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The Order of Starlikeness and Convexity of Confluent Hypergeometric Functions

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ABSTRACT

We determine the conditions for which confluent hypergeometric functions are convex and starlike of order α.

Key Words and Phrases: Confluent hypergeometric functions, convex, starlike of order α, convolution, linear operator, radius of convexity.
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1. INTRODUCTION

Let A denote the class of analytic functions f in the unit disk $E = \{z:|z|<1\}$ with $f(0) = 0$, $f'(0) = 1$. We denote by S the subclass of A consisting of univalent functions. A function $f \in S^*(\alpha)$ S, $0<\alpha<1$, if and only if $\text{Re}\,\frac{zf'(z)}{f(z)} > \alpha$, $z \in E$. We call f a starlike function of order α in E. Also, a function $f \in S$, satisfying $\text{Re}\,\{\frac{(zf'(z))'}{f'(z)}\} > \alpha$, $0<\alpha<1$, $z \in E$, is called a convex function of order α and we denote the class consisting of such functions as $C(\alpha)$.

It is clear that

$$f \in C(\alpha) \text{ if, and only if, } zf' \in S^*(\alpha), \tag{1.1}$$

Let c be a complex numbers with $c \neq 0, -1, -2, \ldots.$, and consider the function defined by

$$\phi(a;c;z) = {}_1F_1(a;c;z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \ldots. \tag{1.2}$$

This function is called Confluent (or Kummer) hypergeometric function and it is analytic in C. It satisfies Kummer's hypergeometric differential equation

$$zw''(z) + (c-z)w'(z) - aw(z) = 0 \tag{1.3}$$

If we let $(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)}$

$$= d(d+1)\ldots.(d+k-1),$$

and $(d)_0 = 1$, then (1.2) can be written as

$$\phi(a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^b}{k!} = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \frac{z^k}{k!} \tag{1.4}$$

It is well-known [1] that

$$c\phi'(a;c;z) = a\phi(a+1;\ c+1;\ z), \tag{1.5}$$

$$\phi(a;a;z) = e^z \tag{1.6}$$

Also, if Re c > Re a > 0, then

$$\phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} e^{tz} dt = \int_0^1 e^{tz} d\mu(t), \tag{1.7}$$

where

$$\mu(t) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(c-a)} t^{a-1} (1-t)^{c-a-1}$$

is a probability measure on [0,1]. In fact

$$\int_0^1 d\mu(t) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(c-a)} \cdot B(a,\ c-a) = 1, \tag{1.8}$$

where B is the beta function.

2. MAIN RESULTS

We shall now determine conditions on a and c so that ϕ belongs to $C(\alpha)$ and $S^*(\alpha)$.

Theorem 2.1

Let a, c and α be real numbers with $a \neq 0$, $0<\alpha<1$ and satisfy $c>N(a,\ \alpha)$, where

$$N(a,\alpha) = \begin{cases} \dfrac{1-2\alpha^2+2\alpha+2|a|}{2(1-\alpha)}, & |\alpha+a| > \dfrac{(1-\alpha)^2}{(3-2\alpha)} \\ \\ \dfrac{4(1-\alpha)^2(1+\alpha)-\alpha^2}{2(3-2\alpha)(1-\alpha)} + \dfrac{(3-2\alpha)(\alpha+a)^2}{2(1-\alpha)}, & |\alpha+a| < \dfrac{(1-\alpha)^2}{(3-2\alpha)} \end{cases} \tag{2.1}$$

Then $\phi(a;c;z)$ is convex of order α in E.

To prove this, we follow the technique of Miller and Mocanu [2] and we need the following result which is a special case of Theorem 1 in [3].

Lemma 2.1

Let D be a set in the complex plane $\mathbb{C}$ and let a function $H:\mathbb{C}^2 \times E \longrightarrow \mathbb{C}$ satisfy the condition

$H(is;t;z) \notin D$ for $z \in E$ and for real s,t with $t < \frac{-(1+s^2)}{2}$. If p is analytic in E with $p(0) = 1$ and $H(p(z); zp'(z):z) \in E$, $z \in E$, then $\text{Re } p(z) > 0$ in E.

