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CERTAIN CLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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ABSTRACT. In this paper we obtain convolution properties and integral transforms of functions in the classes $\Sigma_p^*(\alpha,\beta,A,B)$ ($0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, $0 < B \le 1$) and $\Sigma_p[\alpha,\beta,A,B,\gamma]$ ($0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, $0 < B \le 1$, $\frac{A}{A-B} \le \gamma \le 1$) of moremorphic univalent functions with positive coefficients in $0 < |z| < 1$.

KEY WORDS- Univalent, convolution meromorphic.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $U^* = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. And let Σ_S denote the subclass of Σ consisting of analytic and univalent functions $f(z)$ in U^*. A function $f(z)$ in Σ_S is said to be meromorphically starlike of order α if

$$\mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha, \qquad (z \in U^*) \tag{1.2}$$

for some $\alpha (0 \leq \alpha < 1)$. We denote $\Sigma^*(\alpha)$ the class of all meromorphically starlike functions of order α. A function $f(z)$ in Σ_S is said to be meromorphically convex of order α if

$$\mathrm{Re}\left\{-(1 + \frac{zf''(z)}{f'(z)})\right\} > \alpha, \qquad (z \in U^*) \tag{1.3}$$

for some $\alpha (0 \leq \alpha < 1)$. And we denote by $\Sigma_k(\alpha)$ the class of all meromorphically convex functions of order α. The class $\Sigma^*(\alpha)$ and similar classes have been extensively studied by Pommerenke [10], Clunie [5], Miller [8] and others.

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \qquad (a_n \geq 0) \tag{1.4}$$

that are regular and univalent in U^*.

Definition 1 [1]. A function $f(z)$ in Σ is in the class $\Sigma^*(\alpha,\beta,A,B)$ if it satisfies the condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} + 1}{B\,\dfrac{zf'(z)}{f(z)} + [B + (A-B)(1-\alpha)]} \right| < \beta \qquad (z \in U^*) \tag{1.5}$$

for some $\alpha(0 \le \alpha < 1)$, $\beta(0 < \beta \le 1)$, $-1 \le A < B \le 1$, and $0 < B \le 1$.

Definition 2 [2]. A function $f(z)$ in Σ is in the class $\Sigma[\alpha,\beta,A,B,\gamma]$ if it satisfies the condition

$$\left| \frac{z^2 f'(z) + 1}{[(B-A)\gamma+A]z^2 f'(z)+[(B-A)\gamma\alpha+A]} \right| < \beta \quad (z \in U^*) \tag{1.6}$$

for some $\alpha(0 \le \alpha < 1)$, $\beta(0 < \beta \le 1)$, $-1\le A < B \le 1$, $0 < B \le 1$, and $\gamma(\frac{A}{A-B} \le \gamma \le 1)$.

Let us write

$$\Sigma_p^*(\alpha,\beta,A,B) = \Sigma_p \cap \Sigma^*(\alpha,\beta,A,B)$$

and

$$\Sigma_p[\alpha,\beta,A,B,\gamma] = \Sigma_p \cap \Sigma[\alpha,\beta,A,B,\gamma].$$

The classes $\Sigma_p^*(\alpha,\beta,A,B)$ and $\Sigma_p[\alpha,\beta,A,B,\gamma]$ were introduced and studied by Aouf [1,2].

We note that:

(i) $\Sigma_p^*(\alpha,\beta,-1,1) = \Sigma_p^*(\alpha,\beta)$, is the class of meromorphically starlike functions of order $\alpha(0 \le \alpha < 1)$ and type $\beta(0 < \beta \le 1)$ with positive coefficients, studied by Mogra, Reddy and Juneja [9] and Uralegaddi and Ganigi [12].

(ii) $\Sigma_p^*(\alpha,1,-1,1) = \Sigma_p^*(\alpha)$, is the class of meromorphically starlike functions of order $\alpha(0 \le \alpha < 1)$ with positive coefficients, studied by Juneja and Reddy [7].

(iii) $\Sigma_p[\alpha,\beta,-1,1,\gamma] = \Sigma_p(\alpha,\beta,\gamma)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1,$ and $\frac{1}{2} \le \gamma < 1)$, is the class of meromorphic univalent functions in U^*, studied by Cho, Lee and Owa [4].

