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VARIATIONS ON CALIBRATIONS

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Abstract The theory of calibrations and calibrated foliations is developed in a couple of papers (cf. [1],[2]) by R.Harvey and H.B.Lawson Jr. This article derives some formulas for the first and second variation of a calibration. In particular, we show that on *any* compact oriented minimal submanifold, the first variation of a calibration vanishes. We also demonstrate that a classical result on the vanishing of the Lie derivative of characteristic forms on harmonic foliations also holds for calibrations.

1 Introduction

Let M be a smooth Riemannian manifold. Among the class of all smooth differential forms on M, we are interested in a special subclass of forms called calibrations. A *calibration* ϕ is a closed differential p-form having commas 1, where comass denotes the sup norm

$$\|\phi\|_x^* = \sup\{\phi(\xi) : \xi \text{ is a unit simple } p\text{-vector } \in \wedge^p T_x M\}.$$

(Recall that a p-vector is *simple* if it can be decomposed into a product of 1-vectors.) Calibrations are very useful objects in the theory of minimal surfaces, for the following reasons. A p-dimensional oriented submanifold $N \subseteq M$ is said to be calibrated by ϕ if $\phi(\xi) = 1$ for every tangent unit p-vector ξ on N. In this case, we say that N is a *ϕ-submanifold.* As shown in [2], ϕ-submanifolds are homologically mass-minimizing, i.e. such objects minimizes the volume functional in its homology class. In the case where M is ordinary Euclidean space, ϕ-submanifolds are absolutely volume minimizing surfaces, absolute in the sense that any compact portion of the surface is a solution to the Plateau problem for the portion's boundary (replace the submanifolds in the proof above by compactly-supported pieces). All that we have said so far are also true in the more general setting of *currents*, which will not be discussed here.

2 Variational Formulas

Given a calibration ϕ on M and an oriented submanifold N of dimension p, we would like to investigate the infinitesimal nature of ϕ with respect to a small variation of N. For the purpose of simplifying the proofs, we will restrict ourselves to the case where N is compact, although this restriction can be easily removed by considering compactly-supported variations in the general case. Let η be a normal vector field on N and let $f_t : N \longrightarrow M$ denote a C^∞ variation (f_t is an immersion for each t) with respect to $\eta = f_{0*}(\partial/\partial t)$ for $t \in (-\epsilon, \epsilon)$, with $f_0 = id$. Let $\|\cdot\|$ denote the norm induced by the Riemannian metric on M, and define

$$\phi_t(\xi) = \phi\left(\frac{f_{t*}\xi}{\|f_{t*}\xi\|}\right).$$

Theorem 1 *Let ϕ be a calibration on a C^∞ Riemannian manifold M with metric $\langle\cdot,\cdot\rangle$, and N a compact oriented submanifold of M with mean curvature vector field H. Then*

$$\int_N \frac{d}{dt}\phi_t|_{t=0} = \int_N \langle H, \eta\rangle \phi$$

for all normal vector fields η on N.

Proof. $f_t(N)$ and N are homologous to each other, hence $f_t(N) - N = \partial S_t$ for some $p+1$-dimensional surface S_t. Let $g(t) = \int_{f_t(N)} \phi$. By Stoke's formula and the closure of ϕ, $g(t)$ =constant, and hence $g'(0) = 0$. By a change of variables, observe that

$$g(t) = \int_N \phi(f_{t*}\xi) d\,Vol(N) = \int_N \phi_t(\xi)\|f_{t*}\xi\| d\,Vol(N),$$

where ξ is the unit tangent p-plane field of N, and the conclusion of the theorem follows by straightforward differentiation.

The following corollaries are some immediate consequence of theorem 1.

Corollary 2 *Let ϕ and M be as above, and $N \subset M$ a submanifold with compact support and fixed boundary ∂N. Then the variational formula in Theorem 1 holds for all normal vector fields η which vanishes on ∂N.*

Proof. In this case, we choose a variation f_t with compact support in N and the additional property that $f_t|_{\partial N} = id$ for all t. The rest of the proof is similar to that of theorem 1.

Corollary 3 *Let ϕ and M be as in Theorem 1. Then N is a compact oriented minimal submanifold of M if and only if*

$$\int_N \frac{d}{dt}\phi_t|_{t=0} = 0$$

for all normal vector fields η on N.

Remark It is interesting to note that corollary 3 applies to *any* oriented minimal submanifold, including those that are not calibrated by ϕ. Of course if N is a ϕ-submanifold, then the result is trivial since ϕ_t attains its maximum on the tangent space of N.

Proof. The "only if" part is obvious by theorem 1. For the "if" part, suppose $H \neq 0$. Then it is possible to find a normal vector field η such that the right-hand side of Theorem 1 is positive, a contradiction.

