Nihonkai Math. J.
vol. 3 (1992), 49-66

On the spectrum of the Laplacian in cosymplectic manifolds*

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§1. Introduction

Let (M, g) be an m-dimensional compact orientable Riemannian manifold (connected and C^∞) with metric tensor g. We denote by Δ the Laplacian acting on p-forms on M, $0 \leq p \leq m$. Then we have the spectrum for each p:

$$Spec^P(M, g) := \{0 \leq \lambda_{0,p} \leq \lambda_{1,p} \leq \lambda_{2,p} \leq \cdots \uparrow +\infty\},$$

where each eigenvalue $\lambda_{\alpha,p}$ is repeated as many times as its multiplicity indicates. In order to study the relation between $Spec^p(M, g)$ and the geometry of (M, g) we use the Minakshisundaram - Pleijel - Gaffney's formula. Z. Olszak ([10]), H.K. Pak ([11]), J.S. Pak, J.C. Jeong and W-T. Kim ([12]), S. Yamaguchi and G. Chūman ([18]) and others studied the spectrum of the Laplacian and the curvature of Sasakian manifolds.

The purpose of the present paper is to study cosymplectic analogues for certain results of [1], [10], [12], [13], [14], [15] and [18].

We shall be in C^∞-category. The indices $h, i, j, k, s, t, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$. The Einstein summation convention with respect to those system of indices will be used.

ACKNOWLEDGEMENT. The authors would like to express their hearty thanks to the referee for useful comments.

§2. Preliminaries

By $R = (R_{kji}{}^h), R_1 = (R_{ji})$ and r we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively.

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* This research was supported by TGRC-KOSEF.

For a tensor field T on M, we denote by $\|T\|$ the norm of T with respect to g. Then the Minakshisundaram - Pleijel - Gaffney's formula for $Spec^p(M,g)$ is given by

$$\sum_{\alpha=0}^{\infty} exp(-\lambda_{\alpha,p}t) \sim (4\pi t)^{\frac{-m}{2}} \sum_{\alpha=0}^{\infty} a_{\alpha,p}t^{\alpha} \quad \text{as} \quad t \longrightarrow 0^+,$$

where the constants $a_{\alpha,p}$ are spectral invariants. In the present paper we are interested in the case of $p = 0, 1$ or 2. For $p = 0$, we have (cf. [1])

$$a_{0,0} = \int_M dM = \mathrm{Vol}(M,g), \tag{2.1}$$

$$a_{1,0} = \frac{1}{6}\int_M r dM, \tag{2.2}$$

$$a_{2,0} = \frac{1}{360}\int_M [2\|R\|^2 - 2\|R_1\|^2 + 5r^2] dM, \tag{2.3}$$

where dM denotes the natural volume element of (M,g). For $p = 1$, we have (cf. [18])

$$a_{0,1} = m\ \mathrm{Vol}(M,g), \tag{2.4}$$

$$a_{1,1} = \frac{m-6}{6}\int_M r dM, \tag{2.5}$$

$$a_{2,1} = \frac{1}{360}\int_M [2(m-15)\|R\|^2 - 2(m-90)\|R_1\|^2 + 5(m-12)r^2] dM, \tag{2.6}$$

For $p=2$, we have (cf. [13], [16], [18])

$$a_{0,2} = \frac{1}{2}m(m-1)\mathrm{Vol}(M,g), \tag{2.7}$$

$$a_{1,2} = \frac{1}{12}(m^2-13m+24)\int_M r dM, \tag{2.8}$$

$$a_{2,2} = \frac{1}{720}\int_M [2(m^2-31m+240)\|R\|^2 - 2(m^2-181m+1080)\|R_1\|^2 + 5(m^2-25m+120)r^2]dM. \tag{2.9}$$

§3. Cosymplectic manifolds

Let M be a $(2n+1)$ - dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$ in which there are given a tensor field ϕ of type (1,1), a vector field ξ^h and a 1-form η_h satisfying

$$\phi_j{}^t\phi_t{}^i = -\delta_j^i + \eta_j\xi^i, \quad \phi_t{}^i\xi^t = 0, \eta_t\phi_i{}^t = 0, \quad \eta_t\xi^t = 1. \tag{3.1}$$

Such a set of a tensor field of type (1,1), a vector field and a 1-form is called *almost contact structure* and a manifold with an almost contact structure an *almost contact manifold*.

