Nihonkai Math. J.
vol. 3 (1992), 31-37

SOME PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

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1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k} \quad (p=1,2,\cdots) \tag{1}$$

which are analytic in the unit disc $D=\{z:|z|<1\}$, and $A(1)=A$. Further, we define a function $F_\lambda(z)$ by

$$F_\lambda(z)=(1-\lambda)f(z)+\lambda zf'(z) \tag{2}$$

for $\lambda>0$ and $f(z)\in A(p)$. For

$$f(z)=\sum_{n=0}^{\infty}a_n z^n \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty}b_n z^n ,$$

we define the Hadamard product (or convolution) by

$$f*g(z)=\sum_{n=0}^{\infty}a_n b_n z^n . \tag{3}$$

Let

$$\phi(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1} \qquad z\in D,\ c\neq 0,-1,-2,\cdots ,$$

$$L(a,c)f=\phi(a,c)*f(z) \qquad f(z)\in A , \tag{4}$$

where $(\lambda)_n=\Gamma(n+\lambda)/\Gamma(\lambda)$. It is known by [1] that $L(a,c)$ maps A into itself, and if $c>a>0$, $L(a,c)$ has the integral representation

$$L(a,c)f(z)=\int_0^1 u^{-1}f(uz)d\mu(a,c-a)(u) , \tag{5}$$

where μ is the beta distribution

$$d\mu(a,c-a)(u)=\frac{u^{a-1}(1-u)^{c-a-1}}{B(a,c-a)}du .$$

Clearly $L(a,a)$ is the unit operator and

$$L(a,c)=L(a,b)L(b,c)=L(b,c)L(a,b) \qquad b,c\neq 0,-1,-2,\cdots.$$

Moreover, if $a\neq 0, -1, -2, \cdots$, then $L(a,c)$ has an inverse $L(c,a)$ and is a 1-1 mapping of A into itself.

Recently, Saitoh [2] have studied some propertise of function in the class $A(p)$, and of the function $F_\lambda(z)$ defined by (2). In this paper, we prove a sharp inequality and improve Saitoh's results.

2. Main results

In order to prove our results, we have to recall here the following lemmas.

LEMMA 1. ([3]) Let $p(z)=1+p_1z+p_2z^2+\cdots$ be analytic in the unit disc D. If $\mathrm{Re}\{p(z)\}>0$, then

$$\mathrm{Re}\{p(z)\}\geqslant \frac{1-|z|}{1+|z|}$$

for all $z\in D$.

Taking $(p(z)-\beta)/(1-\beta)$ instead of $p(z)$ in Lemma 1, we have

LEMMA 2. Let $\beta<1$ and $p(z)=1+p_1z+p_2z^2+\cdots$ be analytic in the unit disc D. If $\mathrm{Re}\{p(z)\}>\beta$, then

$$\mathrm{Re}\{p(z)\}\geqslant\frac{1+(2\beta-1)|z|}{1+|z|}$$

for all $z\in D$.

With the aid of the above Lemmas, we prove a important Lemma.

LEMMA 3. Let $\alpha>0$, $\beta<1$ and $p(z)=1+p_1z+p_2z^2+\cdots$ is an analytic function in the unit disc D. If

$$\mathrm{Re}\{p(z)+\alpha zp'(z)\}>\beta, \qquad z\in D \tag{6}$$

then

$$\mathrm{Re}\{p(z)\}>\psi(\alpha,\beta),$$

where

$$\psi(\alpha,\beta)=\frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}\frac{1+(2\beta-1)u}{1+u}\,du. \tag{7}$$

This result is sharp.

Proof. We define the function $q(z)$ by

$$p(z)+\alpha zp'(z)=q(z).$$

Then $q(z)=1+q_1 z+q_2 z^2+\cdots$ is analytic in the unit disc D. By using (4), we obtain

$$\begin{aligned} zq(z)&=(1-\alpha)[zp(z)]+\alpha z[zp(z)]' \\ &=L(\tfrac{1}{\alpha}+1,\tfrac{1}{\alpha})[zp(z)], \end{aligned}$$

that is

$$zp(z)=L(\tfrac{1}{\alpha},\tfrac{1}{\alpha}+1)[zq(z)].$$

It follows from (6) that $\mathrm{Re}\{q(z)\}>\beta$. Therefore, using (5) and Lemma 2, we obtain

$$\begin{aligned} \mathrm{Re}\{p(z)\}&=\mathrm{Re}\{\tfrac{1}{z}L(\tfrac{1}{\alpha},\tfrac{1}{\alpha}+1)[zq(z)]\} \\ &=\frac{1}{B(\frac{1}{\alpha},1)}\int_0^1 u^{\frac{1}{\alpha}-1}\,\mathrm{Re}\{q(uz)\}du \\ &>\frac{1}{\alpha}\int_0^1 u^{\frac{1}{\alpha}-1}\,\frac{1+(2\beta-1)u}{1+u}du. \end{aligned}$$

By considering the function $p(z)=\{L(\frac{1}{\alpha},\frac{1}{\alpha}+1)[(z+(1-2\beta)z^2)/(1-z)]\}/z$, one can show that the result is best possible. Thus we complete the proof of Lemma.