Proof of Theorem 1

Let

$$\left(\frac{z\phi''(z)}{\phi'(z)} + 1\right) = (1-\alpha)\, p(z) + \alpha, \tag{2.2}$$

where $\phi'(z) = \phi'(a;c;z) \neq 0$, see [2]. Clearly the function p(z) is analytic in E with $p(0) = 1$. Since ϕ satisfies the differential equation (1.3), we use (2.2) in (1.3) to have

$$\{zp'(z) + (1-\alpha)p^2(z) + (1-\alpha)(c-z+\alpha-2)\, p(z) - [(1-\alpha)c+z(a+\alpha) - (1-\alpha)^2]\} = 0 \tag{2.3}$$

Let

$$H(w_1; w_2; z) = w_2 + (1-\alpha)\, w_1^2 + (1-\alpha)(c-z+\alpha-2)w_1 - [(1-\alpha)c+z(a+\alpha) - (1-\alpha)^2]$$

and $D = \{0\}$, then (2.3) can be written as

$$H(p(z); zp'(z); z) \in D$$

We shall use Lemma 2.1 to prove that $\text{Re } p(z) > 0$.

Let $z = x + iy$. Then

$$\begin{aligned}\text{Re } H(is;t;z) &= t-(1-\alpha)s^2 + (1-\alpha)ys - (1-\alpha)(c-1+\alpha) - (\alpha+a)x \\ &< -\frac{(1+s^2)}{2} - \frac{2(1-\alpha)s^2}{2} + \frac{2(1-\alpha)ys-2(\alpha+a)x}{2} - \frac{2(1-\alpha)(c-1+\alpha)}{2} \\ &= -\frac{1}{2}\left[(3-2\alpha)s^2-2(1-\alpha)ys + 2(\alpha+a)x+2(1-\alpha)(c-1-\alpha)+1\right] \\ &\equiv Q(s)\end{aligned}$$

Now $Q(s) < 0$ for all real s and $x^2+y^2<1$. In fact the Discriminant Δ of Q(s) is

$$\begin{aligned}\Delta &= (1-\alpha)^2y^2 - (3-2\alpha)[1+2(1-\alpha)(c-1-\alpha)+2(\alpha+a)x] \\ &< (1-\alpha)^2 - (3-2\alpha)[1+2(1-\alpha)(c-1-\alpha) + 2(\alpha+a)x] - (1-a)^2x^2 \\ &\equiv h(x)\end{aligned}$$

If $|\alpha + a| < \frac{(1-\alpha)^2}{(3-2\alpha)}$, then

$$h'(x_0) = 0 \quad \text{for} \quad x_0 = \frac{-(3-2\alpha)(\alpha+a)}{(1-\alpha)^2}$$

and using (2.1), we have

$$h(x) < h(x_0) = (1-\alpha)^2 - (3-2\alpha)[1+2(1-\alpha)(c-1-\alpha)] + \frac{(3-2\alpha)^2(\alpha+a)^2}{(1-\alpha)^2}$$

$$< 0 \qquad \text{for} \quad -1 < x < 1.$$

If $|\alpha+a| > \frac{(1-\alpha)^2}{(3-2\alpha)}$, then $h(x)$ is monotone on $(-1,1)$ and again, from (2.1), we deduce that

$$h(x) < -(3-2\alpha)[1+2(1-\alpha)(c-1-\alpha)] + 2(2-3\alpha)(\alpha+|a|)$$
$$< 0$$

Hence, in both cases, $\Delta<0$ for $x^2+y^2<1$. Also, from (2.1), we have $Q(0) < 0$ and therefore

$$\text{Re } H(is;t;z) < 0, \text{ for } z\varepsilon E \text{ and all real } s \text{ and } t \text{ with } t < -\frac{(1+s^2)}{2}.$$

Hence, from Lemma 2.1, we have Re $p(z) > 0$, $z\varepsilon E$. This proves that $\phi \ \varepsilon \ C(\alpha)$ for $z\varepsilon E$ and $c>N(a,\alpha)$, where $N(a,\alpha)$ is given by (2.1).