If, in an almost contact manifold, there is given a Riemannian metric g_{ji} such that

$$g_{ts}\phi_j{}^t\phi_i{}^s = g_{ji} - \eta_j\eta_i, \quad \eta_i = g_{it}\xi^t, \tag{3.2}$$

then the manifold is called an ***almost contact metric manifold.***

If we put $\phi_{ji} = \phi_j{}^t g_{ti}$, we see from (3.1) and (3.2) that ϕ_{ji} is skew-symmetric.

The almost contact structure is said to be ***normal*** if $[\phi, \phi] + d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denotes the Nijenhuis tensor formed with ϕ and d the operator of the exterior derivative.

A normal almost contact metric structure is said to be ***cosymplectic*** (cf. [2], [3], [4], [5], [7], [8]) if the 2-form ϕ_{ji} and the 1-form η_i are both closed. A manifold with a cosymplectic structure is called a ***cosymplectic manifold.*** It is known in [2] that the cosymplectic structure is characterized by

$$\nabla_k \phi_j{}^i = 0 \text{ and } \nabla_k \eta^i = 0, \tag{3.3}$$

where ∇_k denotes the operator of covariant differentiation with respect to g_{ji}.

If we denote the curvature tensor, Ricci tensor and scalar curvature of a cosymplectic manifold M by $R_{kji}{}^h$, R_{ji} and r respectively, then we have

$$\begin{aligned} &R_{kjit}\xi^t = 0, \quad R_{kjts}\phi_i{}^t\phi_h{}^s = R_{kjih}, \\ &R_{tjis}\phi^{ts} = -R_{jt}\phi_i{}^t, R_{jt}\phi_i{}^t = -R_{it}\phi_j{}^t, \\ &R_{kjts}\phi^{ts} = 2R_{kt}\phi_j{}^t \quad R_{jt}\xi^t = 0, R_{ts}\phi_j{}^t\phi_i{}^s = R_{ji}, \end{aligned} \tag{3.4}$$

where $\phi^{ji} = \phi_t{}^i g^{jt}$, $R_{kjih} = R_{kji}{}^t g_{th}$.

In a cosymplectic manifold M, we call a sectional curvature

$$k = -\frac{g(R(\phi X, X)\phi X, X)}{g(X, X)g(\phi X, \phi X)}$$

determined by two orthogonal vectors X and ϕX the ***ϕ-holomorphic sectional curvature*** with respect to the vector X orthogonal to ξ of M. If the ϕ-holomorphic sectional curvature is always constant with respect to any vector at every point of the manifold M, then we call the manifold M a ***manifold of constant ϕ-holomorphic sectional curvature***. If a cosymplectic manifold has a constant ϕ-holomorphic sectional curvature k at every point, then the components of the curvature tensor of the manifold are of the form ([4], [8])

$$R_{kjih} = \frac{k}{4}(g_{kh}g_{ji} - g_{ki}g_{jh} + \phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih}$$
$$- g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - \eta_k\eta_h g_{ji} + \eta_k\eta_i g_{jh})$$

,where $k = \frac{r}{n(n+1)}$.

Define on M a tensor field $H = (H_{kjih})$ by

$$H_{kjih} = R_{kjih} - \frac{r}{4n(n+1)}(g_{kh}g_{ji} - g_{ki}g_{jh} + \phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh}$$
$$- 2\phi_{kj}\phi_{ih} - g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - \eta_k\eta_h g_{ji} + \eta_k\eta_i g_{jh}). \tag{3.5}$$

By using (3.4) and (3.6), we can easily verify that

$$\|H\|^2 = \|R\|^2 - \frac{2}{n(n+1)}r^2. \tag{3.6}$$

A cosymplectic manifold is of constant ϕ-holomorphic sectional curvature if and only if $H = 0$, provided $n \geq 2$.

