Nihonkai Math. J.
vol. 3 (1992), 103-108

ON A NEW CLASS OF UNIVALENT FUNCTIONS

KHALIDA INAYAT NOOR

ABSTRACT

A well-known linear operator is defined which acts on an analytic function in the open unit disk by forming its convolution with an incomplete beta function. In this paper, using this operator, we define a new class of analytic functions in the unit disk and prove that this class consists entirely of univalent functions. An inclusion result is given. It is shown that it is closed under convolution with convex function and some applications of this result are also discussed.

1. INTRODUCTION

Let A be the class of analytic functions f on the open unit disk $E = \{z : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. The class A is closed under the Hadamard product or convolution

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^{n+1},$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^{n+1}, \quad g(z) = \sum_{n=0}^{\infty} b_n z^{n+1}.$$

In particular, we consider convolution with the function $\phi(a,c)$ defined by

$$\phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad z \in E, \quad c \neq 0, -1, -2, \ldots.$$

where

$$(a)_n = \Gamma(a+n)/\Gamma(a),$$

i.e. $(a)_0 = 1$, $(a)_n = a(a+1) \ldots\ldots\ldots (a+n-1)$, $n > 1$. The function $\phi(a,c)$ is an incomplete beta function, related to the Gauss hypergeometric function by

$$\phi(a,c;z) = z\ {}_2F_1(1,a;c;z)$$

It has an analytic continuation to the z-plane cut along the positive real line from 1 to ∞. We note that $\phi(a,1;z) = \dfrac{z}{(1-z)^a}$ and $\phi(2,1;z)$ is the Koebe function.

Carlson and Shaffer [2] defined a convolution operator on A involving an incomplete beta function as

$$L(a,c)f = \phi(a,c)*f,\ f \in A.$$

If $a=0, -1, -2, \ldots.$, then $L(a,c)f$ is a polynomial. If $a\neq 0, -1, -2, \ldots$, then application of the root test shows that the infinite series for $L(a,c)f$ has the same radius of convergence as that for f because

$$\lim_{n\to\infty} \left|\frac{(a)_n}{(c)_n}\right|^{\frac{1}{n}} = 1$$

Hence $L(a,c)$ maps A into itself. The Ruscheweyh derivatives of f are $L(n+1,1)f$, $n=0,1,2,\ldots.$. $L(a,a)$ is the identity and if $a\neq 0$, $-1, -2, \ldots.$, then $L(a,c)$ has a continuous inverse $L(c,a)$ and is a 1-1 mapping of A onto itself. $L(a,c)$ provides a convenient representation of differentiation and integration. If $g(z) = zf'(z)$, then $g = L(2,1)f$ and $f = L(1,2)g$.

Let P be the class of analytic functions with positive real part in E. Then the class P' is defined to be the class of all functions f such that $f' \in P$.

We now define the following.

Definition 1.1

Let $f\in A$. Then $f \in P'(a,c)$ if and only if, $L(a,c)f \in P'$.

2. PRELIMINARIES

Lemma 2.1 [4].

If $c\neq 0$ and a and c are real and satisfy $a > N(c)$, where

$$N(c) = \begin{cases} |c| + \frac{1}{2}, & \text{if } |c| > \frac{1}{3} \\ \frac{3c^2}{2} + \frac{2}{3}, & \text{if } |c| < \frac{1}{3}, \end{cases} \tag{2.1}$$

then $\phi(c,a;z)$ is convex in E.

Lemma 2.2 [6]

If f is convex in E with $f(0)=0$ and $f'(0)=1$, then $\mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{2}$ for $z \in E$.

Lemma 2.3

If $p(z)$ is analytic in E, $p(0) = 1$ and $\mathrm{Re}\, p(z) > \frac{1}{2}$, $z \in E$, then for any function F, analytic in E and $F(0)=0$, the function $(\frac{1}{2})\, p * F$ takes values of the convex hull of $F(E)$.

This result follows immediately from Herglotz' representation for p.

3. MAIN RESULTS

In the following, we prove that $P'(a,c) \subset P'$.

Theorem 3.1

Let $f \in P'(a,c)$, where a and c satisfy the conditions of Lemma 2.1. Then, for $z \in E$, $f \in P'$. This implies f is close-to-convex and hence univalent in E, see [3].

Proof

Since $L(a,c)f \in P'$, we have

$$\mathrm{Re}\{\phi(a,c;z) * f(z)\}' > 0, \quad z \in E. \tag{3.1}$$

Now

$$f'(z) = \left[\frac{\phi(a,c;z)}{z}\right] * [f'(z)] * \left[\frac{\phi(c,a;z)}{z}\right]$$

$$= [\phi(a,c;z) * f(z)]' * \frac{\phi(c,a;z)}{z}. \tag{3.2}$$

Applying Lemmas 2.1 and 2.2, we see that

$$\mathrm{Re}\,\frac{\phi(c,a;z)}{z} > \frac{1}{2}, \quad z \in E. \tag{3.3}$$

From (3.1), (3.2), (3.3) and Lemma 2.3, we obtain the required result that $\mathrm{Re}\, f'(z) > 0$ for $z \in E$ i.e. $f \in P'$.

