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ON BAZILEVIC FUNCTIONS OF COMPLEX ORDER

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ABSTRACT

Let $\alpha > 0$ and $b \neq 0$ a complex number. Then a function $f \epsilon B(\alpha, b)$ if it is analytic in the unit disc E and $Re\{1 + \frac{1}{b}[\frac{zf'(z)f^{\alpha-1}(z)}{g^{\alpha}(z)} - 1]\} > 0$, for some starlike function $g, z\epsilon E$. The class $B_1(\alpha, b)$ is defined by taking $g(z) = z$ in the same way. We call these functions as Bazilevic functions of complex order b and type α. Arc length coefficient and some other results are solved for these classes.

1. INTRODUCTION

Let S denote the class of all analytic functions f which are univalent in the unit disc $E = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0, f'(0) = 1$. Let K and S^* be the usual subclasses of S consisting of functions which are, respectively, close-to-convex and starlike (w.r. to the origin) in E. Let P denote the class of functions p which are analytic in E and satisfy the conditions $p(0) = 1$ and $Re\, p(z) > 0$ in E.

We define the following.

Definition 1.1

Let $\alpha > 0$ and $b \neq 0$ (complex). Let f be analytic in E and be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Then we say that

(i) $f \epsilon B(\alpha, b)$ if

$$\left\{1 + \frac{1}{b}\left[\frac{zf'(z)f^{\alpha-1}(z)}{g^{\alpha}(z)} - 1\right]\right\} \epsilon P,$$

for some $g \epsilon S^*, z \epsilon E$,

and

(ii) $f \epsilon B_1(\alpha, b)$ if, for $z \epsilon E$,

$$\left\{1 + \frac{1}{b}\left[\frac{zf'(z)f^{\alpha-1}(z)}{z^\alpha} - 1\right]\right\} \epsilon P,$$

We note that $B(\alpha, 1)$ and $B_1(\alpha, 1)$ are the well- known subclasses of the class B of Bazilevic functions, see [1].

Also $B(0, b) = B_1(0, b) = S^*(b)$ is the class of starlike functions of complex order introduced in [2], and $B(1, b) = K(b)$ is the class of close-to-convex functions of complex order defined in [3].

The class $P(b)$ of analytic functions is related with the class P of functions with positive real part and we define it as follows.

Definition 1.2

Let p be an analytic function in E with $p(0) = 1$ and let $b \neq 0$ be a complex number. Then $p \epsilon P(b)$ if, and only if

$$p(z) = bh(z) + (1 - b), \quad where \;\; h \epsilon P. \tag{1.2}$$

2. PRELIMINARY RESULTS

We shall need the following results.

Lemma 2.1

Let N and D be analytic in $E, N(0) = D(0) = 0$ and D be p-valent starlike for every disc $|z| \leq r < 1$. Suppose $\frac{N'(z)}{D'(z)} \epsilon P(b)$. Then $\frac{N(z)}{D(z)} \epsilon (b)$.

Proof

The method of its proof is similar to that of Libera [4].

Lemma 2.2 [1].

If α and c are positive integers and $g \epsilon S^*$, then the function G, defined by

$$G^\alpha(z) = \frac{\alpha + c}{z^\alpha} \int_0^z \xi^{c-1} g^\alpha(\xi) d\xi \tag{2.1}$$

also belongs to S^*.

Lemma 2.3

The class $P(b)$ is a convex set.

Proof

The proof is immediate from the definition (1.2) and the fact that the class P is a convex set.

Lemma 2.4

Let $0 < \alpha \leq 1, \alpha \neq \frac{1}{2}$ and $G \epsilon S^*$. Then g defined as

$$g^\alpha(z) = z^\alpha \left(z^{1-\alpha} G^\alpha(z)\right)' \tag{2.2}$$

is starlike on $|z| < r_0$ and $g(0) = 0 = g'(0) - 1$, where r_0 is given by

$$r_0 = \frac{1}{(\alpha+1) + \sqrt{\alpha^2 + 2}}. \tag{2.3}$$

This result is sharp.

Proof

From (2.2), we can write

$$G^\alpha(z) = \frac{1}{z^{1-\alpha}} \int_0^z \left(\frac{g(\xi)}{\xi}\right)^\alpha d\xi.$$

Taking the logarithemic differentiation, and using the fact that $G \epsilon S^*$, we have

$$\frac{z\left(\frac{g(z)}{z}\right)^\alpha}{\int_0^z \left(\frac{g(\xi)}{\xi}\right)^\alpha d\xi} = \alpha p(z) + (1-\alpha), \quad p(z) = \frac{zG'(z)}{G(z)} \epsilon P.$$