THEOREM 1. Let $f(z)\in A(p)$. If

$$\mathrm{Re}\{\frac{f^{(j)}(z)}{z^{p-j}}\}>\beta \qquad (\beta<p!/(p-j)!;\ z\in D), \tag{8}$$

then we have

$$\mathrm{Re}\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\}>\frac{p!}{(p-j+1)!}\psi(\frac{1}{p-j+1},\frac{(p-j)!}{p!}\beta), \qquad (z\in D) \tag{9}$$

where $1\leq j\leq p$, $\psi(x,y)$ is defined by (7). This result is sharp.

Proof. We define the function $q(z)$ by

$$\frac{(p-j+1)!}{p!}\cdot\frac{f^{(j-1)}(z)}{z^{p-j+1}}=q(z). \tag{10}$$

Then $q(z)=1+q_1 z+q_2 z^2+\cdots$ is analytic in the disc D. Differentiating both sides of (10), we obtain

$$\frac{(p-j)!}{p!}\frac{f^{(j)}(z)}{z^{p-j}} = q(z)+\frac{1}{p-j+1}zq'(z).$$

It follows from (8) that

$$\mathrm{Re}\{q(z)+\frac{1}{p-j+1}zq'(z)\} > \frac{(p-j)!}{p!}\beta .$$

Hence, by using Lemma 3, we have

$$\mathrm{Re}\{q(z)\} > \psi(\frac{1}{p-j+1},\frac{(p-j)!\beta}{p!}),$$

that is

$$\mathrm{Re}\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\} > \frac{p!}{(p-j+1)!}\psi(\frac{1}{p-j+1},\frac{(p-j)!\beta}{p!}).$$

This result is sharp follows from the fact the Lemma 3 is sharp. The proof of Theorem 1 is, therefore, completed.

Taking j=p in Theorem 1, we have

COROLLARY 1. Let $f(z) \in A(p)$ and suppose

$$\mathrm{Re}\{f^{(p)}(z)\} > \beta \qquad (\beta < p!;\ z \in D),$$

then we have

$$\mathrm{Re}\{\frac{f^{(p-1)}(z)}{z}\} > p!(2\beta-1)+2p!(1-\beta)\ln 2 \qquad (z \in D).$$

This result is sharp.

Letting j=1 in Theorem 1, we have

COROLLARY 2. Let $f(z) \in A(p)$ and suppose

$$\mathrm{Re}\{\frac{f'(z)}{z^{p-1}}\} > \beta \qquad (\beta < p,\ z \in D),$$

then we have

$$\mathrm{Re}\{\frac{f(z)}{z^p}\} > \psi(\frac{1}{p},\frac{\beta}{p}). \qquad (z \in D).$$

This result is sharp.

Making p=j=1 in Theorem 1, we have

COROLLARY 3. Let $f(z) \in A$ and suppose

$$\mathrm{Re}\{f'(z)\} > \beta \qquad (\beta < 1,\ z \in D),$$

then we have

$$\mathrm{Re}\{\frac{f(z)}{z}\} > (2\beta-1)+2(1-\beta)\ln 2 \qquad (z\in D).$$

This result is sharp.

REMARK. For $0\leqq\beta<1$, this result has been obtained by Owa [4] using the subordination principle.

THEOREM 2. Let a function $F_\lambda(z)$ defined by (2) for $\lambda>0$ and $f(z)\in A(p)$. If

$$\mathrm{Re}\{\frac{F_\lambda^{(j)}(z)}{z^{p-j}}\} > \beta \qquad (\beta < \frac{p!(1-\lambda+\lambda p)}{(p-j)!};\quad z\in D), \tag{11}$$

then

$$\mathrm{Re}\{\frac{f^{(j)}(z)}{z^{p-j}}\} > \frac{p!}{(p-j)!}\,\psi(\frac{\lambda}{1-\lambda+\lambda p}, \frac{(p-j)!\beta}{(1-\lambda+\lambda p)p!}) \qquad (z\in D),$$

where $0\leqq j\leqq p$, $\psi(x,y)$ is defined by (7). This result is sharp.

