Nihonkai Math. J.
vol. 3 (1992), 1-7

HYPERSURFACES OF THE TWO-DIMENSIONAL COMPLEX PROJECTIVE SPACE

Sharief Deshmukh and M.A. Al-Gwaiz

Abstract : We consider a compact, simply connected, real hypersurface of the two dimensional complex projective space CP^2 of constant holomorphic sectional curvature 4, and obtain a criterion for the hypersurface to be diffeomorphic to S^3 in terms of an inequality which relates the length of the second fundamental form and the mean curvature.

1. Introduction

Real hypersurfaces of the complex projective space CP^n have not been studied as extensively as those of real space forms. One of the reasons for this is that the Codazzi equation is simpler for hypersurfaces in real space forms. To overcome this deficiency, Lawson [2] considered the Riemannian submersion $\pi: S^{2n+1} \rightarrow CP^n$ which gives rise,for a real hypersurface M in CP^n, to a hypersurface $\pi^{-1}(M)$ in S^{2n+1} such that $\pi: \pi^{-1}(M) \rightarrow M$ is a Riemannian submersion with totally geodesic fibers. The properties of a hypersurface $\tilde{M}$ of S^{2n+1}, which are invariant under free S^1 action, can then be projected on the hypersurfaces of CP^n provided that the corresponding S^1 action on S^{2n+1} induces a free S^1 action on $\tilde{M}$. Lawson [2] used this idea to study the minimal hypersurfaces of CP^n, and Okumura [4] to study those with constant mean curvature.

The motivation for the present paper comes from the following example of a real hypersurface of CP^2. Consider the unit sphere S^5 in C^3, and the Riemannian submersion $\pi: S^5 \longrightarrow CP^2$ with totally geodesic fibers given by the integral curves of the unit vector field $\bar{\xi}=-J\bar{N}$ defined on S^5, J being the complex structure on C^3 and $\bar{N}$ the unit normal field to S^5 in C^3. Let $S^1(1/k)$ be the circle of radius k in C^1 and $S^3(1/\sqrt{1-k^2})$ be the 3-sphere of radius $\sqrt{1-k^2}$ in C^2. The product $S^1(1/k)\times S^3(1/\sqrt{1-k^2})$ is then the hypersurface of S^5 whose shape operator $\bar{A}$ is given at a given point of $S^1(1/k)\times S^3(1/\sqrt{1-k^2})$ by

$$\bar{A} = \begin{pmatrix} \sqrt{1-k^2}/k & 0 & 0 & 0 \\ 0 & -k/\sqrt{1-k^2} & 0 & 0 \\ 0 & 0 & -k/\sqrt{1-k^2} & 0 \\ 0 & 0 & 0 & -k/\sqrt{1-k^2} \end{pmatrix} .$$

Since the circle group S^1 acts freely on $S^1(1/k)\times S^3(1/\sqrt{1-k^2})$, the quotient space $S^1(1/k)\times S^3(1/\sqrt{1-k^2})/S^1$ is a real hypersurface of CP^2, and we can define the Riemannian submersion of $S^1(1/k)\times S^3(1/\sqrt{1-k^2})$ onto $S^1(1/k)\times S^3(1/\sqrt{1-k^2})/S^1$, with totally geodesic fibers,as the restriction to $S^1(1/k)\times S^3(1/\sqrt{1-k^2})$ of the Riemannian submersion $\pi : S^5 \longrightarrow CP^2$. Using this we obtain the expression for the shape operator A of the real hypersurface $S^1(1/k)\times S^3(1/\sqrt{1-k^2})/S^1$ in CP^2 as

$$A = -\frac{k}{\sqrt{1-k^2}} I + \frac{\sqrt{1-k^2}}{k}\, \eta\otimes\xi ,$$

$\xi = -JN$ being the unit vector field defined on the hypersurface, J the complex structure on CP^2, N the unit normal field to the hypersurface, and η the 1-form dual to ξ. Denoting the mean curvature of this hypersurface by α , we find that

$$2+3\alpha g(A\xi,\xi)=\mathrm{tr}\cdot A^2 . \tag{1.1}$$

Furthermore, a point of $S^1(1/k)\times S^3(1/\sqrt{1-k^2})/S^1$ is an equivalence class $[(z^1,z^2,z^3)]$, $(z^1,z^2,z^3)\in S^1(1/k)\times S^3(1/\sqrt{1-k^2})$; so if we define a function $f: S^1(1/k)\times S^3(1/\sqrt{1-k^2})/S^1 \longrightarrow S^3$ by

$$f([(z^1,z^2,z^3)])=(z^2/z^1,z^3/z^1) ,$$

then it easily follows that f is a diffeomorphism.