Differentiation and simple computation gives us

$$\frac{zg'(z)}{g(z)} = p(z) + \frac{zp'(z)}{\alpha p(z) + (1-\alpha)}.$$

Now, using the well-known results for $p \epsilon P$, we have

$$\begin{aligned} Re \frac{zg'(z)}{g(z)} &\geq Re\, p(z) \left\{1 - \frac{\frac{2r}{1-r^2}}{[(1-\alpha) + \frac{\alpha(1-r)}{1+r}]}\right\} \\ &= Re\, p(z) \left\{\frac{1 - 2(\alpha+1)r + (2\alpha - 1)r^2}{(1-\alpha)(1-r^2) + \alpha(1-r)^2}\right\}. \end{aligned} \tag{2.4}$$

The right hand side of the inequality (2.4) is positive for $|z| < r_0$, where r_0 is given by (2.3).

The function $g_o(z)$, which corresponds to $G_o(z) = \frac{z}{(1+z)^2} \epsilon S^*$, that is, the function

$$g_o(z) = \frac{z}{(1+z)^2} \left[\frac{1 + (1-2\alpha)z}{1+z} \right]^{\frac{1}{\alpha}}$$

shows that the number r_0 is the best possible one.

3. MAIN RESULTS

Theorem 3.1

Let $f \epsilon B_1(\alpha, b)$ where α is a positive integer. Then $(\frac{f(z)}{z})^\alpha \epsilon P(b)$.

Proof

Since $f \epsilon B_1(\alpha, b)$, we have

$$\frac{zf'(z)}{f(z)^{1-\alpha} z^\alpha} = \frac{d(f^\alpha(z))/dz}{d(z^\alpha)/dz} \epsilon P(b), z \epsilon E.$$

We now apply Lemma 2.1 and obtain the required result.

Theorem 3.2

Let $0 < b_1 < b_2$. Then $B(\alpha, b_1) \subset B(\alpha, b_2)$.

Proof

Let $f \epsilon B(\alpha, b_1)$. Then there exists a starlike function g such that

$$\frac{zf'(z)}{f^{1-\alpha}(z) g^\alpha(z)} = b_1 h(z) + (1 - b_1), \quad h \epsilon P, z \epsilon E,$$

and so

$$1 + \frac{1}{b_2} \left[\frac{zf'(z)}{f^{1-\alpha}(z) g^\alpha(z)} - 1 \right] = \frac{b_1}{b_2} h(z) + \left(1 - \frac{b_1}{b_2} \right).$$

Since $0 < b_1 < b_2$, we have $0 < \frac{b_1}{b_2} < 1$. This means $0 < (1 - \frac{b_1}{b_2}) = \alpha_1 < 1$. Hence

$$1 + \frac{1}{b_2} \left[\frac{zf'(z)}{f^{1-\alpha}(z) g^\alpha(z)} - 1 \right] = (1 - \alpha_1) h(z) + \alpha_1 = H(z),$$

where *Re* $H(z) > 0$ for $z \epsilon E$, and therefore $f \epsilon B(\alpha, b_2)$. This gives us the required result.

Remark 3.1

From theorem 3.2, it is clear that $B(\alpha, b) \subset B(\alpha, 1) = B(\alpha)$ for $0 < b \leq 1$. Hence $f \epsilon B(\alpha, b)$ is univalent in E.

Let $C(r)$ denote the closed curve which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$, and let $L_r(f)$ denote the length of $C(r)$ and $M(r) = max_{|z|=r}|f(z)|$. We prove the following:

Theorem 3.3

Let $f \epsilon B(\alpha, b), 0 < \alpha \leq 1$. Then

$$L_r(f) \leq c(b, \alpha) M^{1-\alpha}(r) \left(\frac{1}{1-r}\right)^{2\alpha},$$

where $c(b, \alpha)$ is a constant depending on b and α only.

Proof

We have

$$\begin{aligned} L_r(f) &= \int_0^{2\pi} |zf'(z)| d\theta, \quad z = re^{i\theta}\ (0 < r < 1) \\ &= \int_0^{2\pi} |f^{1-\alpha}(z) g^{\alpha}(z) h(z)| d\theta, \quad g \epsilon S^*, h \epsilon P(b) \\ &\leq M^{1-\alpha}(r) \int_0^{2\pi} \int_0^r |\alpha g'(z) g^{\alpha-1}(z) h(z) + g^{\alpha}(z) h'(z)| dr\, d\theta \\ &\leq M^{1-\alpha}(r) \left\{ \int_0^{2\pi} \int_0^r \frac{\alpha}{r} |H(z) g^{\alpha}(z) h(z)| dr\, d\theta \right. \\ &\quad \left. + \int_0^{2\pi} \int_0^r \frac{1}{r} |g^{\alpha}(z). zh'(z)| dr\, d\theta \right\}, \end{aligned}$$

where $H(z) = \frac{zg'}{g} \epsilon P$.