Define on M a tensor field $Q = (Q_{ji})$ by

$$Q_{ji} = R_{ji} - \frac{r}{2n}g_{ji} + \frac{r}{2n}\eta_j\eta_i.$$

By a direct calculation, in which we use (3.4), it follows

$$\|Q\|^2 = \|R_1\|^2 - \frac{1}{2n}r^2. \tag{3.7}$$

A cosymplectic manifold is said to be ***η-Einstein*** if $Q = 0$. For any η-Einstein cosymplectic manifold, r is constant, provided $n \geq 2$.

We also consider the so-called *cosymplectic Bochner curvature tensor field* $\bar{B} = (\bar{B}_{kjih})$ defined on M by (cf. [5])

$$(3.8)\quad \bar{B}_{kjih} = R_{kjih} - \frac{1}{2(n+2)}(g_{kh}R_{ji} - g_{jh}R_{ki} + g_{ji}R_{kh} - g_{ki}R_{jh} + \phi_{kh}S_{ji} - \phi_{jh}S_{ki} + \phi_{ji}S_{kh} - \phi_{ki}S_{jh} - 2\phi_{ih}S_{kj} - 2\phi_{kj}S_{ih} - \eta_k\eta_h R_{ji} + \eta_j\eta_h R_{ki} - \eta_j\eta_i R_{kh} + \eta_k\eta_i R_{jh}) + \frac{r}{4(n+1)(n+2)}(g_{kh}g_{ji} - g_{jh}g_{ki} - g_{kh}\eta_j\eta_i + g_{jh}\eta_k\eta_i - g_{ji}\eta_k\eta_h + g_{ki}\eta_j\eta_h + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}),$$

where $S_{ji} = -R_{jt}\phi_i{}^t$ and $S_{ji} = -S_{ij}$.

The tensor field $\bar{B}$ satisfies, among others, the following identities :

$$\bar{B}_{kjih} = \bar{B}_{ihkj},\quad \bar{B}_{kjih} = -\bar{B}_{jkih},\quad \bar{B}_{kjih} = -\bar{B}_{kjhi},$$
$$\bar{B}_{kjih} + \bar{B}_{jikh} + \bar{B}_{ikjh} = 0,$$
$$\bar{B}_{tjis}g^{ts} = 0,\quad \bar{B}_{kjih}\xi^h = 0,\quad \bar{B}_{kjih}\phi^{kh} = 0,\quad \bar{B}_{tsih}\phi^{ts} = 0.$$

A cosymplectic manifold with $\bar{B} = 0$ is said to be *cosymplectic Bochner flat*. Using these identities, (3.4) and (3.8), we can easily check that

$$(3.9)\qquad \|\bar{B}\|^2 = \|R\|^2 - \frac{8}{n+2}\|R_1\|^2 + \frac{2}{(n+1)(n+2)}r^2,$$

$$(3.10)\qquad \|\bar{B}\|^2 = \|H\|^2 - \frac{8}{n+2}\|Q\|^2.$$

Thus we have the following

Theorem 3.1 *Let M be a cosymplectic manifold of dimension ≥ 5. Then M is of constant ϕ-holomorphic sectional curvature if and only if M is η-Einstein and cosymplectic Bochner flat.*

Remark A cosymplectic manifold with vanishing contact Bochner curvature tensor field is said to be *contact Bochner flat.* A cosymplectic manifold is not contact Bochner flat. In fact, we have the equality

$$\|C\|^2 = \|\bar{B}\|^2 + \frac{4n(6n^4 + 15n^3 + 3n^2 - 4n + 4)}{(n+1)(n+2)^2},$$

where C denotes the contact Bochner curvature tensor field due to M. Matsumoto and G. Chūman ([9]).

