'$.

(ii) The constant t' in Theorem 3.2 is optimal for t in general; that is, if $T_{t(s)}(E) = 1$ with some s for a Banach space E of type t, then $s \leqq t'$.

(iii) The constants t' and t in Corollary 3.3 are optimal in the analogous meanings to (i) and (ii).

PROOF. (i) Assume $T_{r(s)}(L_p(L_q)) = 1$ for some $1 \leqq s < \infty$. Then, noting that the 2-dimensional spaces l_p^2 and l_q^2 are isometrically imbedded into $L_p(L_q)$, we have

$$(3.6)\quad \{\frac{1}{2}(\|x + y\|^s + \|x - y\|^s)\}^{1/s} \leqq (\|x\|^r + \|y\|^r)^{1/r}$$

for all x and y in l_p^2 and also in l_q^2. (Here the underlying measure spaces X and Y are assumed to be non-trivial, which means the existence of two disjoint measurable sets of finite positive measure.) Put $x = (1, 0)$, $y = (0, 1)$ and also $x = (1, 1)$, $y = (1, -1)$ in (3.6). Then we have $r \leqq \min\{p, q\}$ and $r \leqq \min\{p', q'\}$, or $r \leqq t$.

(ii) Let E be a Banach space of type t and let $T_{t(s)}(E) = 1$ with some s. Then, the inequality (3.6) with t instead of r holds for any x and y in E. Put here $x = y$. Then, we have $s \leqq t'$.

(iii) is seen analogously to (i) and (ii).

The same are true for L_p and Sobolev spaces $W_p^k(\Omega)$ (cf. [5], [16]), where Ω is an arbitrary domain in $\mathbb{R}^n$:

3.5. COROLLARY. Let $1 < p < \infty$ and let $t = \min\{p, p'\}$. Let E be one of L_p, $l_p^N(L_p)$ and $W_p^k(\Omega)$. Then,

(i) E is of type t and $T_{t(s)}(E) = 1$ for any s with $1 \leqq s \leqq t'$,

(ii) E is of cotype t' and $C_{t'(s)}(E) = 1$ for any s with $t \leqq s < \infty$.

Here, t and t' are optimal in the senses stated in Remark 3. 4.

PROOF. We have (i) and (ii) immediately by Theorem 3. 2 (note that $W_p^k(\Omega)$ is imbedded isometrically into $l_p^N(L_p)$ with a suitable positive integer N). To see that t (resp. t') is optimal in (i) (resp. (ii)) in the sense of Remark 3. 4 (i), one has only to observe that l_p^2 is isometrically imbedded into $W_p^k(\Omega)$: In fact, take an f in $W_p^k(\Omega)$ with support (in Ω) small enough and $\| f \|_{p,k} = 1$. Let g be a translate of f whose support is disjoint with that of f. Then, the correspondence: $(\xi, \eta) \to \xi f + \eta g$ from l_p^2 into $W_p^k(\Omega)$ is an isometry. The constants t' in (i) resp. t in (ii) are also optimal in the sense of Remark 3. 4 (ii) by Remark 3. 4 (ii) and (iii).

3. 6. REMARK. Theorem 3. 2 and Corollary 3. 3 hold without the assumption of finiteness of the measure space (X, M, μ). In fact, for any σ-finite measure μ on M we can take another (finite) measure $\tilde{\mu}$ on M such that $L_p(X, M, \mu ; L_q)$ is isometrically isomorphic to $L_p(X, M, \tilde{\mu} ; L_q)$ (for example, put $\tilde{\mu}(A) := \Sigma_{n=1}^{\infty} 2^{-n} \mu(X_n)^{-1} \mu(A \cap X_n)$, where $X = \Sigma_{n=1}^{\infty} X_n$, $0 < \mu(X_n) < \infty$ $(X_n \in M)$. Then, the correspondence: $f \to \Sigma_{n=1}^{\infty} 2^{n/p} \mu(X_n)^{1/p} \chi_{X_n} f$ is an isometry from $L_p(X, M, \mu ; L_q)$ onto $L_p(X, M, \tilde{\mu} ; L_q)$, where χ_{X_n} is the characteristic function of X_n.) If μ is an arbitrary positive measure on M, we have only to note that the supports of any $f_1, f_2, \dots, f_n$ in $L_p(X, M, \mu ; L_q)$ are σ-finite.

Now, we improve Cobos and Edmunds' results ([6]) on Besov spaces $B^s_{p,q}$ and Triebel-Sobolev spaces $F^s_{p,q}$ (s is a real number):

3.7. COROLLARY. Let $1 < p, q < \infty$ and let $t = \min\{p, q, p', q'\}$. Let E be one of $B^s_{p,q}$ and $F^s_{p,q}$. Then;

(i) E is of type t and $T_{t(s)}(E) = 1$ for any s with $1 \leqq s \leqq t'$,

(ii) E is of cotype t' and $C_{t'(s)}(E) = 1$ for any s with $t \leqq s < \infty$.

Indeed, $B^s_{p,q}$ and $F^s_{p,q}$ are isometrically imbedded into $l_q(L_p(\mathbb{R}^n))$ and $L_p(\mathbb{R}^n; l_q)$, respectively, where on $\mathbb{R}^n$ the Lebesgue measure is equipped (see [6]). Therefore, owing to Remark 3.6 (especially for $F^s_{p,q}$), these assertions (i) and (ii) are direct consequences of Theorem 3.2 and Corollary 3.3.

3.8. REMARK. In Cobos and Edmunds [6], for the spaces $E = B^s_{p,q}$ and $F^s_{q,p}$ it is shown that (i) $T_{p(p')}(E) = 1$ under the condition $1 < p \leqq 2$ and $p \leqq q \leqq p'$ ([6], Theorem 1); (ii) $C_{p(p')}(E) = 1$ under the condition $2 \leqq p < \infty$ and $p' \leqq q \leqq p$ ([6], Theorem 2).

We finally note that the first- and third-named authors [10] have recently characterized those Banach spaces with type (or cotype) constant 1 as those satisfying Clarkson-Boas-type inequalities.

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