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# On $G$-vector bundles with bracket operations and an algebra with universal mapping property

Dedicated to Professor Tsuyoshi Watabe on his 60-th birthday

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## §0. Introduction.

In this paper we shall consider a class of algebras of $G$-invariant smooth sections of $G$-vector bundles over $G$-manifolds with bracket operations for a compact Lie group $G$. The class contains the Lie algebras of $G$-invariant smooth vector fields on $G$-manifolds.

Let $W$ be a Riemannian manifold and $G$ be a compact Lie subgroup of isometries of $W$. For an integer $n$, we shall construct an algebra $\Gamma_n^G(W)$ with a bracket operation which has a universal mapping property. The purpose of this paper is to investigate the geometric properties of the algebra $\Gamma_n^G(W)$.

Let $M$ be an $m$-dimensional $G$-submanifold of $W$ and $h$ be a $G$-invariant smooth section of $G_n(W)|M$, where $G_n(W)$ is the bundle of $n$-planes over $W$. Then $h$ defines a smooth $G$-bundle $\xi_h$ over $M$ and a bracket structure of the set of $G$-invariant smooth sections $\Gamma^G(M,h)$ of $\xi_h$ and induces an epimorphism $\hat{\mu}(h) : \Gamma_n^G(W) \to \Gamma^G(M,h)$. We shall determine the condition that $\hat{\mu}(h)$ is bracket preserving (see Theorem 2.2). Also, by using $\hat{\mu}(h)$, we shall describe the conditions for a $G$-vector bundle $\xi_h$ to be $G$-involutive and to be integrable in the case that $G$ is a finite group (see Theorem 2.3 and Corollary 2.4).

Especially if $h_M : M \to \Gamma_m^G(W)$ is a map associated to the tangent space $\tau(M)$ of $M$, then $\hat{\mu}(h_M)$ is a bracket preserving epimorphism from $\Gamma_m^G(W)$ to the Lie algebra $\mathfrak{X}_G(M)$ of $G$-invariant smooth vector fields on $M$. In the previous paper [1], we studied that the equivalence classes of the orbit spaces $M/G$ of smooth closed $G$-manifolds $M$ are coincide with the isomorphism classes of Lie algebras $\mathfrak{X}(M/G)$ of smooth vector fields of the orbit space $M/G$. If $G$ is a finite group, then the Lie algebra $\mathfrak{X}(M/G)$ is isomorphic to the Lie algebra $\mathfrak{X}_G(M)$ (see [2], §2). This implies that the smooth structure of $M/G$ is induced through the bracket preserving homomorphism $\hat{\mu}(h_M)$ from $\Gamma_n^G(W)$. In [3] we studied that those bracket preserving homomorphisms are closely related to the Riemannian structure of $W$.

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## §1. Construction of the algebra $\Gamma_n^G(W)$.

Let $W$ be an $N$-dimensional Riemannian manifold. Let $G$ be a compact Lie subgroup of isometries of $W$. For an integer $n \leq N$, let $q_n$: $G_n(W) \to W$ denote the bundle of $n$-planes over $W$. Let $p_n$: $E_n(W) \to G_n(W)$ be the canonical $n$-dimensional vector bundle. Let $\mathfrak{F}_n^x(W)$ denote the set of all smooth maps $f$: $U \to G_n(W)$ such that $q_n \circ f = 1_U$ for some open neighborhood $U$ of x. We use the notation $(U, f)$ for a map $f$: $U \to G_n(W)$ in $\mathfrak{F}_n^x(W)$. Let $\hat{\mathfrak{F}}_n^x(W)$ be the set of germs of $\mathfrak{F}_n^x(W)$ at x. Put

$$\mathfrak{F}_n(W) = \bigcup_{x \in W} \mathfrak{F}_n^x(W), \qquad \hat{\mathfrak{F}}_n(W) = \bigcup_{x \in W} \hat{\mathfrak{F}}_n^x(W).$$

We give the sheaf topology on $\hat{\mathfrak{F}}_n(W)$ so that $\hat{\mathfrak{F}}_n(W)$ is a sheaf on $W$ (c.f. [3], Chapter $I$, §1).