(iv) $\Sigma_p[\alpha,1,-1,1,1] = \Sigma_p[\alpha]$, is the class:

$$\Sigma_p[\alpha] = \left\{f \in \Sigma_p : \operatorname{Re}\left\{-\ z^2 f'(z)\right\} > \alpha,\ 0 \le \alpha < 1,\ z \in U^*\right\}.$$

(v) $\Sigma_p^*(0,1,A,B) = \Sigma_p^*(A,B)$, is the class:

$$\Sigma_p^*(A,B) = \left\{f \in \Sigma_p : -\ \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in U^*\right\}.$$

(vi) $\Sigma_p[0,1,A,B,1] = \Sigma_p[A,B]$, is the class:

$$\Sigma_p[A,B] = \left\{f \in \Sigma_p : -\ z^2 f'(z) \prec \frac{1+Az}{1+Bz},\ z \in U^*\right\}.$$

We shall require the following lemmas in our investigation.

<u>Lemma 1 [1]</u>. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ be regular in U^*. If

$$\sum_{n=1}^{\infty} \left\{(1+\beta B)n+\beta[(B-A)\alpha+A]+1\right\} |a_n| \le (B-A)\beta(1-\alpha), \qquad (1.7)$$

$0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, and $0 < B \le 1$, then $f(z) \in \Sigma^*(\alpha,\beta,A,B]$.

<u>Lemma 2 [1]</u>. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \ge 0$, be regular in U^*. Then $f(z) \in \Sigma_p^*(\alpha,\beta,A,B)$ if and only if (1.7) is satisfied.

<u>Lemma 3 [2]</u>. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ be regular in U^*. If

$$\sum_{n=1}^{\infty} n\left[1 + (B - A)\beta\gamma + A\beta\right]|a_n| \le (B - A)\beta\gamma(1 - \alpha), \quad (1.8)$$

for some $\alpha(0 \le \alpha < 1)$, $\beta(0 < \beta \le 1)$, $-1 \le A < B \le 1$, $0 < B \le 1$, and $\gamma(\frac{A}{A-B} \le \gamma \le 1)$, then $f(z)$ is in the class $\Sigma[\alpha,\beta,A,B,\gamma]$.

<u>Lemma 4 [2]</u>. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \ge 0$. be regular in U^*. Then $f(z) \in \Sigma_p[\alpha,\beta,A,B,\gamma]$ if and only if (1.8) is satisfied.

Lemma 5 [1,9,12]. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$, be regular in U^*. Then $f(z) \in \Sigma_p^*(\alpha)$, $0 \leq \alpha < 1$, if and only if

$$\sum_{n=1}^{\infty} (n + \alpha) a_n \leq (1 - \alpha). \tag{1.9}$$

Lemma 6 [2,4]. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $a_n \geq 0$, be regular in U^*. Then $f(z) \in \Sigma_p[\alpha]$ if and only if

$$\sum_{n=1}^{\infty} n a_n \leq (1 - \alpha). \tag{1.10}$$

In the present paper, we extend the study made in [1,2] to obtain convolution properties and integral transforms of functions in the classes $\Sigma_p^*(\alpha,\beta,A,B)$ and $\Sigma_p[\alpha,\beta,A,B,\gamma]$.

3. Integral transforms.

In this section we consider integral transforms of functions in the classes $\Sigma_p^*(\alpha,\beta,A,B)$ and $\Sigma_p[\alpha,\beta,A,B,\gamma]$ of the type considered by Bajpai [3] and Goel and Sohi [6].

Theorem 1. If $f(z)$ is in the class $\Sigma_p^*(\alpha,\beta,A,B)$, then the integral transforms

$$F_c(z) = c \int_0^1 u^c f(uz)\, du, \qquad 0 < c < \infty,$$

are in the class $\Sigma_p^*(\delta)$, $0 \le \delta < 1$, where

$$\delta = \delta(\alpha,\beta,A,B,c) = \frac{\{2+\beta[(B-A)\alpha+(B+A)]\}(2+c)-(B-A)\beta(1-\alpha)c}{\{2+\beta[(B-A)\alpha+(B+A)]\}(2+c)+(B-A)\beta(1-\alpha)c}. \tag{2.1}$$

The result is best possible for the function f(z) given by

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\alpha)}{2+\beta[(B-A)\alpha+(B+A)]}z.$$

<u>Proof</u>. Suppose $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_p^*(\alpha,\beta,A,B)$.
Then we have

$$F_c(z) = c\int_0^1 u^c f(uz)\,du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c\,a_n}{n+c+1} z^n.$$