The above example raises the following question: Is a compact, simply connected, real hypersurface of CP^2 which satisfies (1.1) diffeomorphic to S^3 ? In this paper we shall show that the answer to this question is in the affirmative. In fact, we prove

Theorem: Let M be a compact and simply connected real hypersurface of CP^2, with shape operator A and mean curvature α. If $2+3\alpha g(A\xi,\xi)\geq \mathrm{tr}\cdot A^2$, then M is diffeomorphic to S^3, where $\xi=-JN$, J is the complex structure of CP^2 and N is the unit normal vector field of M in CP^2.

2. Preliminaries

Let J,g and $\bar\nabla$ be, respectively, the complex structure, the hermitian metric and the Riemannian connection on CP^2 . The curvature tensor $\bar R$ of CP^2 is given

$$\bar R(X,Y)Z=g(Y,Z)X-g(X,Z)Y+g(JY,Z)JX-g(JX,Z)JY+2g(X,JY)JZ. \tag{2.1}$$

Let M be a real hypersurface of CP^2 and N be the unit normal vector field to M. We denote by g,∇ and A, respectively, the induced metric, the induced Riemannian connection and the shape operator on M. Consequently, the Gauss and Weingarten formulae are given by

$$(2.2)\qquad \bar\nabla_X Y=\nabla_X Y+g(AX,Y)N,\quad \bar\nabla_X N=-AX,\quad X,Y\in\mathfrak{X}(M)\ ,$$

where $\mathfrak{X}(M)$ is the Lie-algebra of vector fields on M. And we also have the equations of Gauss and Codazzi for the hypersurface,

$$(2.3)\qquad R(X,Y;Z,W)=\bar R(X,Y;Z,W)+g(AY,Z)g(AX,W)-g(AX,Z)g(AY,W),$$

$$(2.4)\qquad \bar R(X,Y)N=(\nabla_Y A)(X)-(\nabla_X A)(Y)\ ,\qquad X,Y,Z,W\in\mathfrak{X}(M)\ ,$$

where R is the curvature tensor of M.

Define a unit vector field ξ on M by $\xi=-JN$, and a (1,1) tensor field ϕ on M by setting $JX=\phi X+\eta(X)N$, $X\in\mathfrak{X}(M)$, where ϕX is the tangential component of JX to M and η is the 1-form dual to ξ. Then it is straightforward to verify that the triplet (ϕ,ξ,η) satisfies

$$(2.5)\qquad \phi^2=-I+\eta\otimes\xi,\quad \phi\xi=0,\quad \eta\circ\phi=0,\quad \operatorname{rank}\phi=2\ ,$$

and that

$$(2.6)\qquad g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\quad g(\phi X,Y)=-g(X,\phi Y),\quad X\in\mathfrak{X}(M).$$

Since $(\bar\nabla_X J)(\xi)=0$ for all $X\in\mathfrak{X}(M)$ we can use equations (2.2), (2.5) and $JX=\phi X+\eta(X)N$ to arrive at

$$(2.7)\qquad \nabla_X\xi=\phi AX\ ,\qquad X\in\mathfrak{X}(M)\ .$$

For a local unit vector field e orthogonal to ξ, we note that $\{e,\phi e,\xi\}$ is a local orthonormal frame on M ; such a frame will be referred to as an adapted frame. Using an adapted frame in (2.3), to compute the Ricci tensor Ric of M, we get

(2.8) $\quad \mathrm{Ric}(X,Y) = 5g(X,Y)-3\eta(X)\eta(Y) + 3\alpha g(AX,Y) - g(AX,AY)$

where α is the mean curvature on M .

3. Proof of the Theorem

Let M be a compact and simply connected real hypersurface of CP^2 . Then, using an adapted frame in (2.7) to compute the divergence of ξ, we obtain $\operatorname{div}\xi = 0$. Next we compute $\|\nabla\xi\|^2$ which, for a local orthonormal frame $\{e_1,e_2,e_3\}$, is given by $\|\nabla\xi\|^2 = \sum_1^3 g(\nabla_{e_i}\xi,\nabla_{e_i}\xi)$. Again using an adapted frame with equation (2.7), we arrive at

$$\|\nabla\xi\|^2 = \mathrm{tr}\cdot A^2 - \|A\xi\|^2 .$$

Now on any compact Riemannian manifold M , and for any $X \in \mathfrak{X}(M)$, we have the following integral formula (cf.[5])

$$\int_M \{\mathrm{Ric}(X,X) + \frac{1}{2}\|L_X g\|^2 - \|\nabla X\|^2 - (\operatorname{div}X)^2\}dv = 0 ,$$

where $L_X g$ is the Lie-derivative of g with respect to X . Applying this formula to the hypersurface M with $X = \xi$, and using equation (2.8) and the hypothesis of the theorem, we get $L_\xi g = 0$, which is equivalent to

(3.1) $\quad g(\nabla_X\xi,Y) + g(\nabla_Y\xi,X) = 0 , \quad X,Y \in \mathfrak{X}(M)$.