For $(U, f) \in \mathfrak{F}_n^x(W)$, let $\Gamma(U, f)$ denote the set of smooth sections of the induced bundle $f^* p_n$. Let $\mathfrak{S}_n^x(W)$ denote the set of sections $\Gamma(U, f)$ for $(U, f) \in \mathfrak{F}_n^x(W)$. Let $\hat{\mathfrak{S}}_n^x(W)$ be the set of germs of those $\Gamma(U, f)$ at x. Put

$$\mathfrak{S}_n(W) = \bigcup_{x \in W} \mathfrak{S}_n^x(W), \qquad \hat{\mathfrak{S}}_n(W) = \bigcup_{x \in W} \hat{\mathfrak{S}}_n^x(W).$$

We give the sheaf topology on $\hat{\mathfrak{S}}_n(W)$ so that $\hat{\mathfrak{S}}_n(W)$ is a sheaf on $\hat{\mathfrak{F}}_n(W)$. Let $f_x$ denote the germ of $f$ at x. and we denote by $\{s\}_{f_x}$ the germ of $s$ at $x$ if $s \in \Gamma(U, f)$. Let $\rho_n : \hat{\mathfrak{S}}_n(W) \to \hat{\mathfrak{F}}_n(W)$ be a map given by $q_n(\{s\}_{f_x}) = f_x$.

We define a $G$-action on $\mathfrak{F}_n(W)$ as follows. For $(U, f) \in \mathfrak{F}_n^x(W)$, let

$$(g \cdot f)(y) = g \cdot f(g^{-1} \cdot y) \qquad \text{for } g \in G,\ y \in g \cdot U.$$

Then $g \cdot f$ is an element of $\mathfrak{F}_n^{g \cdot x}(W)$. We have a $G$-action on $\hat{\mathfrak{F}}_n(W)$ given by

$$g \cdot f_x = (g \cdot f)_{g \cdot x} \qquad \text{for } g \in G,\ f_x \in \mathfrak{F}_n^x(W).$$

Also we define a $G$-action on $\mathfrak{S}_n(W)$ by

$$(g \cdot s)(y) = g \cdot s(g^{-1} \cdot y) \qquad \text{for } g \in G,\ s \in \Gamma(U, f),\ y \in g \cdot U.$$

Then a $G$-action on $\hat{\mathfrak{S}}_n(W)$ is given by

$$g \cdot \{s\}_{f_x} = \{g \cdot s\}_{g \cdot f_x} \qquad \text{for } g \in G,\ \{s\}_{f_x} \in \mathfrak{S}_n^x(W).$$

Note that the map $\rho_n$: $\mathfrak{S}_n(W) \to \mathfrak{F}_n(W)$ is $G$-equivariant. Let $\Gamma_n^G(W)$ be the set of $G$-equivariant continuous sections of $\rho_n$. Let $C_G^\infty(W)$ denote the set of $G$-invariant smooth functions on W. Then $\Gamma_n^G(W)$ is a $C_G^\infty(W)$-module in the natural way.

Now we shall define bracket operations on $\Gamma(U, f)$ and $\Gamma_n^G(W)$. Let $(U, f) \in \mathfrak{F}_n^x(W)$. Note that a section $s \in \Gamma(U, f)$ is regarded as a smooth vector field on U. Define a bracket operation $<\ ,\ >$ on $\Gamma(U, f)$ by

$$< s, t > (y) = p_{f(y)}([s, t](y)) \qquad \text{for } s, t \in \Gamma(U, f),\ y \in U.$$

Here $[s,\ t]$ is the bracket of vector fields $s$ and $t$ on $U$, and $p_{f(y)}$ denote the orthogonal projection from the tangent space $\tau_y(W)$ of $W$ at $y$ to the $n$-plane $f(y)$. Using the above bracket operation, we define a bracket operation $<\ ,\ >$ on $\Gamma_n^G(W)$ by

$$< s, t > (f_x) = \{< \tilde{s}, \tilde{t} >\}_{f_x} \qquad \text{for } s, t \in \Gamma_n^G(W),\ f_x \in \hat{\mathfrak{F}}_n(W),$$

where $\tilde{s}, \tilde{t}$ are elements of $\Gamma(U, f)$ for some open neighborhood $U$ of $x$ such that $s(f_x) = \{\tilde{s}\}_{f_x}$, $t(f_x) = \{\tilde{t}\}_{f_x}$. Note that $\widetilde{< s, t >} = < \tilde{s}, \tilde{t} >$.