On the other hand, the contact conformal curvature tensor field ([6]) and the cosymplectic Bochner curvature tensor field are related by

$$\|C_0\|^2 = \frac{n+2}{n^2}\|\bar{B}\|^2 + \frac{(n+1)(n-2)}{n^2}\|H\|^2 + \frac{4(6n^4 - n^3 + 7n^2 + 8n - 4)}{n(n+1)}$$

So, a cosymplectic manifold cannot be a contact conformal flat

§4. *Spec*^0M and the geometry of M

Assume that M is a compact cosymplectic manifold of dimension $2n+1$ and consider *Spec*^0M. In virtue of (3.7) and (3.9) the coefficient $a_{2,0}$, given by (2.3) may be written as follows:

$$a_{2,0} = \frac{1}{180}\int_M [\|\bar{B}\|^2 + \frac{6-n}{n+2}\|Q\|^2]dM + \frac{C_0(n)}{180}\int_M r^2 dM, \tag{4.1}$$

where $C_0(n)$ is constant depending only on n and $C_0(n) > 0$.

We shall often use the following Lemma 4.1, which is a consequence of the Schwarz inequality (cf. [14], p.394).

Lemma 4.1 *Let (M,g) and (M',g') be compact orientable Riemannian manifolds with* $\mathrm{Vol}(M,g) = \mathrm{Vol}(M',g')$ *and* $\int_M r dM = \int_{M'} r' dM'$. *If* r' *=constant, then* $\int_M r^2 dM \geq \int_{M'} r'^2 dM'$ *with equality if and only if* $r =$ *constant* $= r'$.

Theorem 4.2 *Let M and M' be compact cosymplectic manifolds. Assume that* $Spec^0 M = Spec^0 M'$. *Then* $dim M = dim M' = 2n+1 = m$ *and*

(a) *for* $m \leq 11$, *M is of constant ϕ-holomorphic sectional curvature k if and only if M' is of constant ϕ'-holomorphic sectional curvature $k' = k$,*

(b) *for* $m = 13$, *M is cosymplectic Bochner flat and $r =$ constant if and only if M' is cosymplectic Bochner flat and $r' =$ constant $= r$,*

(c) *if the cosymplectic manifolds are η-Einstein and η'-Einstein, respectively, then M is of constant ϕ-holomorphic sectional curvature k if and only if M' is of constant ϕ'-holomorphic sectional curvature $k' = k$.*

Proof. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $\mathrm{Vol}(M) = \mathrm{Vol}(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover, by virtue of (4.1), $a_{2,0} = a'_{2,0}$ yields

$$(4.2) \qquad \int_M [\|\bar{B}\|^2 + \frac{6-n}{n+2}\|Q\|^2] dM + C_0(n) \int_M r^2 dM$$
$$= \int_{M'} [\|\bar{B}'\|^2 + \frac{6-n}{n+2}\|Q'\|^2] dM' + C_0(n) \int_{M'} r'^2 dM'.$$

(a) If M' is of constant ϕ'-holomorphic sectional curvature, then $\bar{B}' = 0$ and $Q' = 0$. Therefore, (4.2) gives

$$\int_M [\|\bar{B}\|^2 + \frac{6-n}{n+2}\|Q\|^2]dM + C_0(n)(\int_M r^2 dM - \int_{M'} r'^2 dM') = 0,$$

which, by r' = constant , $n \leq 5$ and the Lemma 4.1, yields $\bar{B} = 0$, $Q = 0$ and r = constant = r'.

(b) If $n = 6$ and $\bar{B}' = 0$, it follows from (4.2) that

$$\int_M \|\bar{B}\|^2 dM + C_0(6)(\int_M r^2 dM - \int_{M'} r'^2 dM') = 0,$$

which, by r' = constant and the Lemma 4.1, gives our assertion.