With respect to a smooth variation f_t, recall that the second variational formula for the volume of a minimal submanifold N of M is given by (cf. [4])

$$\left.\frac{d^2V}{dt^2}\right|_{t=0} = \int_N (\|\nabla\eta\|^2 + Ric(\eta,\eta) - \|A\eta\|^2),$$

where Ric is the Ricci curvature tensor of M and A is the second fundamental form of N. Denote by $I(\eta)$ the integrand of the above integral. If $\int_N I(\eta) \geq 0$ for all normal vector fields η, then N is said to be *stable*.

Theorem 4 *Let ϕ be a calibration on a C^∞ Riemannian manifold M, and N a compact oriented minimal submanifold of M. Then*

$$\int_N \frac{d^2}{dt^2}\phi_t|_{t=0} = -\int_N I(\eta)\phi$$

for all normal vector fields η on N.

Proof. As noted in the proof of Theorem 1, the function $g(t) = \int_{f_t(N)} \phi$ =constant. Thus $g''(0) = 0$ and we obtain the conclusion.

Remark As in the first variational formula, Theorem 5 also holds for compactly supported submanifolds with fixed boundary. We note that Theorem 5 does not require that N be calibrated by ϕ. However, if N turns out to be calibrated, then by above, it is stable. This can also be deduced from the area-minimizing property of a ϕ-submanifold.

We note that many examples of calibrations and their calibrated surfaces can be found in the papers [1] and [2].

3 Calibrated Foliations

In the case where M is foliated manifold, Theorem 1 is actually a special case of the following more general result on smooth differential forms. The Lie derivative of ϕ with respect to η will denoted by $\mathcal{L}_\eta\phi$.

Theorem 5 *Let M be a C^∞ foliated Riemannian manifold and ϕ a smooth p-form on M. Let L denote the tangent bundle of the foliation with leaves of dimension p and mean curvature vector field H. Then on L, we have*

$$\frac{d}{dt}\phi_t|_{t=0} = \mathcal{L}_\eta\phi + \langle H, \eta\rangle\phi$$

for all vector fields η transversal to L.

Proof. Let ξ be the unit tangent p-vector field on the foliation $\mathcal{F}$, considered as a section of $\wedge^p L$. Then for a smooth flow f_t with respect to η,

$$f_t^*\phi(\xi) = \phi_t(\xi)\|f_{t*}\xi\|.$$

Differentiating both sides and evaluating at $t = 0$ gives the desired result.

Remark Observe that for $\phi = \chi_{\mathcal{F}}$ =characteristic form of the foliation, then Theorem 4 reduces to a result of Rummler(cf. [5],p.66), since $\chi_{\mathcal{F}}$ attains its maximum on L. Note ϕ is not assumed to be closed.

The following result leads to a necessary and sufficient condition for a calibration to be the characteristic form of the foliation. As usual, we define $ker(\phi) = \{\eta : \iota(\eta) = 0\}$.

Theorem 6 *Let $\mathcal{F}$ be a foliation on a C^∞ Riemannian manifold M with tangent bundle $\mathcal{L}$. Let ϕ be a smooth p-form on M of comass 1 such that $\phi(\xi) = 1$ for the unit tangent p-vector field ξ to the leaves, and assume that $dim(ker(\phi)) = dim\ M - p$. Then the following are equivalent.*
(1) ϕ is a calibration.
(2) $\mathcal{L}_\eta \phi = 0$ for all vector fields $\eta \in ker(\phi)$.
(3) $ker(\phi)$ is involutive.

Proof.
(1)$\Rightarrow$(2): This follows immediately from the rule $\mathcal{L}_\eta = \iota(\eta)d + d\iota(\eta)$.
(2)$\Rightarrow$(3): For any $\zeta, \eta \in ker(\phi)$, we have

$$0 = \iota(\zeta)\mathcal{L}_\eta\phi = \mathcal{L}_\eta\iota(\zeta)\phi - \iota([\eta,\zeta])\phi = -\iota([\eta,\zeta])\phi,$$

hence $[\eta,\zeta] \in ker(\phi)$.
(3)$\Rightarrow$(1): By Frobenius theorem, there exists locally p independent 1-forms $\omega_1, \cdots, \omega_p$ generating a differential ideal whose kernel equals $ker(\phi)$. In particular, in a local neighborhood U of M, $\phi = \omega_1 \wedge \cdots \wedge \omega_p$+linear combination of p-forms each containing some ω_i. Since $\phi = 1$ on the leaves of $\mathcal{F}$, and there is one and only one simple p-form with comass 1 with the same property, the first term is just the characteristic form $\chi_{\mathcal{F}}$. Hence the rest of the terms in the above expression for ϕ disappear, and $\phi = \chi_{\mathcal{F}}$. By prop. 6.4 of [3],$\phi$ is calibration and we're done.

Corollary 7 *Given a calibration ϕ of order p on M calibrating the leaves of a foliation $\mathcal{F}$, then ϕ is the characteristic form of $\mathcal{F}$ if and only if $dim(ker(\phi)) = m - p$.*

Proof. Necessity is obvious, while the above proof shows that ϕ is decomposable, hence the dimension condition is also sufficient.

References

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