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ON THE SPECTRAL GEOMETRY OF CLOSED MINIMAL SUBMANIFOLDS IN A SASAKIAN OR COSYMPLETIC MANIFOLD WITH CONSTANT ϕ - SECTIONAL CURVATURE

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1. INTRODUCTION

The spectral geometry for the second order operators arising in Ri emannian geometry has been studied by many authors [2,5,6,9,10,11]. Among them, the spectral geometry of the normal Jacobi operator for minimal submanifolds was studied by H.Donnelly [2], T.Hasegawa [6]. The normal Jacobi operator arises in the second variation formula for the functional area. This formula can be expressed in terms of an elliptic differential operator J (called the *normal Jacobi operator*) defined on the cross section $\Gamma(NM)$ of the normal bundle of the isometric minimal immersion $f : M \longrightarrow N$, which is defined by $J = \tilde{\Delta} + \tilde{R} - S$, where $\tilde{\Delta}$ is the rough Laplacian on NM and $\tilde{R}$ and S are linear transformations of NM defined by means of a partial *Ricci operator* $\tilde{R}$ of N and of the second fundamental form and its transpose, respectively.

The purpose of the present paper is to study Sasakian and cosympletic analogoues for certain results of [2,6]. The spectral geometry for the Jacobi operator of the energy of a harmonic map was studied by H.Urakawa [11] (for manifolds), and S.Nishikawa, P.Tondeur and L.Vanhecke [9] (for Riemannian foliations).

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2. PRELIMINARIES

Let (ϕ, ξ, η, g) be an almost contact metric structure on a C^∞-mani fold N. This means that

$$
(2.1)\qquad
\begin{aligned}
&\phi^2 = -I + \xi \otimes \eta, && \phi(\xi) = 0,\\
&\eta \circ \phi = 0, && \eta(\xi) = 1,\\
&g(\phi X, Y) = -g(X, \phi Y), && \eta(X) = g(X, \xi),
\end{aligned}
$$

where ϕ is a tensor field of type (1,1), ξ a vector field, η a 1-form, I the identity transformation, g a Riemannian metric and X, Y vector fields on N [cf.12].

Define a 2-form Φ on N by

$$\Phi(X, Y) = g(X, \phi Y)$$

for any vector fields X,Y on N.

If $[\phi, \phi] + d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denotes the Nijenhuis tensor formed with ϕ and d the operator of the exterior derivative, then the almost contact metric structure (ϕ, ξ, η, g) is said to be *normal.* If $\Phi = d\eta$, the almost contact metric structure (ϕ, ξ, η, g) is called a *contact metric structure.*

$\mathcal{N} = (N, \phi, \xi, \eta, g)$ is called a *Sasakian manifold* if a C^∞-manifold N admits a normal contact metric structure (ϕ, ξ, η, g). We note here that in a Sasakian manifold $\mathcal{N}$

$$(2.2)\qquad ({}^N\nabla_X \phi)(Y) = \eta(Y)X - g(X, Y)\xi, \qquad {}^N\nabla_X \xi = \phi X$$

where ${}^N\nabla$ denotes the Levi-Civita connection of g and X, Y vector fields on N.

$\mathcal{N} = (N, \phi, \xi, \eta, g)$ is called a *cosympletic manifold* if a C^∞-manifold N admits a normal almost contact metric structure (ϕ, ξ, η, g) such that Φ is closed and $d\eta = 0$.

It can be shown [1] that the cosympletic structure is characterized by

$$(2.3)\qquad {}^N\nabla_X \phi = 0 \quad \text{and} \quad {}^N\nabla_X \eta = 0,$$

for any vector field X on N. Here and in the sequel, $\mathcal{N} = (N, \phi, \xi, \eta, g)$ will denote either a Sasakian manifold or a cosympletic manifold.

The curvature operator R of g is defined by $R(X,Y)Z = [{}^N\nabla_X, {}^N\nabla_Y]Z - {}^N\nabla_{[X,Y]}Z$ for any vector fields X, Y, Z on N. In $\mathcal{N} = (N, \phi, \xi, \eta, g)$ we call a sectional curvature

$$k = \frac{g(R(X,\phi X)\phi X, X)}{g(X,X)g(\phi X, \phi X)}$$

determined by two orthogonal vectors X and ϕX (which are orthogonal to ξ) the *ϕ-sectional curvature* with respect to X of N. If the ϕ-sectional curvature is always constant with respect to any vector at every point of the manifold N, then we call $\mathcal{N} = (N, \phi, \xi, \eta, g)$ a *manifold of constant ϕ-sectional curvature.* It has been shown [cf.3,7,12] that in $\mathcal{N} = (N, \phi, \xi, \eta, g)$ with constant ϕ-sectional curvature k,

$$\begin{aligned} g(R(X,Y)Z,W) = {} & \alpha\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ & + \beta\{\eta(X)\eta(Z)g(Y,W) + \eta(Y)\eta(W)g(X,Z) \\ & - \eta(X)\eta(W)g(Z,Y) - \eta(Z)\eta(Y)g(X,W) \\ & + \Phi(X,Z)\Phi(W,Y) - \Phi(X,W)\Phi(Z,Y) \\ & - 2\Phi(X,Y)\Phi(Z,W)\}, \end{aligned}$$

where $\alpha = \frac{k+3}{4}, \beta = \frac{k-1}{4}$ in the Sasakian case and $\alpha = \beta = \frac{k}{4}$ in the cosympletic case.