We say $L$ to be an $\mathbf{R}$-algebra if $L$ is a vector space over the field $\mathbf{R}$ of real numbers with a bilinear operation $\beta_L : L \times L \to$ L. By the bracket operations we can regard $\Gamma(U, f)$ and $\Gamma_n^G(W)$ as $\mathbf{R}$-algebras.

**Definition** Let $L$ be an $\mathbf{R}$-algebra. We say that a pair $(L,\ \mu_L)$ has a property (P) if $\mu_L$ satisfies the following conditions:

(a) For any $(U, f) \in \mathfrak{F}_n(W)$, there exists a homomorphism $\mu_L(f)$: $L \to \Gamma(U, f)$ such that

$$\mu_L(\beta_L(v_1, v_2)) = < \mu_L(v_1),\ \mu_L(v_2) > \qquad \text{for } v_1,\ v_2 \in \text{L}.$$

(b) For any $(U, f) \in \mathfrak{F}_n(W)$ and an open set $V$ with $V \subset U$, $\mu_L(f|V) = r_{U,V} \circ \mu_L(f)$, where $r_{U,V}$: $\Gamma(U, f) \to \Gamma(V, f|V)$ is the restriction map.

(c) $g \cdot \mu_L(f)(v) = \mu_L(g \cdot f)(v) \qquad$ for $g \in G$, $(U, f) \in \mathfrak{F}_n(W)$, $v \in L$.

For any $(U, f) \in \mathfrak{F}_n(W)$, define an $\mathbf{R}$-algebra homomorphism $\hat{\mu}(f)$ :

$\Gamma_n^G(W) \to \Gamma(U,f)$ as follows. For any $s \in \Gamma_n^G(W)$ and $x \in U$, there exists $\tilde{s} \in \Gamma(U,f)$ such that $s(f_x) = \{\tilde{s}\}_{f_x}$. Let $\hat{\mu}(f)(s)(x) = \tilde{s}(x)$. It is easy to see that $\hat{\mu}(f)$ is well defined.

**Theorem 1.1** (1) *The* $\mathbf{R}$*-algebra* $(\Gamma_n^G(W),\ \hat{\mu})$ *has the property* (P).

(2) *If* $(L,\ \mu_L)$ *is a pair which has the property* (P), *then there exists a unique* $\mathbf{R}$*-algebra homomorphism* $\varphi\colon L \to \Gamma_n^G(W)$ *such that*

$$\hat{\mu}(f) \circ \varphi = \mu_L(f) \qquad \textit{for any } \ (U,f) \in \mathfrak{F}_n(W).$$

*Proof.* (1) Since $s(f_x) = \{\hat{\mu}(f)(s)\}_{f_x}$ for $s \in \Gamma_n^G(W)$, $(U,f) \in \mathfrak{F}_n(W)$, $x \in U$, we have that $\hat{\mu}(f)$ is bracket-preserving, and $\hat{\mu}$ satisfies the condition (a). It is clear that $\hat{\mu}$ satisfies the condition (b). If $g \in G$, $(U,f) \in \mathfrak{F}_n(W)$, $x \in U$, then

$$s((g\cdot f)_{g\cdot x}) = g\cdot s(f_x) = \{g\cdot \hat{\mu}(f)(s)\}_{(g\cdot f)_{g\cdot x}}.$$

Thus

$$(g\cdot \hat{\mu}(f)(s))(g\cdot x) = g\cdot \hat{\mu}(f)(s)(x) = \hat{\mu}(g\cdot f)(s)(g\cdot x).$$

This implies that $\hat{\mu}$ has the property (P).

(2) Let $(L,\ \mu_L)$ be a pair which has the property (P). Put

$$\varphi(v)(f_x) = \{\mu_L(f)(v)\}_{f_x} \qquad \text{for } \ v \in L,\ f_x \in \mathfrak{F}_n^x(W).$$

Then for $g \in G$,

$$\begin{aligned} g\cdot(\varphi(v)(f_x)) &= g\cdot\{\mu_L(f)(v)\}_{f_x} \\ &= \{\mu_L(g\cdot f)(v)\}_{(g\cdot f)_{g\cdot x}} = \varphi(v)(g\cdot f_x). \end{aligned}$$

Thus $\varphi$: $L \to \Gamma_n^G(W)$ is a well defined homomorphism. If $v_1, v_2 \in L$, $f_x \in \hat{\mathfrak{F}}_n^x(W)$, then

$$\begin{aligned} \varphi(\beta_L(v_1,v_2))(f_x) &= \{< \mu_L(f)(v_1),\ \mu_L(f)(v_2) >\}_{f_x} \\ &=< \varphi(v_1),\ \varphi(v_2) > (f_x). \end{aligned}$$

Therefore $\varphi$ is an $\mathbf{R}$-algebra homomorphism.