Theorem 2.2

Let $a\neq 1$ and $c > 1+N(a-1,\alpha)$, $0<\alpha<1$, where $N(a,\alpha)$ is as defined in (2.1). Then $z\phi(a;c;z) \ \varepsilon \ S^*(\alpha)$ for $z\varepsilon E$.

Proof: Its proof follows immediately from relations (1.1), (1.5) and Theorem 2.1.

Remark

For $\alpha=0$, we obtain the results proved in [2].

3. APPLICATIONS

To illustrate some of the applications of our main result, we need the following concepts.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then the Hadamard product (also called the convolution) of f and g is defined as

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

It is known [4] that if $f\varepsilon C(\alpha)$, $0<\alpha<1$ and g is convex then $f*g\,\varepsilon C(\alpha)$.

Let μ_i, $0 \leq i \leq 5$ be the linear operators defined on A by the equations below.

$$\mu_0(f(z)) = zf'(z), \quad \mu_1(f(z)) = [f(z) + zf'(z)]/2$$

$$\mu_2(f(z)) = \int_0^z \frac{f(\xi)}{\xi} d\xi, \quad \mu_3(f(z)) = \frac{2}{z} \int_0^z f(\xi) d\xi,$$

$$\mu_4(f(z)) = \int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi} d\xi, \quad |x| \leq 1, \quad x \neq 1$$

$$\mu_5(f(z)) = \frac{1+\gamma}{z^\gamma} \int_0^z \xi^{\gamma-1} f(\xi) d\xi, \quad \text{Re } \gamma > 0$$

Each of these operators can be written, (see [5]), as a convolution operator given by $\mu_i(f) = \psi_i * f$, $0 \leq i \leq 5$, where

$$\psi_0(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2},$$

$$\psi_1(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - \frac{z^2}{2}}{(1-z)^2},$$

$$\psi_2(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = - \text{Log}(1-z)$$

$$\psi_3(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n + \frac{-2[z + \log(1-z)]}{z}$$

$$\psi_4(z) = \sum_{n=1}^{\infty} \frac{1 - x^n}{(1 - x)^n} z^n = \frac{1}{1-x} \text{ Log } \left[\frac{1-xz}{1-z}\right], \quad |z|=1, \quad x \neq 1$$

$$\psi_5(z) = \sum_{n=1}^{\infty} \frac{1+\gamma}{n+\gamma} z^n, \quad \text{Re } \gamma > 0$$

For a given subclass M of S, let $r_c[M]$ denote the minimum radius of convexity over all functions f in M. It is not difficult to find the radius of convexity of each of the functions ψ_i, $0 \leq i \leq 5$, that is

$$r_c(\psi_0) = 2 - \sqrt{3}, \quad r_c(\psi_1) = \frac{1}{2}$$

and

$$r_c(\psi_2) = r_c(\psi_3) = r_c(\psi_4) = r_c(\psi_5) = 1$$

These facts together with (Theorem 1.2) yield the following result as a consequence.

Theorem 2.3

Let a,c and α be real numbers with $a\neq 0$, $0<\alpha<1$ and $c>N(a,\alpha)$ where $N(a,\alpha)$ is defined by (2.1). Then $\mu_i(\phi(z)) = \phi*\psi_i \in C(\alpha)$ up to $r_c(\psi_i)$ for each i, $0<i<5$. Here $\phi(z) = \phi(a;c;z)$.

It is known [6] that $f\in C(\alpha)$ implies that $f\in S^*(\beta)$ where

$$\beta(\alpha) = \begin{cases} \dfrac{2\alpha-1}{2(1-2^{1-2\alpha})}, & \alpha \neq \dfrac{1}{2} \\[2ex] \dfrac{1}{2\log 2}, & \alpha = \dfrac{1}{2} \end{cases} \tag{2.4}$$

and it is a sharp result.

Using this and Theorem 1, we immediately have the following result.

Theorem 2.4

Let a,c and α be as defined in Theorem 2.1. Then $\phi(a;c;z) \in S^*(\beta)$ where β is given by (2.4).

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