Throughout this paper, $\mathcal{N}^S(k)$($\mathcal{N}^C(k)$ resp.) will denote a $(2n+1)$-dimensional Sasakian manifold(cosympletic manifold resp.) of constant ϕ-sectional curvature k.

For a Riemannian manifold M which is isometrically immersed in a Riemannian manifold N, the formulas of Gauss and Weingarten are respectively given by

$${}^N\nabla_X Y = \nabla_X Y + B(X,Y), \qquad {}^N\nabla_X V = -A^V X + D_X V$$

for vector fields X, Y tangent to M and a normal vector field V, where ∇ be the Levi-Civita connection on M, and A and B are called the *second fundamental forms* of M, which are related by $g(V, B(X,Y)) = g(A^V(X), Y)$.

Furthermore, we can consider A as a cross section of the Riemannian vector bundle $Hom(NM, SM)$, where SM is the bundle of symmetric transformations of the tangent bundle TM and NM is the normal bundle of M in N. Then $S := {}^tA \circ A \in \Gamma(Hom(NM, NM))$, where $\Gamma(\bullet)$ denotes the space of smooth sections of $\bullet$. Henceforth we adopt the following notations ;

σ := the trace of ${}^tA \circ A$(*i.e.*, the square norm of A),

l_n := the trace of $S \circ S$(*i.e.*, the square norm of S),

k_n := the square norm of the curvature tensor of the normal connection,

t := the square norm of the covariant derivative of the second fundamental form A.

A $(2m+1)$-dimensional submanifold M of N is said to be ***invariant*** if the structure vector field ξ is tangent to M everywhere on M and ϕX is tangent to M for any tangent vector field X on M [cf.7,12]. We easily see that any invariant submanifold M with induced structure tensors which will be denoted by the same letters (ϕ, ξ, η, g) as in N, is also a Sasakian manifold or cosympletic manifold according as $\mathcal{N} = (N, \phi, \xi, \eta, g)$ is a Sasakian manifold or cosympletic manifold. Both the invariant submanifolds will be denoted by $\mathcal{M} = (M, \phi, \xi, \eta, g)$ unless otherwise stated.

Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ be an invariant submanifold of a Sasakian or cosympletic manifold $\mathcal{N} = (N, \phi, \xi, \eta, g)$. Then we have from (2.1), (2.2) and (2.3)

$$B(X, \xi) = 0, \quad A^V(\xi) = 0, \tag{2.4}$$

$$B(X, \phi Y) = B(\phi X, Y) = -\phi B(X, Y), \tag{2.5}$$

$$\phi A^V(X) = -A^V(\phi X) = A^{\phi V}(X) \tag{2.6}$$

for any vector fields X,Y tangent to M. It is clear from (2.5) and (2.6) that any invariant submanifold $\mathcal{M}$ of $\mathcal{N}$ is minimal [cf.12].

Let $\phi = (\phi_i{}^j)$, $\xi = (\xi^i)$, $\eta = (\eta_i)$ and $g = (g_{ij})$ be the components of the tensor fields ϕ, ξ, η and g, respectively, with respect to a local coordinate system $(x^1, \cdots, x^{2m+1})$ on $\mathcal{M} = (M, \phi, \xi, \eta, g)$. And also denote by $R = (R_{ijkl})$, $\rho = (R_{ij}) = (g^{kl}R_{kijl})$ and $\tau = (R_{ij}g^{ij})$ the corresponding curvature tensor, Ricci tensor and scalar curvature, where $(g^{ij}) = (g_{ij})^{-1}$.

Now we consider the so-called *contact Bochner curvature tensor* $B^S = (B^S_{kjih})$ and *η-Einstein tensor* $Q^S = (Q^S_{ij})$ defined on the invariant submanifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ of a Sasakian manifold $\mathcal{N}$ respectively by [cf. 8]

$$\begin{aligned} B^S_{kjih} = R_{kjih} &- \frac{1}{2m+4}(g_{kh}R_{ji} - g_{ki}R_{jh} - g_{jh}R_{ki} + g_{ji}R_{kh} \\ &- \phi_{kh}R_{jl}\phi_i{}^l + \phi_{ki}R_{jl}\phi_h{}^l - \phi_{ji}R_{kl}\phi_h{}^l + \phi_{jh}R_{kl}\phi_i{}^l \\ &+ 2\phi_{kj}R_{il}\phi_h{}^l + 2\phi_{ih}R_{kl}\phi_j{}^l - R_{kh}\eta_j\eta_i + R_{ki}\eta_j\eta_h \\ &- R_{ji}\eta_k\eta_h + R_{jh}\eta_k\eta_i) + \frac{r-4}{2m+4}(g_{kh}g_{ji} - g_{ki}g_{jh}) \\ &+ \frac{r+2m}{2m+4}(\phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih}) \\ &- \frac{r}{2m+4}(g_{kh}\eta_j\eta_i - g_{ki}\eta_j\eta_h + g_{ji}\eta_k\eta_h - g_{jh}\eta_k\eta_i), \end{aligned}$$

$$Q^S_{ij} = R_{ij} - ag_{ij} - b\eta_i\eta_j,$$

where $r = \frac{\tau+2m}{2m+2}$, $\phi_i{}^k g_{kj} = \phi_{ij}$, $a = \frac{\tau}{2m} - 1$ and $b = 2m+1-\frac{\tau}{2m}$.