Using well-known distortion theorems for the starlike function g, and then applying Schwarz's inequality, we have

$$L_r(f) \leq M^{1-\alpha}(r) \int_0^r r^{\alpha-1} (\frac{1}{1-r})^{2\alpha} \left[\alpha \left(\int_0^{2\pi} |H(z)|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} |h(z)|^2 d\theta \right)^{\frac{1}{2}} \right.$$

$$+ \int_0^{2\pi} |zh'(z)|d\theta\Big] \, dr.$$

$$\leq \; M^{1-\alpha}(r).2\pi \int_0^r r^{\alpha-1}(\frac{1}{1-r})^{2\alpha} \left[\alpha\left(\frac{1+3r^2}{1-r^2}\right)^{\frac{1}{2}}\left(\frac{1+\{4|b|^2-1\}r^2}{1-r^2}\right)^{\frac{1}{2}}\right.$$

$$+ \frac{2|b|r}{1-r^2}]dr, \qquad (3.1)$$

where we have used in (3.1) a result due to Pommerenke [5] for $h \epsilon P$ and the results for $h \epsilon P(b)$, see [6].

Thus

$$L_r(f) \;\leq C_1(b,\alpha)M^{1-\alpha}(r)\left(\tfrac{1}{1-r}\right)^{2\alpha} + C_2(b,\alpha)M^{1-\alpha}(r)\left(\tfrac{1}{1-r}\right)^{2\alpha}$$

$$= c(b,\alpha)M^{1-\alpha}(r)\left(\tfrac{1}{1-r}\right)^{2\alpha}, 0 < \alpha \leq 1,$$

and $c(b,\alpha)$ is a constant depending on α and b only.

Theorem 3.4

Let $f \epsilon B(\alpha, b), 0 < \alpha \leq 1$, and be given by (1.1). Then, for $n \geq 2$,

$$|a_n| \leq c(b,\alpha)M^{1-\alpha}(r)n^{2\alpha-1},$$

where $c(b,\alpha)$ is a constant which depends only on b and α.

Proof

With $z = re^{i\theta}$, Cauchy's theorem gives

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi} \bar{e}^{in\theta} z f'(z)\, d\theta.$$

Thus

$$n|a_n| \;\leq \tfrac{1}{2\pi r^n} \;\int_0^{2\pi}|zf'(z)|d\theta$$

$$= \tfrac{1}{2\pi r^n} L_r(f)$$

Using theorem 3.3, and putting $r = (1-\frac{1}{n})$, we obtain the required result.

Theorem 3.5

Let α be a positive integer and $f \epsilon B_1(\alpha, b)$. Then the function F_1 defined by

$$F_1^{\alpha+\beta}(z) = z^{\beta}\, f^{\alpha}(z) \qquad (3.2)$$

belongs to $B_1(\alpha+\beta, b)$ for any $\beta \geq 0$.

Proof

From (3.2), we have

$$\frac{(\alpha+\beta)F_1'(z)}{F_1^{1-(\alpha+\beta)}(z)} = \beta z^{\beta-1} f^{\alpha}(z) + \alpha \frac{z^{\beta} f'(z)}{f^{1-\alpha}(z)},$$

and therefore

$$\begin{aligned}\frac{zF_1'(z)}{F_1^{1-(\alpha+\beta)}(z) z^{\alpha+\beta}} &= \frac{\beta}{(\alpha+\beta)} \left(\frac{f(z)}{z}\right)^{\alpha} + \left(\frac{\alpha}{\alpha+\beta}\right) \frac{zf'(z)}{f^{1-\alpha}(z) z^{\alpha}} \\ &= \frac{\beta}{(\alpha+\beta)} p_1(z) + \frac{\alpha}{(\alpha+\beta)}\ p_2(z),\end{aligned}$$

where $p_1, p_2 \epsilon P(b)$, when we use theorem 3.1 and the fact that $f \epsilon B_1(\alpha, b)$. We now use lemma 2.3 and obtain the desired result.

Theorem 3.6

Let α and c be positive integers and $f \epsilon B(\alpha, b)$. Then the function F be defined by

$$F^{\alpha}(z) = \frac{\alpha+c}{z^c} \int_0^z \xi^{c-1} f^{\alpha}(\xi) d\xi \tag{3.3}$$

also belongs to $B(\alpha, b)$.

Proof

Since $f \epsilon B(\alpha, b)$, there exists a starlike function g such that

$$\frac{zf'(z) f^{\alpha-1}(z)}{g^{\alpha}(z)} \epsilon P(b)$$

Let G be defined by (2.1). Then $G \epsilon S^*$ for $z \epsilon E$.

