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REVERSE INEQUALITIES OF ARAKI, CORDES AND LÖWNER-HEINZ INEQUALITIES

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ABSTRACT. In this paper, we show reverse inequality to Araki's inequality and investigate the equivalence among reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities. Among others, we show that if A and B are positive operators on a Hilbert space H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$K(m, M, p)\|BAB\|^p \leq \|B^pA^pB^p\| \qquad \text{for all } 0 < p < 1,$$

where $K(m, M, p)$ is a generalized Kantorovich constant by Furuta.

1. INTRODUCTION

Let A and B be positive operators on a Hilbert space H. The equivalence among Cordes and Löwner-Heinz inequalities was discussed by many authors. In [10], Furuta showed that the Cordes inequality for the operator norm

$$\|A^pB^p\| \leq \|AB\|^p \qquad \text{for all } 0 < p < 1 \tag{1}$$

is equivalent to the Löwner-Heinz inequality (cf.[16])

$$A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \quad \text{for all } 0 < p < 1 \tag{2}$$

(cf. [7]). In [1], Araki showed a trace inequality which entailed the following inequality:

$$\|B^pA^pB^p\| \leq \|BAB\|^p \quad \text{for all } 0 < p < 1. \tag{3}$$

Moreover, it was shown in [8, 2] that the Cordes inequality (1) is equivalent to Araki's inequality (3).

On the other hand, Furuta [11] showed the following Kantorovich type inequalities of the Löwner-Heinz inequality (2): If A and B are positive operators such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then

$$A \geq B \geq 0 \quad \text{implies} \quad K(m, M, p)A^p \geq B^p \quad \text{for all } p > 1, \tag{4}$$

where a generalized Kantorovich constant $K(m, M, p)$ [5, 9, 13] is defined as

(5)
$$K(m, M, p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^p \qquad \text{for all real numbers } p.$$

We here cite Furuta's textbook [12] as a pertinent reference to Kantorovich inequalities.

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In this note, we show reverse inequalities to Araki's inequality (3) and the Cordes inequality (1): If A and B are positive operators such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then the following inequalities hold

$$K(m,M,p)\|BAB\|^p \leq \|B^pA^pB^p\| \quad \text{for all } 0<p<1, \tag{6}$$

$$K(m^2,M^2,p)^{1/2}\|AB\|^p \leq \|A^pB^p\| \quad \text{for all } 0<p<1, \tag{7}$$

respectively. We moreover show that reverse inequalities (4), (6) and (7) are mutually equivalent.

2. Preliminary

Let A be a positive operator on a Hilbert space H and x a unit vector in H. Then it follows from Hölder-McCarthy inequality that

$$(Ax,x) \leq (A^px,x)^{\frac{1}{p}} \quad \text{for all } p>1. \tag{8}$$

By using the Mond-Pečarić method [14, 15], we have the following reverse inequality of (8) [17, 6]:

Lemma 1. *If A is a positive operator on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then for each $\alpha > 0$*

$$(A^px,x)^{\frac{1}{p}} \leq \alpha(Ax,x) + \beta(m,M,p,\alpha) \quad \textit{for all } p>1 \tag{9}$$

holds for every unit vector $x \in H$, where

(10)

$$\beta(m,M,p,\alpha) = \begin{cases} \frac{p-1}{p}\left(\frac{M^p-m^p}{\alpha p(M-m)}\right)^{\frac{1}{p-1}} + \frac{\alpha(Mm^p-mM^p)}{M^p-m^p} & \textit{if } \frac{M^p-m^p}{pM^{p-1}(M-m)} \leq \alpha \leq \frac{M^p-m^p}{pm^{p-1}(M-m)}, \\ (1-\alpha)M & \textit{if } 0<\alpha \leq \frac{M^p-m^p}{pM^{p-1}(M-m)}, \\ (1-\alpha)m & \textit{if } \alpha \geq \frac{M^p-m^p}{pm^{p-1}(M-m)}. \end{cases}$$