Then we have

$$\begin{aligned} (2.7) \qquad |B^S|^2 = |R|^2 &- \frac{8}{m+2}|\rho|^2 + \frac{2}{(m+1)(m+2)}\tau^2 \\ &+ \frac{4(3m^2+3m-2)}{(m+1)(m+2)}\tau - 24m^2 + 36m - 56 \\ &+ \frac{8(13m+14)}{(m+1)(m+2)}, \end{aligned}$$

$$|Q^S|^2 = |\rho|^2 - \frac{1}{2m}\tau^2 + 2\tau - 2m(2m+1). \tag{2.8}$$

A Sasakian manifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ is a called *η−Einstein* if Q^S vanishes identically. It is well known that a Sasakian manifold of constant ϕ-sectional curvature is η-Einstein. For any η-Einstein manifold of dimension ≥ 5, the scalar curvature is necessarily constant. Any 3-dimensional Sasakian manifold is η-Einstein, but in this case τ may not be constant. Moreover, it may be easily seen that $Q^S = 0$ and $B^S = 0$ hold if and only if a Sasakian manifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ has a constant ϕ-sectional curvature.

Next define the so-called *cosympletic Bochner curvature tensor* $B^C = (B^C_{kjih})$ and *η-Einstein tensor* $Q^C = (Q^C_{ij})$ on the invariant submanifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ of a cosympletic manifold $\mathcal{N}$ respectively by [cf. 10]

$$\begin{aligned} B^C_{kjih} = R_{kjih} &- \frac{1}{2(m+2)}(g_{kh}R_{ji} - g_{jh}R_{ki} + g_{ji}R_{kh} - g_{ki}R_{jh} \\ &- \phi_{kh}S_{ji} - \phi_{jh}S_{ki} + \phi_{ji}S_{kh} - \phi_{ki}S_{jh} - \phi_{ki}S_{jh} - 2\phi_{ih}S_{kj} \\ &- 2\phi_{kj}S_{ih} - \eta_k\eta_h R_{ji} - \eta_j\eta_h R_{ki} - \eta_j\eta_i R_{kh} + \eta_k\eta_i R_{jh}) \\ &+ \frac{\tau}{4(m+1)(m+2)}(g_{kh}g_{ji} - g_{jh}g_{ki}) \\ &- \frac{\tau}{4(m+1)(m+2)}(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{jh}\eta_k\eta_i - g_{ki}\eta_j\eta_h) \\ &+ \frac{\tau}{4(m+1)(m+2)}(\phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}), \end{aligned}$$

$$Q^C_{ij} = R_{ij} - \frac{\tau}{2m}g_{ij} + \frac{\tau}{2m}\eta_i\eta_j,$$

where $S_{ij} = -R_{jk}\phi^k_i$ and $S_{ji} = -S_{ij}$.

Then we also obtain

$$|B^C|^2 = |R|^2 - \frac{8}{m+2}|\rho|^2 + \frac{2}{(m+1)(m+2)}\tau^2, \tag{2.9}$$

$$|Q^C|^2 = |\rho|^2 - \frac{1}{2m}\tau^2. \tag{2.10}$$

A cosympletic manifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ is said to be *cosympletic Bochner flat* (*η-Einstein* resp.) if $B^C = 0 (Q^C = 0\ resp.)$. A cosympletic manifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ has a constant ϕ-sectional curvature if and only if $B^C = 0$ and $Q^C = 0$ hold. For any η-Einstein cosympletic manifold, τ is constant.

Let $\tilde{\mathcal{R}}$ be the partial Ricci transformation, which is defined by

$$\tilde{\mathcal{R}}(V) := \sum_{i=1}^{2m+1} \{R(e_i, V)e_i\}^{\perp},$$

where V is a normal vector field, $\{e_i : i = 1, \cdots, 2m+1\}$ an orthnormal basis of the tangent space T_xM at $x \in M$ and $\perp$ denotes the normal part of a vector with respect to the metric g.

Now we consider the differential operator J, which is usually called the *normal Jacobi operator*, defined by

$$J = \tilde{\Delta} + \tilde{R} - S,$$

where $\tilde{\Delta} = -\sum_{i=1}^{2m+1}(D_{e_i}D_{e_i} - D_{\nabla_{e_i}e_i})$.

Throughout this paper M will denote a closed (compact without boundary) manifold. In fact the operator J arising from the second variation formula of M is self-adjoint, elliptic of second order, and has a discrete spectrum as consequence of compactness of M.