If $\psi$: $L \to \Gamma_n^G(W)$ is an $\mathbf{R}$-algebra homomorphism such that

$$\hat{\mu}(f)\circ\psi = \mu_L(f) \qquad \text{for any } \ (U,f) \in \mathfrak{F}_n(W),$$

then

$$\{\hat{\mu}(f)(\psi(v))\}_{f_x} = \varphi(v)(f_x) \qquad \text{for } \ v \in L,\ f_x \in \hat{\mathfrak{F}}_n^x(W).$$

Thus $\psi(v)(f_x) = \varphi(v)(f_x)$, and this completes the proof of Theorem 1.1.

**Corollary 1.2** *Any pair* $(L,\ \mu_L)$ *which has the property* (P) *is isomorphic to* $(\Gamma_n^G(W),\ \hat{\mu})$.

## §2. Properties of bracket operations

Let $M$ be an $m$-dimensional connected smooth closed $G$-submanifold of $W$. Let $\Gamma_n^G(M)$ denote the set of smooth $G$-maps $h$: $M \to G_n(W)$ such that $q_n \circ h = 1_M$. For $h \in \Gamma_n^G(M)$ let $\xi_h$ denote the $G$-bundle induced from $p_n : E_n(W) \to G_n(W)$ by $h$. Let $\Gamma(M, h)$ denote the set of smooth sections of the bundle $\xi_h$, and let $\Gamma^G(M, h)$ denote the subset of $\Gamma(M, h)$ whose elements are $G$-invariant.

Take a $G$-invariant $\varepsilon$-tubular neighborhood $U_\varepsilon$ of $M$ in $W$. Let $\pi$: $U_\varepsilon \to M$ be the natural projection and let

$$P_{u,\pi(u)}: \tau_u(W) \to \tau_{\pi(u)}(W) \qquad (u \in U_\varepsilon)$$

denote the parallel translation along the geodesic joining $u$ and $\pi(u)$. Define a smooth $G$-map $h^\sharp$: $U_\varepsilon \to G_n(W)$ by

$$h^\sharp(u) = P^{-1}_{u,\pi(u)}(h(\pi(u))) \qquad \text{for } u \in U_\varepsilon.$$

Then $(U_\varepsilon, h^\sharp) \in \mathfrak{F}_n^u(W)$ for $u \in U$. For $s \in \Gamma(M, h)$, put

$$s^\sharp(u) = P^{-1}_{u,\pi(u)}(s(\pi(u))) \qquad \text{for } u \in U_\varepsilon.$$

Then $s^\sharp \in \Gamma(U_\varepsilon, h^\sharp)$.

Now define a bracket operation $< , >$ on $\Gamma^G(M, h)$ by

$$< s, t > (x) = < s^\sharp, t^\sharp > (x) \qquad \text{for } s, t \in \Gamma^G(M, h),\ x \in \mathrm{M}.$$

Let $\hat{\mu}(h)$: $\Gamma_n^G(W) \to \Gamma^G(M, h)$ be a map defined by

$$\hat{\mu}(h)(s)(x) = \hat{\mu}(h^\sharp)(s)(x) \qquad \text{for } s \in \Gamma_n^G(W),\ x \in \mathrm{M}.$$

**Proposition 2.1** *$\hat{\mu}(h)$ is epimorphic.*

*Proof.* Let $\xi$ be a real valued $G$-invariant smooth function on $W$ such that

$$\xi(u) = 1 \qquad \text{if } \|u - \pi(u)\| \le \varepsilon/3,$$

$$= 0 \qquad \text{if } \|u - \pi(u)\| \ge 2\varepsilon/3.$$

For any $s \in \Gamma^G(M,h)$, $(U,f) \in \mathfrak{F}_n^y(W)$ $(y \in U_\epsilon)$ put

$$s^f(z) = p_{f(z)}(\xi(z)s^\sharp(z)) \qquad \text{if } z \in U_\epsilon \cap U.$$

Then $s^f \in \Gamma(U_\epsilon \cap U, f)$. Define

$$\begin{aligned} \hat{s}(f_y) &= \{s^f\}_{f_y} && \text{for } y \in U_\epsilon \\ &= 0 && \text{for } y \notin U_\epsilon \end{aligned}$$

Then $\hat{s}$ is a well defined section of $\rho_n$. Since

$$p_{g\cdot f(y)}(v) = g \cdot p_{f(y)}(g^{-1} \cdot v) \qquad \text{for } g \in G,\ v \in p_n^{-1}(g \cdot f(y)),$$

we see that $\hat{s} \in \Gamma_n^G(W)$ and $\hat{\mu}(h)(\hat{s}) = s$. This completes the proof of Proposition 2.1.