But, in view of (2.7), the skew symmetry of ϕ and the symmetry of A, equation (3.1) leads to $A\phi = \phi A$. And from this it follows that $\phi A\xi = o$. Since rank $\phi = 2$ and $\phi\xi = 0$, we conclude that $A\xi = \mu\xi$, where μ is a smooth function on M . Suppose e is a unit vector field orthogonal to ξ, and is an eigenvector of A , that is, $Ae = \lambda e$ for some smooth function λ on M . Then the equation $A\phi = \phi A$ implies that $A\phi e = \lambda\phi e$, thus A is diagonalized in the adapted frame $\{e,\phi e,\xi\}$, with at most two distinct eigenvalues. Now equation (2.7) gives the local equations

$$(3.2) \qquad \nabla_e\xi = \lambda\phi e\ , \quad \nabla_{\phi e}\xi = -\lambda e\ , \quad \nabla_\xi\xi = 0\ ,$$

and the Codazzi equation (2.4) written in the form

$$(\nabla_{\phi e}A)(e) - (\nabla_e A)(\phi e) = \bar{R}(e,\phi e)J\xi\ ,$$

yields

$$(\phi e\cdot\lambda)e + \lambda\nabla_{\phi e}e - A(\nabla_{\phi e}e) - (e\cdot\lambda)\phi e - \lambda\nabla_e\phi e + A(\nabla_e\phi e) = 2\xi\ .$$

Taking the inner product of each term in the above equation with e, ϕe and ξ , and using equations (3.2) together with $g(\nabla_e\phi e,\xi) = -g(\phi e,\nabla_e\xi)$, etc , we obtain

$$(3.3) \qquad e\cdot\lambda = 0\ , \quad \phi e\cdot\lambda = 0 \quad \text{and} \quad \lambda^2 - \lambda\mu = 1\ .$$

Similarly the Codazzi equation

$$(\nabla_\xi A)(e) - (\nabla_e A)(\xi) = \bar{R}(e,\xi)J\xi$$

gives

$$(3.4) \qquad \xi\cdot\lambda = 0\ .$$

The equations (3.3) and (3.4) imply that λ is a constant, and $\lambda^2 - \lambda\mu = 1$ implies that $\lambda \neq 0$ and that μ is also a constant. With these results the Ricci curvatures of M may now be computed from (2.8) . We get

$$(3.5) \qquad \begin{aligned} &\mathrm{Ric}(e,e) = \mathrm{Ric}(\phi e,\phi e) = 4 + 2\lambda^2\ , \quad \mathrm{Ric}(\xi,\xi) = \lambda^2\ , \\ &\mathrm{Ric}(e,\xi) = \mathrm{Ric}(\phi e,\xi) = \mathrm{Ric}(e,\phi e) = 0\ , \end{aligned}$$

where λ is a non-zero constant. Thus, if ω^1, ω^2 are the 1-form duals of e, ϕe respectively then any $X \in \mathfrak{X}(M)$ has the local representation $X = \omega^1(X)e + \omega^2(X)\phi e + \eta(X)\xi$. Therefore, when $X \neq 0$,

$$Ric(X,X) = [\omega^1(X)]^2 Ric(e,e) + [\omega^2(X)]^2 Ric(\phi e, \phi e) + [\eta(X)]^2 Ric(\xi,\xi) > 0 .$$

Hence M has strictly positive Ricci curvature. Being compact and simply connected, M is therefore dffeomorphic to S^3 by Hamilton's theorem (cf. [1]).

REFERENCES

[1] R.S. Hamilton, "Three manifolds with positive Ricci curvature", *J. Diff. Geom.* 17(1982),255-306.

[2] H.B. Lawson, Jr., "Rigidity theorem in rank-1 symmetric spaces", *J. Diff. Geom.* 4(1970), 349-357.

[3] M. Okumura, "On some real hypersurfaces of a complex projective space", *Trans. Amer. Math. Soc.* 212(1975),355-364.

[4] M. Okumura, "Compact real hypersurfaces with constant mean curvature of a complex projective space", *J.Diff.Geom.* 13(1978), 43-50.

[5] K. Yano, "On harmonic and Killing vector fields", *Ann. of Math.* 55(1952), 38-45.

Received Jul. 15, 1991

Department of Mathematics, College of Science, King Saud University, P.O.Box, Riyadh 11451, Saudi Arabia.