(c) Let $Q = 0$ and $Q' = 0$. Then from (4.2), we have

$$\int_M \|\bar{B}\|^2 dM + C_0(n) \int_M r^2 dM = \int_{M'} \|\bar{B}'\|^2 dM' + C_0(n) \int_{M'} r'^2 dM'.$$

If M' is of constant ϕ'-holomorphic sectional curvature, then $\bar{B}' = 0$ and r' = constant. Then from the above equation and Lemma 4.1, we obtain $\bar{B} = 0$ and r = constant. But $\bar{B} = 0$ and $Q = 0$ imply $H = 0$. This completes the proof of our Theorem 4.2.

We say that two Riemannian manifolds (M, g) and (M', g') are $\alpha-$*isospectral* if $\lim_{n\to\infty}\sup|\lambda_{n,0} - \lambda'_{n,0}|n^{-\alpha} = C < \infty$ ([11], [17]).

We first introduce the following Lemma 4.3 due to H.K. Pak ([11]) and J.Y. Wu ([17]).

Lemma 4.3 *Let (M, g) and (M', g') be two compact α-isospectral Riemannian manifolds.*

(a) *If $\alpha = -\frac{4}{m}$ and $m \geq 4$, then $a_{i,0} = a'_{i,0}$,* $\quad i = 0, 1, 2,$

(b) *If* $\alpha = -1$, *then* $a_{i,0} = a'_{i,0}$ *for all* $i \leq [\frac{m}{2}]$.

From the Lemma 4.3, we have the following

Corollary 4.4 *Let* M *and* M' *be compact* α*-isospectral cosymplectic manifolds. Assume that* $\alpha = -\frac{4}{m}$ *or* -1 . *Then* $dimM = dimM' = 2n+1 = m$ *and*

(a) *for* $5 \leq m \leq 11$, M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k = k'$,

(b) *for* $m = 13$, M *is cosymplectic Bochner flat and* r =*constant if and only if* M' *is cosymplectic Bochner flat and* r'=*constant* $= r$,

(c) *if the cosymplectic manifolds are* η*-Einstein and* η'*-Einstein , respectively, and* $m \geq 5$, *then* M *is of constant* ϕ*-holomorphic sectional curvatur* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$.

§5. $Spec^1 M$ and the geometry of M

Assume that M is a compact cosymplectic manifold of dimension $2n+1$ and consider $Spec^1 M$. In virtue of (3.7) and (3.9) the coefficient $a_{2,1}$, given by (2.6), reduces to

$$a_{2,1} = \frac{1}{180}\int_M [2(n-7)\|\bar{B}\|^2 - \frac{2n^2-101n-66}{n+2}\|Q\|^2]dM + \frac{C_1(n)}{360}\int_M r^2 dM, \tag{5.1}$$

where $C_1(n)$ is constant depending only on n and $C_1(n) = \frac{1}{n(n+1)}(n-3)(10n^2-17n-11)$.

Theorem 5.1 *Let* M *and* M' *be compact cosymplectic manifolds. Assume that* $Spec^1 M = Spec^1 M'$. *Then* $dimM = dimM' = 2n+1 = m$ *and*

(a) *for* $17 \leq m \leq 103$, M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$,

(b) *for* $m = 15$, M *is* η*-Einstein with constant scalar curvature* r *if and only if* M' *is* η'*-Einstein with constant scalar curvature* $r' = r$,

(c) *if the cosymplectic manifolds are* η*-Einstein and* η'*-Einstein, respectively, and* $m = 7$ *or* $m \geq 17$*, then* M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$,

(d) *if the cosymplectic manifolds are both cosymplectic Bochner flat and* $m \leq 103$*, then* M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$.