Let $G_x$ $(x \in M)$ denote the isotropy subgroup of $G$ at $x$, and put

$$h(x)^{G_x} = \{v \in h(x)\ ;\ g \cdot v = v \quad \text{for } g \in G_x\}.$$

**Theorem 2.2** *The homomorphism $\hat{\mu}(h)$ is bracket preserving if and only if $h(x)^{G_x}$ is contained in $\tau_x(M)$ for $x \in M$.*

*Proof.* Assume that $h(x)^{G_x}$ is contained in $\tau_x(M)$ for $x \in$ M. If $s$, $t \in \Gamma_n^G(W)$ and $x \in M$, there exist $\tilde{s}, \tilde{t} \in \Gamma(U, h^\sharp)$ such that

$$s(h_x^\sharp) = \{\tilde{s}\}_{h_x^\sharp}, \quad t(h_x^\sharp) = \{\tilde{t}\}_{h_x^\sharp}.$$

Then

$$< s, t > (h^\sharp) = \{< \tilde{s}, \tilde{t} >\}_{h_x^\sharp} \qquad \text{for } x \in \text{M}.$$

Since $h(x)^{G_x}$ is contained in $\tau_x(M)$ for $x \in M$,

$$\hat{\mu}(h)(s)(x),\ \hat{\mu}(h)(t)(x) \in \tau_x(M).$$

Then, for $x \in M$,

$$\begin{aligned} \hat{\mu}(h)(< s,\ t >)(x) &= \hat{\mu}(h^\sharp)(< s,\ t >)(x) \\ &= < \tilde{s}, \tilde{t} > (x) \\ &= p_{h(x)}([\tilde{s},\ \tilde{t}](x)) \\ &= p_{h(x)}([\hat{\mu}(h)(s),\ \hat{\mu}(h)(t)](x)) \\ &= < \hat{\mu}(h)(s),\ \hat{\mu}(h)(t) > (x). \end{aligned}$$

Thus $\hat{\mu}(h)$ is bracket preserving.

Conversely, assume that $h(q)^{G_q}$ is not contained in $\tau_q(M)$ for some $q \in$ M. Then there exists $s \in \Gamma^G(M, h)$ such that $s(q)$ is not a vector in $\tau_q(M)$ for some $q \in$ M. By the differentiable slice theorem (c.f. [5], Chapter I, §1.3), we have a $G$-invariant neighborhood of $q$ in $W$ which is $G$-equivariantly diffeomorphic to a differentiable $G$-bundle $G \times_{G_q} (S_q \times \nu_q(M))$ over $G/G_q$. Here $S_q$ is a linear slice at $q$ in $M$ and $\nu_q(M)$ is the normal space of $M$ in $W$ at q. Then we have a local coordinate system

$$\{x_1, ..., x_m, y_1., ..., y_{N-m}\} \quad (\dim W = N)$$

of $W$ on an open neighborhood $V_1$ of $q$ in $U_{\varepsilon/3}$ such that

(1) $x_i(q) = 0 \ (1 \leq i \leq m), \ \ y_j(q) = 0 \ (1 \leq j \leq N - m)$.

(2) The set $V_0 = \{p \in V \ ; \ y_1(q) = ... = y_{N-m}(q) = 0\ \}$ together with the restriction of $\{x_1, ..., x_m\}$ to $V_0$ form a local chart on $M$ at x.

(3) $x_i(v) = x_i(\pi(v)) \quad$ for $\ v \in V_1, \ i = 1, ..., m$.

(4) $y_j(g \cdot v) = y_j(v) \ (1 \leq j \leq \ell)$ for $g \in G$, if $v, \ g \cdot v \in V_1$,

where $\ell$ is the dimension of the fixed point set $\nu_q(M)^{G_q}$ of the normal space $\nu_q(M)$ under the isotropy subgroup $G_q$ of $G$ at $q$.