In view of Lemma 5, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{n+\delta}{1-\delta}\,\frac{c\,a_n}{n+c+1} \le 1. \tag{2.2}$$

Since $f(z) \in \Sigma_p^*(\alpha,\beta,A,B)$, by Lemma 2, we have

$$\sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} a_n \le 1. \tag{2.3}$$

Thus (2.2) will be satisfied if, for each n,

$$\frac{(n+\delta)c}{(1-\delta)(n+c+1)} \le \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)},$$

or

$$\delta \leq \frac{\{(1+\beta B)n+\beta[(B-A)\alpha+A]+1\}(n+c+1)-(B-A)\beta(1-\alpha)cn}{\{(1+\beta B)n+\beta[(B-A)\alpha+A]+1\}(n+c+1)+(B-A)\beta(1-\alpha)cn}. \quad (2.4)$$

Since the right hand side of (2.4) is an increasing function of n, putting $n = 1$ in (2.4) we get

$$\delta \leq \frac{\{2+\beta[(B-A)\alpha+(B+A)]\}(2+c)-(B-A)\beta(1-\alpha)c}{\{2+\beta[(B-A)\alpha+(B+A)]\}(2+c)+(B-A)\beta(1-\alpha)c}.$$

Hence the theorem.

Remark 1. It is interesting to note that for $c = 1$ and $(\alpha,\beta) = (0,1)$, Theorem 1 gives that if $f(z) \in \Sigma_p^*(A,B)$, then

$$F_1(z) = \int_0^1 u\, f(uz)\, du$$

is in the class $\Sigma_p^*(\frac{3 + B + 2A}{3 + 2B + A})$.

Using arguments similar to those in the proof of Theorem 1, using Lemmas 4 and 6, we can obtain the following result.

Theorem 2. If $f(z)$ is in the class $\Sigma_p[\alpha,\beta,A,B,\gamma]$, then the integral transforms

$$F_c(z) = c\int_0^1 u^c\, f(uz)\, du, \qquad 0 < c < \infty,$$

are in the class $\Sigma_p[\delta]$, $0 \leq \delta < 1$, where

$$\delta = \delta(\alpha,\beta,A,B,\gamma,c) = \frac{(1+(B-A)\beta\gamma+A\beta)(2+c)-(B-A)\beta\gamma(1-\alpha)c}{(1+(B-A)\beta\gamma+A\beta)(2+c)}.$$

The result is best possible for the function f(z) given by

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-\alpha)}{1+(B-A)\beta\gamma+A\beta}\, z.$$

<u>Remark 2</u>. It is interesting to note that for c = 1 and $(\alpha,\beta,\gamma) = (0,1,1)$, Theorem 1 gives that if $f(z) \in \Sigma_p[A,B]$, then

$$F_1(z) = \int_0^1 u\, f(uz)\, du$$

is in the class $\Sigma_p[\frac{3+2B+A}{3+3B}]$.

3. <u>Convolution properties.</u>

Robertson [11] has shown that if $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in the class Σ_s then so is their convolution $f(z)*g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$. We prove the following theorems for functions in the classes $\Sigma_p^*(\alpha,\beta,A,B)$ and $\Sigma_p[\alpha,\beta,A,B,\gamma]$.

Theorem 3. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in the class $\Sigma_p^*(\alpha,\beta,A,B)$ then, $f(z)*g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$ is in the class $\Sigma_p^*(\varphi,\beta,A,B)$, where

$$\varphi = 1 - \frac{2(B-A)\beta(1+\beta B)(1-\alpha)^2}{\{2+\beta[(B-A)\alpha+(B+A)]\}^2 + (B-A)^2\beta^2(1-\alpha)^2}. \tag{3.1}$$

The result is best possible for the functions

$$f(z) = g(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\alpha)}{2+\beta[(B-A)\alpha+(B+A)]} z .$$

Proof. In order to complete our theorem, we have to find the largest $\varphi = \varphi(\alpha,\beta,A,B)$ such that

$$\sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\varphi+A]+1}{(B-A)\beta(1-\varphi)} a_n b_n \leq 1 \tag{3.2}$$

for $f(z)$ and $g(z)$ in the class $\Sigma_p^*(\alpha,\beta,A,B)$. Since $f(z)$ and $g(z)$ are in the class $\Sigma_p^*(\alpha,\beta,A,B)$, in view of Lemma 2, we see that

$$\sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} a_n \leq 1, \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} b_n \leq 1. \tag{3.4}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} \sqrt{a_n b_n} \le 1. \tag{3.5}$$