Proof. For the sake of reader's convenience, we give a proof. Put $\beta = \beta(m,M,p,\alpha)$ and

$$f(t) = (at+b)^{\frac{1}{p}} - \alpha t \quad \text{for} \quad a = \frac{M^p-m^p}{M-m} \quad \text{and} \quad b = \frac{Mm^p-mM^p}{M-m}.$$

Then it follows that

$$f'(t) = \frac{a}{p}(at+b)^{\frac{1}{p}-1} - \alpha$$

and the equation $f'(t)=0$ has exactly one solution

$$t_0 = \frac{1}{a}\left(\frac{\alpha p}{a}\right)^{\frac{p}{1-p}} - \frac{b}{a}.$$

If $m \leq t_0 \leq M$, then we have $\beta = \max_{m\leq t\leq M} f(t) = f(t_0)$ since

$$f''(t) = \frac{a^2(1-p)}{p^2}(at+b)^{\frac{1}{p}-2} < 0$$

and the condition $m \leq t_0 \leq M$ is equivalent to the condition

$$\frac{M^p-m^p}{pM^{p-1}(M-m)} \leq \alpha \leq \frac{M^p-m^p}{pm^{p-1}(M-m)}.$$

If $M \le t_0$, then $f(t)$ is increasing on $[m, M]$ and hence we have $\beta = \max_{m\le t\le M} f(t) = f(M) = (1-\alpha)M$. Similarly, we have $\beta = \max_{m\le t\le M} f(t) = f(m) = (1-\alpha)m$ if $t_0 \le m$. Hence it follows that

$$(at+b)^{\frac{1}{p}} - \alpha t \le \beta \qquad \text{for all } t \in [m, M].$$

Since t^p is convex for $p > 1$, it follows that $t^p \le at + b$ for $t \in [m, M]$. By the spectral theorem, we have $A^p \le aA + b$ and hence $(A^p x, x) \le a(Ax, x) + b$ for every unit vector $x \in H$. Therefore we have

$$\begin{aligned}(A^p x, x)^{\frac{1}{p}} - \alpha(Ax, x) &\le (a(Ax, x) + b)^{\frac{1}{p}} - \alpha(Ax, x) \\ &\le \max_{m\le t\le M} f(t) = \beta(m, M, p, \alpha).\end{aligned}$$

□

As a complementary result, we state the following lemma.

Lemma 2. *If A is a positive operator on H such that $0 < mI \le A \le MI$ for some scalars $m < M$, then for each $\alpha > 0$*

(11) $$(A^p x, x)^{\frac{1}{p}} \ge \alpha(Ax, x) + \overline{\beta}(m, M, p, \alpha) \qquad \text{for all } 0 < p < 1$$

holds for every unit vector $x \in H$, where

(12)

$$\overline{\beta}(m, M, p, \alpha) = \begin{cases} \frac{p-1}{p}\left(\frac{M^p - m^p}{\alpha p(M-m)}\right)^{\frac{1}{p-1}} + \frac{\alpha(Mm^p - mM^p)}{M^p - m^p} & \text{if } \frac{M^p - m^p}{pm^{p-1}(M-m)} \le \alpha \le \frac{M^p - m^p}{pM^{p-1}(M-m)}, \\ (1-\alpha)M & \text{if } \alpha \ge \frac{M^p - m^p}{pM^{p-1}(M-m)}, \\ (1-\alpha)m & \text{if } 0 < \alpha \le \frac{M^p - m^p}{pm^{p-1}(M-m)}. \end{cases}$$

By Lemmas 1 and 2, we have the following estimates of both the difference and the ratio in the inequality (8).

Lemma 3. *If A is a positive operator on H such that $0 < mI \le A \le MI$ for some scalars $m < M$, then*

(13) $$(A^p x, x)^{\frac{1}{p}} \le K(m, M, p)^{\frac{1}{p}}(Ax, x) \qquad \text{for all } p > 1$$

and

(14) $$K(m, M, p)^{\frac{1}{p}}(Ax, x) \le (A^p x, x)^{\frac{1}{p}} \qquad \text{for all } 0 < p < 1$$

hold for every unit vector $x \in H$, where a generalized Kantorovich constant $K(m, M, p)$ is defined as (5) in §1.