Now we state the following Simon's type formula on $\mathcal{M} = (M, \phi, \xi, \eta, g)$

$$\frac{1}{2}\tilde{\Delta}\sigma = t - \tilde{k}_n - l_n + \gamma \cdot \sigma, \tag{2.11}$$

where γ denotes $\frac{m(k+3)+2k}{2}$ (or $\frac{(m+2)k}{2}$) according as $\mathcal{M}$ is an invariant submanifold of $\mathcal{N}^S(k)$ (or $\mathcal{N}^C(k)$), and $\tilde{k}_n := -\sum_{a,b} Tr([A^a, A^b]^2)$, $A^a := A^{e_a}, \{e_a : a = 2m+2, \cdots, 2n+1\}$ an orthonormal basis of the normal space N_xM at $x \in M, [A^a, A^b] = A^a \circ A^b - A^b \circ A^a$. And also using (2.4) and (2.6) we can calculate the following identities (2.12) $\sim$ (2.15). If $\mathcal{M}$ is an invariant submanifold of $\mathcal{N}^S(k)$, then

$$\begin{aligned}\tilde{k}_n = {} & m(m+1)(k-1)\{4 + (k-1)(m+1) \\ & + 8m\} + 8m^2(2m+1) - 2\{(k-1)(m+1) \\ & + 4m\}\tau + 2|\rho|^2,\end{aligned} \tag{2.12}$$

$$l_n = m^2(k^2 + 6k - 3) + m(k^2 + 2k - 1) - 2k\tau + \frac{1}{2}|R|^2. \tag{2.13}$$

If $\mathcal{M}$ is an invariant submanifold of $\mathcal{N}^C(k)$, then

$$\tilde{k}_n = m(m+1)^2 k^2 - 2(m+1)k\tau + 2|\rho|^2, \tag{2.14}$$

$$l_n = m(m+1)k^2 - 2k\tau + \frac{1}{2}|R|^2. \tag{2.15}$$

A Riemannian manifold (M, g) which is isometrically immersed in $\mathcal{N} = (N, \phi, \xi, \eta, g)$ is called a *normal anti-invariant submanifold* of $\mathcal{N} = (N, \phi, \xi, \eta, g)$ if ξ is normal to M and $\phi(TM) \subset NM$ [cf.12]. On a Sasakian manifold $\mathcal{N} = (N, \phi, \xi, \eta, g)$, if ξ is normal to M, then $\phi(TM) \subset NM$(*i.e.*, M is an anti-invariant submanifold of $\mathcal{N} = (N, \phi, \xi, \eta, g)$). Assume that $\dim M = n$. Then the following identities hold when the ambient manifold is a $(2n+1)$-dimensional Sasakian manifold $\mathcal{N} = (N, \phi, \xi, \eta, g)$;

$$D_X(\phi Y) = \phi \nabla_X Y - g(X, Y)\xi, \tag{2.16}$$

$$\phi B(X, Y) = -A^{\phi Y} X, \tag{2.17}$$

$$g(B(X, Y), \xi) = 0, \tag{2.18}$$

$$g(B(X, Y), \phi Z) = g(B(X, Z), \phi Y) \tag{2.19}$$

for any vector fields X, Y, Z tangent to M, where (2.16) and (2.17) follow from (2.2), (2.18) from (2.16), and (2.19) from (2.17).

In the case of a $(2n+1)$-dimensional cosympletic manifold $\mathcal{N} = (N, \phi, \xi, \eta, g)$, (2.17) $\sim$ (2.19) still holds.

If a normal anti-invariant submanifold M of $\mathcal{N}^S(k)$(*or* $\mathcal{N}^C(k)$) is minimal, then the Simon's type formula is given by

$$\frac{1}{2}\tilde{\Delta}\sigma = t - \tilde{k}_n - l_n + \delta\sigma \tag{2.20}$$

with the aid of (2.17) $\sim$ (2.19), where δ denotes $\alpha n+\beta$.

Finally we introduce the *Wely conformal curvature tensor* $C=(C_{kjih})$ and the *Einstein tensor* $G=(G_{ij})$ on M, which are respectively defined by

$$C_{kjih}=R_{kjih}-\frac{1}{n-2}(g_{kh}\rho_{ji}-g_{jh}\rho_{ki}+g_{ji}\rho_{kh}-g_{ki}\rho_{jh}$$
$$+\frac{1}{(n-1)(n-2)}(g_{jk}g_{il}-g_{jl}g_{ik})\tau,$$
$$G_{ij}=\rho_{ij}-\frac{\tau}{n}g_{ij}.$$

Then we have

$$|C|^2=|R|^2-\frac{4}{n-2}|\rho|^2+\frac{2}{(n-1)(n-2)}\tau^2, \tag{2.21}$$

$$|G|^2=|\rho|^2-\frac{1}{n}\tau^2. \tag{2.22}$$

$G=0$ holds if and only if M is Einstein. $C=0$ and $G=0$ hold if and only if M has a constant sectional curvature ($n\geq 4$).

3. The caculation of spectral invariants

In this section we apply the normal Jacobi operator J acting on $\Gamma(NM)$ to the Gilkey's results and obtain Sasakian or cosympletic spectral invariants.