Next, we prove an inclusion result.

Theorem 3.2

Let $a \neq 0$, c and d be real and $c > N(d)$, where $N(d)$ is defined as in (2.1). Then

$$P'(a,d) \subset P'(a,c).$$

Proof

Let $f \in P'(a,d)$. This implies that

$$(\phi(a,d)*f)' = p \in P, \qquad \text{for } z \in E. \tag{3.4}$$

Now

$$\phi(a,c) = \phi(a,d) * \phi(d,c).$$

So

$$\begin{aligned}(\phi(a,c) * f)' &= \frac{\phi(a,c)}{z} * f' \\ &= \frac{1}{z}\,[\phi(a,d) * \phi(d,c)] * f' \\ &= \left(\frac{\phi(a,d)}{z} * f'\right) * \frac{\phi(d,c)}{z} \\ &= (\phi(a,d) * f)' * \frac{\phi(d,c)}{z}.\end{aligned}$$

Using Lemmas 2.1 and 2.2 along with (3.4), we have $(\phi(a,c)*f)' \in P$, and consequently $f \in P'(a,c)$. This proves our theorem.

We now prove that the class $P'(a,c)$ is closed under convolution with convex univalent functions.

Theorem 3.3

Let $a \neq 0$ and c be real and satisfy $c > N(a)$, where $N(a)$ is defined in the similar way of (2.1). Let ψ be a convex univalent function in E. If $f \in P'(a,c)$, then $\psi * f \in P'(a,c)$.

Proof

We want to show that

$$(\phi(a,c) * (\psi*f))' \in P.$$

Let $\quad \dfrac{\psi(z)}{z} = \psi_1(z).$ Then

$$\begin{aligned}(\phi(a,c) * (\psi*f))' &= (\phi(a,c) * (f * z\psi_1))' \\ &= \frac{\phi(a,c)}{z} * (f * z\psi_1)'\end{aligned}$$

$$= \left(\frac{\phi(a,c)}{z} * f'\right) * \psi_1$$

$$= (\phi(a,c)*f)' * \psi_1.$$

Using the Lemmas 2.1, 2.2, 2.3 and the fact that $f \in P'(a,c)$, we obtain the desired result.

We shall now give some applications of theorem 3.3.

Theorem 3.4

Let $f \in P'(a,c)$, where a and c satisfy the conditions of Theorem 3.3. Let, for $b > 0$

$$F(z) = \frac{b+1}{z^b} \int_0^z t^{b-1} f(t)dt. \tag{3.5}$$

Then $F \in P'(a,c)$.

The operator (3.5) for $b = 1,2,3,\ldots.$ was studied by Bernardi [1].

Proof

Let

$$\psi_b(z) = \sum_{j=1}^{\infty} \frac{b+1}{b+j} z^j , \quad 1+\gamma = \frac{1}{b}, \quad b > 0$$

Then ψ_b is convex for $z \in E$ (see [5]). Setting

$$F(z) = (\psi_b * f)(z)$$

and using Theorem 3.3 we get the required result.

Theorem 3.5

Let $f \in P'(a,c)$ with a and c satisfying the conditions of Theorem 3.3. For $0 < \lambda < 1$, let

$$F_\lambda(z) = (1-\lambda) f(z) + \lambda z f'(z). \tag{3.6}$$

Then $F_\lambda \in P'(a,c)$ for $|z| < r_0$, where

$$r_0 = \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}} . \tag{3.7}$$

Proof

We can write (3.6) as follows

$$F_\lambda(z) = (\psi_\lambda * f)(z), \tag{3.8}$$

where

$$\psi_\lambda(z) = (1-\lambda)\frac{z}{1-z} + \lambda\frac{z}{(1-z)^2}, \quad 0 < \lambda < 1$$

$$= z + \sum_{n=2}^{\infty} [1+(n-1)\lambda]z^n.$$

The function ψ_λ is convex for $|z| < r_0$, where r_0 is given by (3.7) and this radius is best possible. Therefore applying Theorem 3.3 for (3.8) we see that $F_\lambda \in P'(a,c)$ for $|z| < r_0$ where r_0 is given by (3.7). This completes the proof.

REFERENCES

1. Bernardi, S.D., Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135(1969), 429-446.

2. Carlson, B.C. and Shaffer, D.B., Starlike and Prestarlike Hypergeometric Functions, SIAM J. Math. Anal. 15(1984), 737-745.

3. Goodman, A.W., Univalent Functions, Vol. I, Vol. II, Polygonal Publishing House, Washington, New Jersey, 1983.

4. Miller, S.S., and Mocanu, P.T., Univalence of Gaussian and Confluent Hypergeometric Functions, Proc. Amer. Math. Soc. 110(1990), 333-342.

5. Ruscheweyh, S., New criteria for univalent functions, Proc. Amer. Math. Soc. 49(1975), 109-115.

6. Strohhacker, Beitrage Zur Theorie dee schlichten Funktionen, Math. Z. 37 (1933), 356-380.

Mathematics Department
College of Science
King Saud University
P.O. Box 2455
Riyadh 11451, Saudi Arabia

Received Sep. 13,1991 Revised Oct.13,1992