Proof. Because of (2.4) and (2.5), $a_{0,1} = a'_{0,1}$ and $a_{1,1} = a'_{1,1}$ imply $\mathrm{Vol}(M) = \mathrm{Vol}(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover, by virtue of (5.1), $a_{2,1} = a'_{2,1}$ yields

$$(5.2) \int_M [\, 2(n-7)\|\bar{B}\|^2 - \frac{2n^2 - 101n - 66}{n+2}\|Q\|^2 \,]dM + C_1(n)\int_M r^2 dM$$

$$= \int_{M'} [2(n-7)\|\bar{B}'\|^2 - \frac{2n^2 - 101n - 66}{n+2}\|Q'\|^2]dM' + C_1(n)\int_{M'} r'^2 dM'.$$

Using (5.2) and the Lemma 4.1, we easily obtain our assertions.

Theorem 5.2 *Let* M *and* M' *be compact cosymplectic manifolds. Assume that* $Spec^0 M = Spec^0 M'$ *and* $Spec^1 M = Spec^1 M'$*. Then* $\dim M = \dim M' = 2n+1 = m$ *and*

(a) M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$,

(b) *M is η-Einstein with constant scalar curvature r if and only if M' is η'-Einstein with constant scalar curvature $r' = r$,*

(c) *M is cosymplectic Bochner flat with constant scalar curvature r if and only if M' is cosymplectic Bochner flat with constant scalar curvature $r' = r$.*

Proof. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $\mathrm{Vol}(M) = \mathrm{Vol}(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover, by virtue of (2.3) and (2.6), $a_{2,0} = a'_{2,0}$ and $a_{2,1} = a'_{2,1}$ yield

$$\int_M [5\|R\|^2 + 13r^2] dM = \int_{M'} [5\|R'\|^2 + 13r'^2] dM', \tag{5.3}$$

$$\int_M [10\|R_1\|^2 + r^2] dM = \int_{M'} [10\|R'_1\|^2 + r'^2] dM'. \tag{5.4}$$

(a) By (3.6), relation (5.3) may be written as

$$\int_M \|H\|^2 dM - \int_{M'} \|H'\|^2 dM' + \frac{13n^2 + 13n + 10}{5n(n+1)} \left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0.$$

Let $H' = 0$ and r'=constant. Then, by the Lemma 4.1, the last identity leads to $H = 0$ and r = constant = r'.

(b) By (3.7), relation (5.4) may be written as

$$\int_M \|Q\|^2 dM - \int_{M'} \|Q'\|^2 dM' + \frac{n+5}{10n} \left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0.$$

Let $Q' = 0$ and r'=constant. Then, by the Lemma 4.1, the last equality leads to $Q = 0$ and r = constant = r'.

(c) Using (3.9), we rewrite (5.3) in the form

$$\int_M [5\|\bar{B}\|^2 + \frac{40}{n+2}\|R_1\|^2 + \frac{13n^2+39n+16}{(n+1)(n+2)}r^2]dM$$
$$= \int_{M'} [5\|\bar{B}'\|^2 + \frac{40}{n+2}\|R_1'\|^2 + \frac{13n^2+39n+16}{(n+1)(n+2)}r'^2]dM'.$$

This equality, by (5.4), gives

$$\int_M \|\bar{B}\|^2 dM - \int_{M'} \|\bar{B}'\|^2 dM' + \frac{13n^2+35n+12}{5(n+1)(n+2)}(\int_M r^2 dM - \int_{M'} r'^2 dM') = 0.$$

Assume that $\bar{B}' = 0$ and $r' =$ constant. In view of the Lemma 4.1, the last relation yields now $\bar{B} = 0$ and $r =$ constant $= r'$. This completes the proof of the Theorem.

§6. *Spec*^2M and the geometry of M

Assume that M is a compact cosymplectic manifold of dimension $2n+1$ and consider *Spec*^2M. With the help of (3.7) and (3.9) the coefficient $a_{2,2}$, given by (2.9), may be written as follows :

$$(6.1) \qquad a_{2,2} = \frac{1}{180}\int_M [(n-7)(2n-15)\|\bar{B}\|^2 + \frac{-2n^3+191n^2-324n-60}{n+2}\|Q\|^2]dM + \frac{1}{180}\int_M \frac{10n^4-107n^3+310n^2-147n-30}{2n(n+1)}r^2 dM.$$