Now consider the semigroup e^{-tJ} given by

$$e^{-tJ}V(x)=\int_M K(t,x,y,J)V(y)dv_g(y),$$

where $K(t,x,y,J)\in Hom(N_yM,N_xM)$ is the kernel function and dv_g denotes the volume element of M with respect to g. Then we have asymptotic expansions for L^2-trace

$$Tr(e^{-tJ})=\sum_{i=1}^{\infty}e^{-t\lambda_i}\sim(4\pi t)^{-\frac{2m+1}{2}}\sum_{j=0}^{\infty}t^ja_j(J)\quad(t\downarrow 0^+),$$

where each $a_j(J)$ is the spectral invariants of J, which depends only on the discrete spectrum ;

$$Spec(M, J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \cdots \uparrow +\infty\}$$

Applying the normal Jacobi operator J to the Gilkey's results [4,p.327] , we obtain

Theorem [cf. 5,6].

(i) $$a_0(J) = q \cdot Vol(M, g),$$

(ii) $$a_1(J) = \frac{q}{6}\int_M \tau dv_g + \int_M Tr(E) dv_g,$$

(iii) $$a_2(J) = \frac{q}{360}\int_M (5\tau^2 - 2|\rho|^2 + 2|R|^2) dv_g + \frac{1}{360}\int_M \{-30k_n + Tr(60\tau E + 180E^2)\} dv_g,$$

where q is the codimension $2(n-m)$ and $E := -\tilde{R} + S$.

If $\mathcal{M} = (M, \phi, \xi, \eta, g)$ is a $(2m+1)$-dimensional invariant submanifold of $\mathcal{N}^S(k)$ with dimension $2n+1$, then we obtain

$$\tau = (k-1)m(m+1) + 2m(2m+1) - \sigma \tag{3.1}$$

$$Tr(E) = km(n+1) + 2n + 3mn - m - \tau \tag{3.2}$$

(3.3)
$$\begin{aligned} Tr(E^2) &= \frac{1}{2}\{2 + (k+3)m\}^2(n-m) + \{2(k+3)m\}\sigma + l_n \\ &= \frac{1}{2}\{2 + (k+3)m\}^2(n-m) + \{2 + (k+3)m\} \\ &\quad \{(k-1)m(m+1) + 2m(2m+1)\} + m^2(k^2 + 6k - 3) \\ &\quad + m(k^2 + 2k - 1) - \{2(k+1) + (k+3)m\}\tau + \frac{1}{2}|R|^2 \end{aligned}$$

$$
\begin{aligned}
(3.4) \qquad k_n &= \tilde{k}_n + (k-1)^2 m(n-m) + 2(k-1)\sigma \\
&= m(m+1)(k-1)\{4 + (k-1)(m+1) + 8m\} \\
&+ 8m^2(2m+1) + (k-1)^2 m(n-m) \\
&+ 2(k-1)\{(k-1)m(m+1) + 2m(2m+1)\} \\
&- 2\{(c-1)(m+2) + 4m\}\tau + 2|\rho|^2,
\end{aligned}
$$

where (3.1) follows from the equation of Gauss, (3.2) from the definition of E, (3.3) from (2.13) and (3.1), and (3.4) from the equation of Ricci, (3.1) and (2.2)

Next, if $\mathcal{M} = (M, \phi, \xi, \eta, g)$ is a $(2m+1)$-dimensional invariant submanifold of $\mathcal{N}^C(k)$ with dimension $(2n+1)$, then we also have

$$
(3.5) \qquad \tau = m(m+1)k - \sigma,
$$

$$
(3.6) \qquad Tr(E) = m(n+1)k - \tau,
$$

$$
\begin{aligned}
(3.7) \qquad Tr(E^2) &= \frac{1}{2}(n-m)m^2k^2 + mk\sigma + l_n \\
&= \frac{m}{2}(m^2 + mn + 4m + 2)k^2 - (2+m)k\tau \\
&+ \frac{1}{2}|R|^2,
\end{aligned}
$$

$$
\begin{aligned}
(3.8) \qquad k_n &= m(n-m)k^2 + 2k\sigma + \tilde{k}_n \\
&= \{m(n-m) + 2m(m+1) + m(m+1)^2\}k^2 \\
&- 2(m+2)k\tau + 2|\rho|^2.
\end{aligned}
$$

Substituting (3.1) $\sim$ (3.4) into Theorem, we obtain

Theorem 1. *Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ be a $(2m+1)$- dimensional compact invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\mathcal{N}^S(k)$ with constant ϕ-sectional curvature k. Then the coefficients*

$a_0(J), a_1(J)$ and $a_2(J)$ of the asymptotic expansion for the normal Jacobi operator J are respectively given by

$$a_0(J) = qVol(M,g), \tag{3.9}$$

(3.10)
$$a_1(J) = \{km(n+1) + 2n + 3mn - m\}Vol(M,g) + \frac{q-6}{6}\int_M \tau dv_g$$

(3.11)
$$a_2(J) = \frac{1}{180}\int_M [(q+45)|R|^2 - (q+30)|\rho|^2 + (\frac{5}{2}q - 30)\tau^2]dv_g + b_0\int_M \tau dv_g + b_1 Vol(M,g),$$

where b_0 and b_1 are constants determined by m, n and k.