Now, from (3.3), we have

$$\begin{aligned}\frac{zF'(z)}{F^{1-\alpha}(z) G^{\alpha}(z)} &= \frac{\frac{1}{\alpha}\left[z^c f^{\alpha}(z) - c \int_0^z \xi^{c-1} f^{\alpha} d\xi\right]}{\int_0^z [\xi^{c-1} g^{\alpha}(\xi)] d\xi} \\ &= \frac{N(z)}{D(z)},\end{aligned}$$

where $D(z) = \int_0^z \xi^{c-1} g^{\alpha} d\xi$ is $(\alpha+c)$-valently starlike for $z \epsilon E$. Also

$$\frac{N'(z)}{D'(z)} = \frac{zf'(z) f^{\alpha-1}(z)}{g^{\alpha}(z)} \epsilon P(b).$$

Thus, using Lemma 2.1, we obtain the required result that $F \epsilon B(\alpha, b)$ for $z \epsilon E$.

We note that theorem 3.6 remains true if $B(\alpha, b)$ is replaced by $B_1(\alpha, b)$.

Theorem 3.7

Let $F \epsilon B(\alpha, b), 0 < \alpha \leq 1, \alpha \neq \frac{1}{2}$. Let f be defined as

$$f^\alpha(z) = z^\alpha \left(z^{1-\alpha} F^\alpha(z)\right)'. \tag{3.4}$$

Then $f \epsilon B(\alpha, b)$ for $|z| < r_0$, where r_0 is given by (2.3).

Proof

From (3.4), we have

$$F^\alpha(z) = z^{\alpha-1} \int_0^z \left(\frac{f(\xi)}{\xi}\right)^\alpha d\xi.$$

So

$$\alpha F'(z) F^{\alpha-1}(z) = (\alpha - 1) z^{\alpha-2} \int_0^z \left(\frac{f(\xi)}{\xi}\right)^\alpha d\xi + \frac{z^{\alpha-1} f^\alpha(z)}{z^\alpha}. \tag{3.5}$$

Now, since $F \epsilon B(\alpha, b)$, there exists a $G \epsilon S^*$ such that $\frac{zF'(z)F^{\alpha-1}(z)}{G^\alpha(z)} \epsilon P(b), z \epsilon E$, and from lemma 2.4, it follows that the function g defined by (2.2) also belongs to S^* for $|z| < r_0$, where r_0 is given by (2.3). Thus, from (3.4), we have $p \epsilon P(b)$,

$$\frac{\alpha z F'(z)}{F^{1-\alpha}(z) G^\alpha(z)} = \frac{(\alpha - 1) \int_0^z (\frac{f(\xi)}{\xi})^\alpha d\xi}{\int_0^z (\frac{g(\xi)}{\xi})^\alpha d\xi} + \frac{z(\frac{f(z)}{z})^\alpha}{\int_0^z (\frac{g(\xi)}{\xi})^\alpha d\xi} = \alpha p(z).$$

Differentiating, we obtain

$$\frac{z f'(z)}{f^{1-\alpha}(z) g^\alpha(z)} = p(z) + \frac{\{p'(z) \int_0^z (\frac{g(\xi)}{\xi})^\alpha d\xi\}}{(\frac{g(z)}{z})^\alpha},$$

or we can write, for $h \epsilon P$,

$$1 + \frac{1}{b}\left\{\frac{z f'(z)}{f^{1-\alpha}(z) g^\alpha(z)} - 1\right\} = h(z) + \frac{h'(z) \int_0^z (\frac{g(\xi)}{\xi})^\alpha d\xi}{(\frac{g(z)}{z})^\alpha}. \tag{3.6}$$

Now

$$\begin{aligned} Re\left(\frac{z(\frac{g(z)}{z})^\alpha}{\int_0^z(\frac{g(\xi)}{\xi})^\alpha d\xi}\right) &= Re\left(\frac{z(z^{1-\alpha}G^\alpha(z))'}{z^{1-\alpha}G^\alpha(z)}\right) \\ &= Re\left(\alpha\frac{zG'(z)}{G(z)}+(1-\alpha)\right) \\ &\geq \frac{1-(2\alpha-1)r}{1+r}. \end{aligned} \tag{3.7}$$

Hence, from (3.6), (3.7) and a well-known result for $h\epsilon P(|h'(z)| \leq \frac{2Re\, h(z)}{1-r^2})$, we have

$$Re\left[1+\frac{1}{b}\left\{\frac{zf'(z)}{f^{1-\alpha}(z)g^\alpha(z)}-1\right\}\right] \geq Re\; h(z)\left[\frac{1-2(\alpha+1)r+(2\alpha-1)r^2}{(1-r)\{1-(2\alpha-1)r\}}\right]. \tag{3.8}$$

The right hand side of the inequality (3.8) is positive for $|z| < r_0$, where r_0 is given by (2.3). This proves our result.

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