Theorem 6.1 *Let M and M' be compact cosymplectic manifolds. Assume that Spec$^2(M)$= Spec$^2M'$. Then $dimM = dimM' = 2n+1 = m$ and* (a)*for* $m = 5,7,9$ *and* 13 *or* $17 \leq m \leq 187$, M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$,

(b) *for* $m = 15$, M *is* η*-Einstein with constant scalar curvature* r *if and only if* M' *is* η'*-Einstein with constant scalar curvature* $r' = r$,

(c) *if the cosymplectic manifolds are* η*-Einstein and* η'*-Einstein, respectively, and* $m \neq 11$ *and* 15, *then* M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$,

(d) *if the cosymplectic manifolds are both cosymplectic Bochner flat, and* $5 \leq m \leq 9$ *or* $13 \leq m \leq 187$, *then* M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$.

Proof. The proof is based on the equalities $a_{0,2} = a'_{0,2}, a_{1,2} = a'_{1,2}$ and $a_{2,2} = a'_{2,2}$, where the coefficients are given by (2.7), (2.8) and (6.1). The idea of the proof is similar to that of Theorem 5.1. Therefore, we shall omit the details.

Theorem 6.2 *Let* M *and* M' *be compact cosymplectic manifolds. Assume that* $Spec^0 M = Spec^0 M'$ *and* $Spec^2 M = Spec^2 M'$. *Then* $dim M = dim M'$ $= 2n + 1 = m$ *and*

(a) *for* $m \geq 7$, M *is of constant* ϕ*-holomorphic sectional curvature* k *if and only if* M' *is of constant* ϕ'*-holomorphic sectional curvature* $k' = k$,

(b) *for* $m \geq 15$, M *is* η*-Einstein with constant scalar curvature* r *if and only if* M' *is* η'*-Einstein with constant scalar curvature* $r' = r$,

(c) *for* $m \geq 7, M$ *is cosymplectic Bochner flat with constant scalar curvature* r *if and only if* M' *is cosymplectic Bochner flat with constant scalar curvature* $r' = r$.

Proof. Because of (2.1) and (2.2), $a_{0,0} = a'_{0,0}$ and $a_{1,0} = a'_{1,0}$ imply $\mathrm{Vol}(M) = \mathrm{Vol}(M')$ and $\int_M r dM = \int_{M'} r' dM'$. Moreover by virtue of (2.3) and (2.9), $a_{2,0} = a'_{2,0}$ and $a_{2,2} = a'_{2,2}$ yield

(6.2)
$$\int_M [(10n-23)\|R\|^2 + (26n-67)r^2]dM = \int_{M'} [(10n-23)\|R'\|^2 + (26n-67)r'^2]dM',$$

(6.3)
$$\int_M [2(10n-23)\|R_1\|^2 + (2n-19)r^2]dM = \int_{M'} [2(10n-23)\|R'_1\|^2 + (2n-19)r'^2]dM'.$$

(a) By (3.6), relation (6.2) may be written as

$$\int_M (10n-23)\|H\|^2 dM - \int_{M'} (10n-23)\|H'\|^2 dM' + \frac{26n^3 - 41n^2 - 47n - 46}{n(n+1)}\left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0.$$

Let $H' = 0$, r'=constant and $n \geq 3$. Then, in view of the Lemma 4.1, the last identity leads to $H = 0$ and r = constant $= r'$.

(b) By (3.7), relation (6.3) may be written as

$$\int_M (10n-23)\|Q\|^2 dM - \int_{M'} (10n-23)\|Q'\|^2 dM'$$
$$+ \frac{2n^2-9n-23}{2n}\left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0.$$

Let $Q' = 0$, $r' =$ constant and $n \geq 7$. Then by the Lemma 4.1, our last equality leads to $Q = 0$ and $r =$ constant $= r'$.