In the cosympletic case, substituting (3.5) $\sim$ (3.8) into Theorem, we also obtain

Theorem 2. *Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ be a $(2m+1)$-dimensional compact invariant submanifold of a $(2n+1)$-dimensional cosympletic manifold $\mathcal{N}^C(k)$ with constant ϕ-sectional curvature k. Then the coefficients $a_0(J), a_1(J)$ and $a_2(J)$ of the asymptotic expansion for the normal Jacobi operator J are respectively given by*

$$a_0(J) = qVol(M,g), \tag{3.12}$$

$$a_1(J) = (n+1)mkVol(M,g) + \frac{q-6}{6}\int_M \tau dv_g, \tag{3.13}$$

(3.14)
$$a_2(J) = \frac{1}{180}\int_M [(q+45)|R|^2 - (q+30)|\rho|^2 + (\frac{5}{2}q - 30)\tau^2]dv_g + c_0\int_M \tau dv_g + c_1 Vol(M,g),$$

where c_0 and c_1 are constants determined by m, n and k.

From now on we denote both the contact Bochner curvature tensor B^S (η-Einstein tensor Q^C resp.) and the cosympletic Bochner curvature tensor B^C (η-Einstein tensor Q^S resp.) by the same letter B (Q resp.)

Corollary 1. *Under the same situations as stated in Theorem 1 or 2, the following quantities are its spectral invariants when the codimension q is not equal to 6.*

(1) $dimM,\ Vol(M,g),\ \int_M \tau dv_g,$

(2) $\int_M \sigma dv_g,$

(3) $\int_M [(q+45)|R|^2 - (q+30)|\rho|^2 + (\frac{5}{2}q - 30)\tau^2] dv_g,$

(4) $\int_M [(q+45)|B|^2 + \frac{q(6-m) - 30m + 300}{m+2}|Q|^2 + b_2\tau^2] dv_g,$

(5) $\int_M [\frac{1}{2}|B|^2 + \frac{2m+8}{m+2}|Q|^2 + \frac{m+2}{m(m+1)}\tau^2 - t] dv_g,$

(6) $\int_M [(5q+210)|Q|^2 + \frac{5q(1-m) + 30(2m+7)}{2m}\tau^2 - 2(q+45)t] dv_g.$

where $b_2 := \frac{1}{m(m+1)}\{(n-m)(5m^2+4m+3) - 30m^2 - 45m + 75\}$.

Proof. We prove for the Sasakian case. (1) and (3) are clear. (2) follows from (3.1) and (3.10). Substituting (2.7) and (2.8) into (3), we have (4). (5) follows from (2.11) $\sim$ (2.13). Eliminating B from (4) and (5), we obtain (6).

If M is an n-dimensional, minimal, normal anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\mathcal{N}^S(k)$ or cosympletic manifold $\mathcal{N}^C(k)$ with constant ϕ-sectional curvature k, then we also have

$$\tau = \alpha n(n-1) - \sigma \tag{3.15}$$

$$Tr(E) = 2\alpha n^2 + 2\beta n - \tau \tag{3.16}$$

(3.17)
$$Tr(E^2) = (\alpha n + 3\beta)^2(n+1) - 2\beta(\alpha n + 3\beta)(n+3) + \beta^2(n+3)^2 + 2(\alpha n + 3\beta)\sigma + l_n$$

(3.18)
$$k_n = 2\beta^2(n^2 - n) - 4\alpha\beta n(n-1) + 4\beta\tau + \tilde{k}_n,$$

where $\alpha = \frac{k+3}{4}, \beta = \frac{k-1}{4}$ in the Sasakian case and $\alpha = \beta = \frac{k}{4}$ in the cosympletic case.

Substituting (3.1) $\sim$ (3.18) into Theorem, we get

Theorem 3. *Let M be n-dimensional compact, minimal, normal anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\mathcal{N}^S(k)$ or cosympletic manifold $\mathcal{N}^C(k)$ with constant ϕ-sectional curvature k. Then the coefficients $a_0(J), a_1(J)$ and $a_2(J)$ of the asymptotic expansion for the normal Jacobi operator J are respectively given by*

(3.19)
$$a_0(J) = (n+1)Vol(M, g),$$

(3.20)
$$a_1(J) = \frac{n-5}{6}\int_M \tau dv_g + (2\alpha n^2 + 2\beta n)Vol(M),$$

(3.21)
$$a_2(J) = \frac{1}{360}\int_M [2(n+1)|R|^2 - 2(n+1)|\rho|^2 + 5(n-11)\tau^2 - 30\tilde{k}_n + 180 l_n]dv_g + \frac{1}{3}\int_M [\alpha n^2 + (\beta - 3\alpha)n - 10\beta]\tau dv_g + d_0 Vol(M),$$

where d_0 is a number determined by n and k.