Proof. By the differentiation of $F_\lambda(z)$, we obtain

$$F_\lambda^{(j)}(z)=(1-\lambda+\lambda j)f^{(j)}(z)+\lambda zf^{(j+1)}(z). \tag{12}$$

We define the function $q(z)$ by

$$\frac{(p-j)!}{p!}\cdot\frac{f^{(j)}(z)}{z^{p-j}} = q(z). \tag{13}$$

Then $q(z)=1+q_1z+q_2z^2+\cdots$ is analytic in D. Making the differentiation in (13), we have

$$q(z)+\frac{\lambda}{1-\lambda+\lambda p}zq'(z)=\frac{(p-j)!}{(1-\lambda+\lambda p)p!}\cdot\frac{F_\lambda^{(j)}(z)}{z^{p-j}}.$$

By using Lemma 3 and (11), we have

$$\mathrm{Re}\{q(z)\} > \psi(\frac{\lambda}{1-\lambda+\lambda p}, \frac{(p-j)!\beta}{(1-\lambda+\lambda p)p!}).$$

It follows from (13) that Theorem 2 holds.

REMARK. Theorem 1 and Theorem 2 are the improvement and the extension of the result by Saitoh [2].

Taking $j=0$ in Theorem 2, we have

COROLLARY 4. Let a function $F_\lambda(z)$ defined by (2) for $\lambda>0$ and $f(z)\in A(p)$. If

$$\mathrm{Re}\{\frac{F_\lambda(z)}{z^p}\} > \beta \qquad (\beta < 1-\lambda+\lambda p;\ z \in D),$$

then

$$\mathrm{Re}\{\frac{f(z)}{z^p}\} > \psi(\frac{\lambda}{1-\lambda+\lambda p}, \frac{\beta}{1-\lambda+\lambda p}) \qquad (z \in D).$$

This result is sharp.

Putting j=p in Theorem 2, we have

COROLLARY 5. Let a function $F_\lambda(z)$ defined by (2) for $\lambda > 0$ and $f(z) \in A(p)$. If

$$\mathrm{Re}\{F_\lambda^{(p)}(z)\} > \beta \qquad (\beta < p!(1-\lambda+\lambda p);\ z \in D),$$

then

$$\mathrm{Re}\{f^{(p)}(z)\} > p!\ \psi(\frac{\lambda}{1-\lambda+\lambda p}, \frac{\beta}{(1-\lambda+\lambda p)p!}) \qquad (z \in D).$$

This result is sharp.

Taking j=1 in Theorem 2, we have

COROLLARY 6. Let a function $F_\lambda(z)$ defined by (2) for $\lambda > 0$ and $f(z) \in A(p)$. If

$$\mathrm{Re}\{\frac{F_\lambda'(z)}{z^{p-1}}\} > \beta \qquad (\beta < p(1-\lambda+\lambda p);\quad z \in D),$$

then

$$\mathrm{Re}\{\frac{f'(z)}{z^{p-1}}\} > p\ \psi(\frac{\lambda}{1-\lambda+\lambda p}, \frac{\beta}{(1-\lambda+\lambda p)p}) \qquad (z \in D).$$

This result is sharp.

Making p=1 and j=0 in Theorem 2, we have

COROLLARY 7. Let a function $F_\lambda(z)$ defined by (2) for $\lambda > 0$ and $f(z) \in A$. If

$$\mathrm{Re}\{\frac{F_\lambda(z)}{z}\} > \beta \qquad (\beta < 1; z \in D),$$

then

$$\mathrm{Re}\{\frac{f(z)}{z}\} > \psi(\lambda, \beta) \qquad (z \in D).$$

This result is sharp.

Making p=j=1 in Theorem 2, we have

COROLLARY 8. Let a function $F_\lambda(z)$ defined by (2) for $\lambda > 0$ and $f(z) \in A$. If

$$\mathrm{Re}\{F'_\lambda(z)\} > \beta \qquad (\beta < 1;\ z \in D),$$

then

$$\mathrm{Re}\{f'(z)\} > \psi(\lambda, \beta) \qquad (z \in D).$$

This result is sharp.

REMARK. Corollary 7 and Corollary 8 are the improvement and the extension of the results by Owa [5].

REFERENCES

[1] B.C.Carlson and D.B.Shaffer; Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal., 15(1984), 737-745.

[2] H.Saitoh; Properties of certain analytic functions, Proc. Japan Acad. Ser.A, 65(1989), 131-134.

[3] T.H.Macgregor; Functions whose derivative has a positive real part, Trans. Amer. Math. Soc., 104(1962), 532-537.

[4] S.Owa, W.Ma and L.Liu; On a class of analytic functions satisfying $\mathrm{Re}\{f'(z)\} > \alpha$, Bull. Korean Math. Soc., 25. No.2(1988), 211-214.

[5] S.Owa and M.Nunokawa; Propertise of certain analytic functions, Math. Japanica, 33. No.4(1988), 577-582.

Received Dec. 6, 1991

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