Then the section $s$ is described on $V_1$ as follows:

$$\begin{aligned} s \ &= \ \sum_{i=1}^{m} a_i(x_1, ..., x_m, y_1, ..., y_{N-m}) \frac{\partial}{\partial x_i} \\ &+ \ \sum_{j=1}^{N-m} b_j(x_1, ..., x_m, y_1, ..., y_{N-m}) \frac{\partial}{\partial y_j}. \end{aligned}$$

Since $h(q)^{G_q}$ is not contained in $\tau_q(M)$, there exists a number $k \ (1 \leq k \leq \ell)$ with $b_k(0) \neq 0$. As in the proof of Proposition 2.1, we have $\hat{s} \in \Gamma_n^G(W)$ such that

$$\hat{s}(f_y) = \{s^f\}_{f_y} \qquad \text{for } \ y \in V_1, \ f_y \in \hat{\mathfrak{F}}_n^y(W).$$

Let $V_2$ be a $G_q$-invariant open neighborhood of $q$ in $W$ such that the closure $\bar{V}_2$ of $V_2$ is contained in $V_1$. There exists $\hat{y}_k \in C_G^\infty(W)$ such that $\hat{y}_k = y_k$ on $V_2$. Put

$$\hat{t}(f_y) = \hat{y}_k(y)\hat{s}(f_y) \qquad \text{for } \ y \in W, \ f_y \in \hat{\mathfrak{F}}_n^y(W).$$

Then $\hat{t} \in \Gamma_n^G(W)$, and we see that, for $u \in U_{\varepsilon/3} \cap V_2$,

$$\hat{\mu}(h^\sharp)(\hat{s})(u) = s^\sharp(u),$$
$$\hat{\mu}(h^\sharp)(\hat{t})(u) = \hat{y}_k(u) s^\sharp(u).$$

Thus

$$\hat{\mu}(h)(<\hat{s},\ \hat{t}>)(q) = <s^{\sharp},\ \hat{y}_k \cdot s^{\sharp}> (q) = s^{\sharp}(\hat{y}_k)(q)s^{\sharp}(q).$$

Since the $(m+k)$-th component of $s^{\sharp}(\hat{y}_k)(q)s^{\sharp}(q)$ is $b_k(0)^2$, we have

$$\hat{\mu}(h)(<\hat{s},\ \hat{t}>)(q) \neq 0.$$

Since $\hat{\mu}(h)(\hat{t}) = 0$ on a neighborhood of $q$ in $M$,

$$<\hat{\mu}(h)(\hat{s}),\ \hat{\mu}(h)(\hat{t})> (q) = 0.$$

This completes the proof of Theorem 2.2.

Let $\mathfrak{X}_G(M)$ denote the Lie algebra of $G$-invariant smooth vector fields on M. Let $h_M$: $M \to G_m(W)$ denote a $G$-map given by

$$h_M(x) = \tau_x(M) \qquad \text{for } x \in M.$$

Note that $\mathfrak{X}_G(M) = \Gamma^G(M,\ h_M)$. For $h \in \Gamma^G_n(M)$, let $\hat{\eta}(h) : \Gamma^G_n(W) \to \mathfrak{X}_G(M)$ be a homomorphism defined by

$$\hat{\eta}(h)(s)(x) = p_{h_M(x)}(\hat{\mu}(h)(s)(x)) \qquad \text{for } x \in M,\ s \in \Gamma^G_n(W).$$

We say that $h$ is $G$-involutive if $h(x)^{G_x}$ is contained in $\tau_x(M)$ for any $x \in M$ and

$$[s,\ t] \in \Gamma^G(M,h) \qquad \text{for any } s,t \in \Gamma^G(M,h).$$

**Theorem 2.3** (1) $\hat{\eta}(h) = 0$ *if and only if* $h(x)^{G_x}$ *is contained in the normal space* $\nu_x(M)$ *at* $x$ *in* $W$ *for any* $x \in M$.

