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ON THE REPRESENTATIONS OF AN INTEGER AS A SUM OF TWO OR FOUR TRIANGULAR NUMBERS

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ABSTRACT: In this note we show how Ramanujan's ${}_1\psi_1$-summation formula can be employed to obtain formulas for the number of representations of an integer $N \geq 1$ as a sum of two or four triangular numbers.

1. INTRODUCTION

. The representations of an integer N as a sum of k squares is one of the most beautiful problem in the theory of numbers. The study of representations of an integer as sums of squares is treated in depth in the book of E. Grosswald [4] and are useful in lattice point problems, crystallography and certain problems in mechanics. q-hypergeometric functions have played an important role in the theory of representations of the numbers as sums of squares. For instance Jacobi's two and four square theorems have been proved by S. Bhargava and Chandrashekar Adiga [2] on using the following "${}_1\psi_1$ summation formula" of Ramanujan [9]:

$$\frac{(-qz;q^2)_\infty\ (-q/z;q^2)_\infty\ (q^2;q^2)_\infty\ (\alpha\beta q^2;q^2)_\infty}{(-\alpha qz;q^2)_\infty\ (-\beta q/z;q^2)_\infty\ (\alpha q^2;q^2)_\infty\ (\beta q^2;q^2)_\infty}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(1/\alpha;q^2)_k\ (-\alpha q)^k}{(\beta q^2;q^2)_k} z^k + \sum_{k=1}^{\infty} \frac{(1/\beta;q^2)_k\ (-\beta q)^k}{(\alpha q^2;q^2)_k} z^{-k} \qquad (1)$$

where

$$|q| < 1,\ |\beta q| < |z| < |1/\alpha q|,$$

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty} (1-aq^k)$$

and

$$(a)_k = (a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

This formula was first brought before the mathematical world by G.H. Hardy [5. pp.222,223] who described it as "a remarkable formula with many parameters". A number of direct and elementary proofs of (1) can be found in the literature (see for example [1], [8]). Letting $\alpha \to 0$, $\beta \to 0$ in (1) we get the well-known Jacobi's triple product identity:

$$(-qz;q^2)_\infty\ (-q/z;q^2)_\infty\ (q^2;q^2)_\infty = \sum_{k=-\infty}^{\infty} q^{k^2} z^k \qquad (2)$$

where q,a,z are complex numbers with $|q| < 1$ and $z \neq 0$. M.D. Hirschhorn [6], [7] has given alternate proofs of Jacobi's two and four theorems on using (2).

In chapter 16 of his second notebook [9] Ramanujan introduces the function

$$f(a,b) = 1 + \sum_{k=1}^{\infty} (ab)^{k(k-1)/2} (a^k+b^k), \quad |ab| < 1. \tag{3}$$

This in fact corresponds (putting $a = qe^{2iz}$, $b = qe^{-2iz}$ and $q = e^{i\pi\tau}$ where z is complex and Im $(\tau) > 0$) to the classical theta function [10] given by

$$\vartheta_3 (z,\tau) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nz.$$

Ramanujan then develops the theory of theta-functions using the function f(a,b) and its restrictions

$$\phi(q) \equiv f(q,q) = 1+2 \sum_{n=1}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty}, \tag{4}$$

$$\psi(q) \equiv f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{5}$$

and

$$f(-q) \equiv f(-q,-q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(3n-1)/2} + \sum_{n=1}^{\infty} (-1)^n q^{n(3n+1)/2}$$

$$= (q;q)_\infty . \tag{6}$$

As direct consequences of the definition (3) Ramanujan notes that if ab = cd, then

$$f(a,b)\, f(c,d) + f(-a,-b)\, f(-c,-d)$$
$$= 2f(ac,bd)\, f(ad,bc) \tag{7}$$

and

$$f(a,b)\ f(c,d) - f(-a,-b)\ f(-c,-d)$$
$$= 2a\ f(b/c,\ ac^2d)\ \ f(b/d,acd^2)\ . \tag{8}$$

For details of the proofs of the above identities, one may refer [1]. Putting a=b=c=d=q in (7) and (8) we get

$$\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2) \tag{9}$$

and

$$\phi^2(q) - \phi^2(-q) = 8q\psi^2(q^4)\ . \tag{10}$$

An integer n is said to be a triangular if $n = a(a+1)/2$, $a \in Z$. The purpose of this note is to show how Ramanujan's ${}_1\psi_1$-summation formula can be used to obtain formulas for the number of representations of an integer $N \geq 1$ as a sum of two or four triangular numbers.

2. MAIN THEOREMS

THEOREM 1. If $N \geq 1$, the number of representations of N in the form $n(n+1)/2 + m(m+1)/2$; $n,m \geq 0$ is

$$T_2(N) = d_1(4N+1) - d_3(4N+1)$$

where $d_i(4N+1)$ is the number of divisors d of 4N+1, $d \equiv i \pmod 4$.

For a proof of this Theorem one may refer N.J. Fine [3, p.73]. However, Theorem 1 also follows from (1), (4), (5) and (10).

THEOREM 2. If $N \geq 1$, the number of representations of N in the form $n(n+1)/2 + m(m+1)/2 + i(i+1)/2 + k(k+1)/2$; $n,m,i,k \geq 0$ is

$$T_4(N) = \sum_{d|2N+1} d .$$

PROOF. Putting $\alpha = \beta = -1$ in (1) and after some simplification of the right side we get

$$\frac{(-qz;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty^2}{(qz;q^2)_\infty (q/z;q^2)_\infty (-q^2;q^2)_\infty^2}$$

$$= \frac{1+qz}{1-qz} - 2 \sum_{n=1}^{\infty} \frac{q^{2n} [(zq)^n-(zq)^{-n}]}{1+q^{2n}} .$$

Dividing both sides by 1+qz/1-qz and letting $z \to -1/q$ and then changing q^2 to $-q$ and using (4), we get

$$\phi^4(q) = 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1+(-q)^n} . \tag{11}$$

Multiplying (9) and (10) and using (4) and (5) we have

$$\phi^4(q) - \phi^4(-q) = 16q\, \psi^4(q^2) . \tag{12}$$

using (11) in (12) we get

$$q\psi^4(q^2) = \sum_{n=0}^{\infty} \frac{(2n+1)\, q^{2n+1}}{1-q^{4n+2}} .$$

Changing q to $q^{1/2}$ in this and using (5) we have

$$\left[\sum_{n=0}^{\infty} q^{n(n+1)/2}\right]^4 = \sum_{n=0}^{\infty} \frac{(2n+1)q^n}{1-q^{2n+1}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (2n+1)\, q^{2nm+m+n}$$

$$= \sum_{N=0}^{\infty} D_N q^N \qquad (13)$$

where $D_N = \sum_{2N+1 = (2n+1)(2m+1)} (2n+1) = \sum_{d|2N+1} d$.

Now, comparing the coefficients of q^N on both sides of (13) we get the required result.

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