This implies that we need only to show that

$$\frac{(1+\beta B)n+\beta[(B-A)\varphi+A]+1}{(1-\varphi)} a_n b_n \le \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(1-\alpha)} \sqrt{a_n b_n} \tag{3.6}$$

or

$$\sqrt{a_n b_n} \le \frac{(1-\varphi)\{(1+\beta B)n+\beta[(B-A)\alpha+A]+1\}}{(1-\alpha)\{(1+\beta B)n+\beta[(B-A)\varphi+A]+1\}} \quad (n \ge 1). \tag{3.7}$$

Hence, by the inequality (3.5), it is sufficient to prove that

$$\frac{(B-A)\beta(1-\alpha)}{\{(1+\beta B)n+\beta[(B-A)\alpha+A]+1\}} \le \frac{(1-\varphi)\{(1+\beta B)n+\beta[(B-A)\alpha+A]+1\}}{(1-\alpha)\{(1+\beta B)n+\beta[(B-A)\varphi+A]+1\}} \quad (n \ge 1).. \tag{3.8}$$

It follows from (3.8) that

$$\varphi \le 1 - \frac{(1+n)(B-A)\beta(1+\beta B)(1-\alpha)^2}{\{(1+\beta B)n+\beta[(B-A)\alpha+A]+1\}^2 + (B-A)^2\beta^2(1-\alpha)^2} \quad (n \ge 1). \tag{3.9}$$

Defining the function $\psi(n)$ by

$$\psi(n) = 1 - \frac{(1+n)(B-A)\beta(1+\beta B)(1-\alpha)^2}{\{(1+\beta B)n+\beta[(B-A)\alpha+A]+1\}^2+(B-A)^2\beta^2(1-\alpha)^2} \quad (n \geq 1), \tag{3.10}$$

we see that $\psi(n)$ is increasing of n. Therefore, we conclude that

$$\varphi \leq \psi(1) = 1 - \frac{2(B-A)\beta(1+\beta B)(1-\alpha)^2}{\{2+\beta[(B-A)\alpha+(B+A)]\}^2+(B-A)^2\beta^2(1-\alpha)^2}$$

which completes the assertion of Theorem 3.

Using arguments similar to those in the proof of Theorem 3., we can obtain the following results.

<u>Theorem 4</u>. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_p^*(\alpha,\beta,A,B)$, and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \Sigma_p^*(\varphi,\beta,A,B)$, then $f(z)*g(z) \in \Sigma_p^*(\tau,\beta,A,B)$, where

$$\tau = 1 - \frac{2(B-A)\beta(1+\beta B)(1-\alpha)(1-\varphi)}{\{2+\beta[(B-A)\alpha+(B+A)]\}\{2+\beta[(B-A)\varphi+(B+A)]\}+(B-A)^2\beta^2(1-\alpha)(1-\varphi)}.$$

The result is best possible for the functions

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\alpha)}{2+\beta[(B-A)\alpha+(B+A)]} z$$

and

$$g(z) = \frac{1}{z} + \frac{(B-A)\beta(1-\varphi)}{2+\beta[(B-A)\varphi+(B+A)]} z.$$

Theorem 5. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in the class $\Sigma_p[\alpha,\beta,A,B,\gamma]$, then $f(z)*g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n \in \Sigma_p[\delta,\beta,A,B,\gamma]$, where

$$\delta = 1 - \frac{(B-A)\beta\gamma(1-\alpha)^2}{1+(B-A)\beta\gamma+A\beta}.$$

The result is best possible for the functions

$$f(z) = g(z) = \frac{1}{z} + \frac{(B - A)\beta\gamma(1 - \alpha)}{1 + (B - A)\beta\gamma + A\beta} z.$$

Theorem 6. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_p[\alpha,\beta,A,B,\gamma]$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \Sigma_p[\eta,\beta,A,B,\gamma]$, then $f(z)*g(z) \in \Sigma_p[\tau,\beta,A,B,\gamma]$, where

$$\tau = 1 - \frac{(B-A)\beta\gamma(1-\alpha)(1-\eta)}{1+(B-A)\beta\gamma+A\beta}.$$

The result is best possible for the functions

$$f(z) = \frac{1}{z} + \frac{(B - A)\beta\gamma(1 - \alpha)}{1 + (B - A)\beta\gamma + A\beta} z$$

and

$$g(z) = \frac{1}{z} + \frac{(B - A)\beta\gamma(1 - \eta)}{1 + (B - A)\beta\gamma + A\beta} z.$$

Theorem 7. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_p^*(\alpha,\beta,A,B)$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $|b_n| \leq 1$, $n = 1, 2, \ldots$, then $f(z)*g(z) \in \Sigma^*(\alpha,\beta,A,B)$.