Corollary 2. *Under the same situations as stated in Theorem 3, the following quantities are its spectral invariants when n is not equal to 5.*

(1) $dimM$, $Vol(M,g)$, $\int_M \tau dv_g$, $\int_M (\tilde{k}_n + l_n - t)dv_g$,

(2) $\int_M \sigma dv_g$,

(3) $\frac{n+1}{180}\int_M (|R|^2 - |\rho|^2)dv_g + \frac{n-11}{72}\int_M \tau^2 dv_g + \frac{1}{12}\int_M (6l_n - \tilde{k}_n)dv_g$,

(4) $\frac{n+1}{180}\int_M (|C|^2 + \frac{6-n}{n-2}|G|^2)dv_g + d_1\int_M \tau^2 dv_g + \frac{1}{12}\int_M (6l_n - \tilde{k}_n)dv_g$,

(5) $\frac{n+1}{180}\int_M (|C|^2 + \frac{6-n}{n-2}|G|^2)dv_g + d_1\int_M \tau^2 dv_g + \frac{1}{12}\int_M (6t - 7\tilde{k}_n)dv_g$,

where $d_1 = \frac{5n^3-62n^2+59n+6}{360n(n-1)}$.

4. Some applications

In this section, by using the Sasakian or cosympletic spectral invariants, we obtain some spectral properties.

Denote $\mathcal{M} = (M, \phi, \xi, \eta, g)$ and $\mathcal{M}' = (M', \phi', \xi', \eta', g')$ by $(2m+1)$-dimensional invariant submanifolds of $\mathcal{N}^S(k)$ or $\mathcal{N}^C(k)$. Then $\mathcal{M}$ is totally geodesic if and only if $\mathcal{M}$ is of constant ϕ-sectional curvature k. And if $n - m < \frac{m(m+1)}{2}$, then any invariant submanifolds of $\mathcal{N}^S(k)$ or $\mathcal{N}^C(k)$ with constant ϕ-sectional curvature are also totally geodesic [cf.12]. Now we assume that the codimension is not equal to 6.

First, from (2) in Corollary 1 we obtain

Proposition 1. *Assume that $Spec(\mathcal{M}, J) = Spec(\mathcal{M}', J')$, Then if $\mathcal{M}$ is totally geodesic, so does $\mathcal{M}'$.*

Proposition 2. *Assume that $m = 1$ $(n \geq 2)$ or $m = 2(n \geq 7)$ or $m = 3(n \geq 9)$ or $m = 4(n \geq 10)$ or $m = 5(n \geq 12)$ or $m = 6(n \geq 13)$ or $m = 7(14 \leq n \leq 51)$ or $m = 8(15 \leq n \leq 22)$, and $Spec(\mathcal{M}, J) =$*

Spec$(\mathcal{M}', J')$, Then $\mathcal{M}$ has a constant ϕ-sectional curvature $\tilde{k}$ if and only if $\mathcal{M}'$ has the same constant ϕ'-sectional curvature $\tilde{k}$.

Proof. Under the assumption the coefficient of $|Q|^2 > 0$, and $b_2 > 0$ in (4) of Corollary 1. Hence if $\mathcal{M}$ has a constant ϕ-sectional curvature $\tilde{k}$, then we get

$$b_2 \int_M \tau^2 dv_g = \int_{M'} (q+45)|B'|^2 dv_{g'} + \int_{M'} \frac{q(6-m) - 30m + 300}{m+2} |Q'|^2 dv_{g'}$$
$$+ b_2 \int_{M'} \tau'^2 dv_{g'} \geq b_2 \int_{M'} \tau'^2 dv_{g'}.$$

On the other hand,

$$\int_M \tau^2 dv_g \leq \int_{M'} \tau'^2 dv_{g'}$$

because $\int_M \tau dv_g = \int_{M'} \tau' dv_{g'}$, $\tau =$ constant and $\int_M dv_g = \int_{M'} dv_{g'}$. Therefore $B' = 0 = Q'$. *Q.E.D.*

The following Propositions 3, 4 and 5 are due to (4) of Corollary 1.

Proposition 3. *Assume that $m \leq 6$ or $m = 7(8 \leq n \leq 51)$ or $m = 8(9 \leq n \leq 22)$ or $m = 9(10 \leq n \leq 13)$, and Spec$(\mathcal{M}, J) =$ Spec$(\mathcal{M}', J')$.*
If $\mathcal{M}$ has a constant ϕ-sectional curvature $\tilde{k}$ and $\int_M \tau^2 dv_g \geq \int_{M'} \tau'^2 dv_{g'}$, then $\mathcal{M}'$ has the same constant ϕ'-sectional curvature $\tilde{k}$.