(c) Using (3.9), we rewrite (6.2) in the form

$$\int_M [(10n-23)\|\bar{B}\|^2 + \frac{8(10n-23)}{n+2}\|R_1\|^2 + \frac{26n^3+11n^2-169n-88}{(n+1)(n+2)} r^2] dM$$
$$= \int_{M'} [(10n-23)\|\bar{B}'\|^2 + \frac{8(10n-23)}{n+2}\|R_1'\|^2 + \frac{26n^3+11n^2-169n-88}{(n+1)(n+2)} r'^2] dM'.$$

This equality, by (6.3), gives

$$\int_M (10n-23)\|\bar{B}\|^2 dM - \int_{M'} (10n-23)\|\bar{B}'\|^2 dM'$$
$$+ \frac{26n^3+3n^2-101n-12}{(n+1)(n+2)}\left(\int_M r^2 dM - \int_{M'} r'^2 dM'\right) = 0.$$

Let $\bar{B}' = 0$, $r' =$ constant and $n \geq 3$. Then, by the Lemma 4.1, the last relation yields $\bar{B} = 0$ and r=constant=r'. This completes the proof of the Theorem.

References

[1] M. Berger, P. Gauduchon et E. Mazet, *Le Spectre d' une Variété Riemanniene*, Lecture Notes in Mathematics 194, Springer-Verlag 1971.

[2] D.E. Blair, *The theory of quasi-Sasakian structure*, J. Differential Geometry, 1 (1967), 331-345.

[3] D.E. Blair and G.D. Ludden, *Hypersurfaces in almost contact manifold*, Tohoku Math. J. 21(1969), 354-362.

[4] S.S. Eum, *Cosymplectic manifolds of constant ϕ-holomorphic sectional curvature*, Bull. of Korean Math. Soc. 9(1972), 1-7.

[5] S.S. Eum, *On the cosymplectic Bochner curvature tensor*, J. of Korean Math. Soc. 15(1978), 29-37.

[6] J.C. Jeong, J.D. Lee, G.H. Oh and J.S. Pak, *On the contact conformal curvature tensor*, Bull. Korean Math. Soc. 27(1990), 133-142.

[7] U.K. Kim, *On anti-invariant submanifolds of cosymplectic manifolds*, J. Korean Math. Soc. 20(1983), 9-29.

[8] G.D. Ludden, *Submanifolds of cosymplectic manifolds*, J. Differential Geometry, 4(1970), 237-244.

[9] M. Matsumoto and G. Chūman, *On the C - Bochner curvature tensor*, TRU Math. 5(1969), 21-30.

[10] Z. Olszak, *The spectrum of the Laplacian and the curvature of Sasakian manifolds*, Lecture Notes in Mathematics 838, Springer - Verlag, 1981, 221-228.

[11] H.K. Pak, *On the spectral rigidity of almost isospectral manifolds*, preprint.

[12] J.S. Pak, J.C. Jeong and W-T. Kim, *The contact conformal curvature tensor field and the spectrum of the Laplacian*, J. Korean Math. Soc. 28(1991), 267-274.

[13] V.K. Patodi, *Curvature and the fundamental solution of the heat operator*, J. Indian Math. Soc. 34(1970), 269-285.

[14] S. Tanno, *Eigenvalues of the Laplacian of Riemannian manifolds*, Tohoku Math. J. 25(1973), 391-403.

[15] S. Tanno, *The Spectrum of the Laplacian for 1-forms*, Proc. Amer. Math. Soc. 45(1974), 125-129.

[16] Gr. Tsagas, ***On the Spectrum of the Laplace operator for the exterior 2-forms***, Tensor N.S. 33(1979), 94-96.

[17] J.Y.Wu, ***On almost isospectral manifolds***, Indiana Univ. Math. J. 39(1990), 1373-1381.

[18] S. Yamaguchi and G. Chūman, ***Eigenvalues of the Laplacian of Sasakian manifolds***, TRU Math. 15-2(1979), 31-41.

Received Oct. 4, 1991 Revised Apr. 8, 199

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