Proof. Since

$$\sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} |a_n b_n|$$

$$= \sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} a_n |b_n|$$

$$\leq \sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} a_n \leq 1,$$

by Lemma 1, it follows that $f(z)*g(z) \in \Sigma^*(\alpha,\beta,A,B)$.

Corollary 1. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_p^*(\alpha,\beta,A,B)$ and

$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $0 \le b_n \le 1$, $n = 1, 2, \ldots$, then $f(z) * g(z) \in \Sigma_p^*(\alpha, \beta, A, B)$.

Using arguments similar to those in the proof of Theorem 7, and using Lemmas 3, and 4, we can obtain the following result.

Theorem 8. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_p[\alpha, \beta, A, B, \gamma]$, and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $|b_n| \le 1$, $n = 1, 2, \ldots$, then $f(z) * g(z) \in \Sigma\ [\alpha, \beta, A, B, \gamma]$.

Corollary 2. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \Sigma_p[\alpha, \beta, A, B, \gamma]$, and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $0 \le b_n \le 1$, $n = 1, 2, \ldots$, then $f(z) * g(z) \in \Sigma_p[\alpha, \beta, A, B, \gamma]$.

Theorem 9. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in the class $\Sigma_p^*(\alpha, \beta, A, B)$ and

$$2 + 3(B - A)\alpha\beta + (3A - B)\beta \ge 0,$$

then

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) z^n$$

also belongs to the class $\Sigma_p^*(\alpha,\beta,A,B)$.

<u>Proof</u>. Since $f(z) \in \Sigma_p^*(\alpha,\beta,A,B)$, using Lemma 2, we have

$$\sum_{n=1}^{\infty} \frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} a_n \leq 1.$$

and so

$$\sum_{n=1}^{\infty} \left[\frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)}\right]^2 a_n^2 \leq 1.$$

Similarly, since $g(z) \in \Sigma_p^*(\alpha,\beta,A,B)$,

$$\sum_{n=1}^{\infty} \left[\frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)}\right]^2 b_n^2 \leq 1.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{2}\left[\frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)}\right]^2 \left(a_n^2 + b_n^2\right) \leq 1.$$

In view of Lemma 2, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left[\frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)}\right] \left(a_n^2 + b_n^2\right) \leq 1. \tag{3.11}$$

Thus the inequality (3.11) will be satisfied if, for $n = 1,2,\ldots$

$$\frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)} \le \frac{1}{2}\left[\frac{(1+\beta B)n+\beta[(B-A)\alpha+A]+1}{(B-A)\beta(1-\alpha)}\right]^2, \quad (3.12)$$

or if

$$(1 + \beta B)n + 3(B - A)\alpha\beta + (3A - 2B)\beta + 1 \ge 0, \qquad (3.13)$$

for $n = 1, 2, \ldots$. The left hand side of (3.13) is an increasing function of n, hence (3.13) is satisfied for all n if

$$2 + 3(B - A)\alpha\beta + (3A - B)\beta \ge 0,$$

which is true by our assumption. Hence the theorem.

Using arguments similar to those in the proof of Theorem 9, and using Lemma 4, we can obtain the following result.

Theorem 10. If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in the class $\Sigma_p[\alpha,\beta,A,B,\gamma]$ and

$$1 + 2\gamma(B - A)\alpha\beta - ((B - A)\gamma - A)\beta \ge 0,$$

then

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) z^n$$

also belongs to the class $\Sigma_p[\alpha,\beta,A,B,\gamma]$.

Remark 3.

(i) Putting A = −1 and B = 1, in Theorems 1,3,4,7 and 9, we get the corresponding results obtained in [9].

(ii) Putting A = −1 and B = 1, in Theorems 2,5,6,8 and 10, we get the corresponding results for the class $\Sigma_p(\alpha,\beta,\gamma)$ defined by Cho, Lee and Owa [4].

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