Proof. It is clear from the fact that

$$(n-m)(6-m) - 15m + 150 > 0. \qquad Q.E.D.$$

Proposition 4. *Assume that $m = 7(n \geq 53)$ or $m = 8(n \geq 24)$ or $m = 9(n \geq 16)$, and Spec$(\mathcal{M}, J) =$ Spec$(\mathcal{M}', J')$. If $\mathcal{M}$ is η-Einstein, the contact Bochner curvature tensor (or the cosympletic Bochner curvature tensor) of $\mathcal{M}'$ vanishes and $\int_M \tau^2 dv_g \geq \int_{M'} \tau'^2 dv_{g'}$, then $\mathcal{M}$ and $\mathcal{M}'$ have the same constant ϕ-sectional curvature and ϕ'-sectional curvature respectively.*

Proof. It is clear from the fact that

$$(n-m)(6-m) - 15m + 150 < 0 \quad \text{and} \quad b_2 > 0. \qquad Q.E.D.$$

Proposition 5. *Let $\mathcal{M}$ and $\mathcal{M}'$ be η-Einstein and η'-Einstein respectively, $Spec(\mathcal{M}, J) = Spec(\mathcal{M}', J')$. Then $\mathcal{M}$ has a constant ϕ-sectional curvature $\tilde{k}$ if and only if $\mathcal{M}'$ has the same constant ϕ'-sectional curvature $\tilde{k}$.*

Proof. If $\mathcal{M}$ and $\mathcal{M}'$ are η-Einstein and η'-Einstein respectively, then τ and τ' are constants. So $\int_M \tau^2 dv_g = \int_{M'} \tau'^2 dv_{g'}$. *Q.E.D.*

Proposition 6. *Assume that $Spec(\mathcal{M}, J) = Spec(\mathcal{M}', J')$. If $\mathcal{M}$ is of constant ϕ-sectional curvature $\tilde{k}$, then $\int_M t dv_g \leq \int_{M'} t' dv_{g'}$, and the equality holds if and only if $\mathcal{M}'$ is of constant ϕ'-sectional curvature $\tilde{k}' = \tilde{k}$.*

Proof. From (5) of Corollary 1 we obtain

$$\begin{aligned}
&\frac{m+2}{m(m+1)} \int_M \tau^2 dv_g - \int_M t dv_g \\
&= \int_{M'} [\frac{1}{2}|B'|^2 + \frac{2m+8}{m+2}|Q'|^2 + \frac{m+2}{m(m+1)}\tau'^2] dv_{g'} - \int_{M'} t' dv_{g'} \\
&\geq \frac{m+2}{m(m+1)} \int_{M'} \tau'^2 dv_{g'} - \int_{M'} t' dv_{g'} \\
&\geq \frac{m+2}{m(m+1)} \int_M \tau^2 dv_g - \int_{M'} t' dv_{g'} \qquad Q.E.D.
\end{aligned}$$

Proposition 7. *Suppose that $n \leq m + 6 + \frac{27}{m-1}$ and $Spec(\mathcal{M}, J) = Spec(\mathcal{M}', J')$. If $\mathcal{M}$ is η-Einstein, then $\int_M t dv_g \leq \int_{M'} t' dv_{g'}$ holds and the equality holds if and only if $\mathcal{M}'$ is η'-Einstein. Furthermore, if $\mathcal{M}$ is η-Einstein and the second fundamental form of $\mathcal{M}'$ is parallel, then $\mathcal{M}'$ is also η'-Einstein and the second fundamental form of $\mathcal{M}$ is parallel.*

Proof. It is obvious from (6) of Corollary 1. *Q.E.D.*

From now on, we consider n ($\neq 5$)-dimensional, compact, miniaml, normal anti-invariant submanifolds M and M' of $\mathcal{N}^S(k)$ or $\mathcal{N}^C(k)$ with dimension $2n+1$.

First of all we have from (2) of Corollary 2.

Proposition 8. *Assume that $Spec(M, J) = Spec(M', J')$. Then if M is totally geodesic, so does M'.*

Proposition 9. *Assume that $Spec(M, J) = Spec(M', J')$ and $\int_M (l_n - t) dv_g \leq \int_{M'} (l'_n - t') dv_{g'}$. Then the second fundamental forms on M commute each other if and only if the second fundamental forms on M' commute each other and $\int_M (l_n - t) dv_g = \int_{M'} (l'_n - t') dv_{g'}$.*

Proof. This follows from (1) of Corollary 2. *Q.E.D.*

We get from (5) of Corollary 2.

Proposition 10. *Let M and M' be Einstein. Assume that $Spec(M, J) = Spec(M', J')$, $\int_M (6t - 7\tilde{k}_n) dv_g \leq \int_{M'} (6t' - 7\tilde{k}'_n) dv_{g'}$. Then M has a constant curvature $\tilde{k}$ if and only if M' has the same constant curvature $\tilde{k}$ and $\int_M (6t - 7\tilde{k}_n) dv_g = \int_{M'} (6t' - 7\tilde{k}'_n) dv_{g'}$.*

Proposition 11. *Assume that $Spec(M, J) = Spec(M', J')$. If M has a constant curvature $\tilde{k}$, and M' is Einstein and if $\int_M (6l_n - \tilde{k}_n) dv_g \leq \int_{M'} (6l'_n - \tilde{k}'_n) dv_{g'}$, then M' has the same constant curvature $\tilde{k}$ and $\int_M (6l_n - \tilde{k}_n) dv_g = \int_{M'} (6l'_n - \tilde{k}'_n) dv_{g'}$.*

Proof. It follows from (4) of Corollary 2. *Q.E.D.*

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