(2) *Assume* $\hat{\eta}(h) \neq 0$. *Then* $\hat{\eta}(h)$ *is bracket preserving if and only if* $h$ *is* $G$*-involutive.*

*Proof.* (1) Assume $\hat{\eta}(h) = 0$. Let $x \in$ M. For any $v \in h(x)^{G_x}$, there exists $s \in \Gamma^G(M,h)$ such that $s(x) =$ v. By Proposition 2.1, there exists $\hat{s} \in \Gamma^G_n(W)$ such that $\hat{\mu}(h)(\hat{s}) = s$. Then $\hat{\eta}(h)(\hat{s})(x) = p_{h_M(x)}(v) = 0$, and $v \in \nu_x(M)$. Conversely assume that $h(x)^{G_x}$ is contained in $\nu_x(M)$ for any $x \in$ M. Since $\hat{\mu}(h)(s)(x) \in h(x)^{G_x}$ for $s \in \Gamma^G_n(W)$, $x \in M$, we have $\hat{\eta}(h) = 0$.

(2) If $\hat{\mu}(h)$ is bracket preserving, then it follows from Theorem 2.2 that $\hat{\eta}(h)$ is bracket preserving if and only if $h$ is $G$-involutive. Assume that $\hat{\eta}(h) \neq 0$ and $\hat{\mu}(h)$ is not bracket preserving. Since $M$ is connected, from Theorem 2.2 there exist a point $q \in M$ and a vector $v \in h(q)^{G_q}$ such that $v \notin \tau_q(M)$ and $v \notin \nu_q(M)$. We can choose $s \in \Gamma^G(M,h)$ such that $s(q) = v$.

Let $\{x_1, ..., x_m, y_1., ..., y_{N-m}\}$ be a local coordinate system as in the proof of Theorem 2.2. Then $s$ is described on a neighborhood of $q$ as follows:

$$\begin{aligned} s &= \sum_{i=1}^{m} a_i(x_1, ..., x_m, y_1, ..., y_{N-m})\frac{\partial}{\partial x_i} \\ &+ \sum_{j=1}^{N-m} b_j(x_1, ..., x_m, y_1, ..., y_{N-m})\frac{\partial}{\partial y_j}. \end{aligned}$$

Since $s(q) \notin \tau_q(M)$ and $s(q) \notin \nu_q(M)$, there exist a number $i$ $(1 \leq i \leq m)$ with $a_i(0) \neq 0$ and a number $k$ $(1 \leq k \leq \ell)$ with $b_k(0) \neq 0$. With the same argument as in the proof of Theorem 2.2, we define $\hat{s}$ and $\hat{t}$. Then we have

$$\hat{\mu}(h)(< \hat{s},\ \hat{t} >)(q) = s^{\sharp}(\hat{y}_k)(q)s^{\sharp}(q).$$

Note that the $i$-th component of $s^{\sharp}(\hat{y}_k)(q)s^{\sharp}(q)$ is $b_k(0)a_i(0)$. Thus

$$\hat{\eta}(h)(< \hat{s},\ \hat{t} >)(q) \neq 0.$$

Since $\hat{\mu}(h)(t) = 0$ on a neighborhood of $q$ in $M$,

$$< \hat{\eta}(h)(\hat{s}),\ \hat{\eta}(h)(\hat{t}) > (q) = 0.$$

This completes the proof of Theorem 2.3.

$h$ is said to be integrable if the induced bundle $\xi_h$ is integrable.

**Corollary 2.4** *If $G$ is a finite group and $\hat{\eta}(h) \neq 0$, then $\hat{\eta}(h)$ is bracket-preserving if and only if $h$ is integrable.*

*Proof.* Assume that $\hat{\eta}(h)$ is bracket preserving. By Theorem 2.3, we see that

$$< s, t >= [s, t] \qquad \text{for } s, t \in \Gamma^G(M, h).$$

Put $M_0 = \{x \in M\ ;\ G_x = \{1\}\}$. Let $s, t \in \Gamma(M, h)$. For $x \in M_0$, there exist $s_1$, $t_1 \in \Gamma^G(M, h)$ such that $s_1 = s$ and $t_1 = t$ on a neighborhood $V$ of $x$ in M. Then for $x \in V$

$$< s, t > (x) \ = < s_1, t_1 > (x) \ = \ [s_1, t_1](x) \ = \ [s, t](x).$$

Since $M_0$ is open dense in $M$, we see that $< s,\ t >= [s,\ t]$. Then, by Frobenius'theorem (c.f. [5], Theorem 2.4.5), $h$ is integrable. If $h$ is integrable, then $\hat{\eta}(h)$ is bracket preserving by Theorem 2.3 (2), and Corollary 2.4 follows.

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