Proof. For $p > 1$, if we put $\beta(m, M, p, \alpha) = 0$ in Lemma 1, then it follows that

$$\frac{p-1}{p}\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{1}{p-1}} + \alpha^{\frac{p}{p-1}} \frac{(Mm^p - mM^p)}{M^p - m^p} = 0$$

and hence

$$\alpha^{\frac{p}{p-1}} = -\frac{p-1}{p}\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{1}{p-1}} \frac{M^p - m^p}{Mm^p - mM^p}.$$

Therefore, we have

$$\begin{aligned}\alpha^p &= \frac{M^p - m^p}{p(M-m)}\left(\frac{p-1}{p}\frac{M^p - m^p}{mM^p - Mm^p}\right)^{p-1} \\ &= K(m, M, p)\end{aligned}$$

and we obtain the desired inequality (13). We similarly have the inequality (14) by Lemma 2. □

We remark that $K(m, M, 2)$ coincides with the Kantorovich constant $\frac{(M+m)^2}{4Mm}$.

Lemma 4. *If A is a positive operator on H such that $0 < mI \le A \le MI$ for some scalars $m < M$, then*

(15) $$(A^p x, x)^{\frac{1}{p}} - (Ax, x) \le -C(m^p, M^p, \frac{1}{p}) \qquad \text{for all } p > 1$$

and

(16) $$-C(m^p, M^p, \frac{1}{p}) \le (A^p x, x)^{\frac{1}{p}} - (Ax, x) \qquad \text{for all } 0 < p < 1$$

hold for every unit vector $x \in H$, where the constant $C(m, M, p)$ [14, 18] is defined as

(17) $$C(m, M, p) = (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M-m}.$$

Proof. For $p > 1$, if we put $\alpha = 1$ in Lemma 1, then it follows that

$$\begin{aligned}-C(m^p, M^p, \frac{1}{p}, 1) &= (1 - \frac{1}{p})\left(\frac{M-m}{\frac{1}{p}(M^p - m^p)}\right)^{\frac{\frac{1}{p}}{\frac{1}{p}-1}} - \frac{M^p m - m^p M}{M-m} \\ &= \beta(m, M, p, 1)\end{aligned}$$

and we obtain the desired inequality (15). We similarly have the inequality (16) by Lemma 2. □

We summarize some important properties of a generalized Kantorovich constant [5, 13, 15].

Lemma 5. *Let $m < M$ be given. Then a generalized Kantorovich constant $K(m, M, p)$ has the following properties.*

(i) *$K(m, M, p) = K(M, m, p)$ for all $p \in \mathbb{R}$.*
(ii) *$K(m, M, p) = K(m, M, 1-p)$ for all $p \in \mathbb{R}$.*
(iii) *$K(m, M, 0) = K(m, M, 1) = 1$ for all $p \in \mathbb{R}$.*
(iv) *$K(m, M, p)$ is increasing for $p > \frac{1}{2}$ and decreasing for $p < \frac{1}{2}$.*
(v) *$K(m^r, M^r, \frac{p}{r})^{\frac{1}{p}} = K(m^p, M^p, \frac{r}{p})^{-\frac{1}{r}}$ for $pr \ne 0$.*
In particular, $K(m, M, p) = K(m^p, M^p, \frac{1}{p})^{-p}$ for $p \ne 0$.

3. Reverse inequality of Araki, Cordes and Löwner-Heinz inequalities

First of all, we show the following reverse inequality to Araki's inequality (3).

Theorem 6. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then for each $\alpha > 0$*

$$||BAB||^p \leq \alpha ||B^pA^pB^p|| + \beta(m^p, M^p, \frac{1}{p}, \alpha)||B||^{2p} \qquad \text{for all } 0 < p < 1, \tag{18}$$

or equivalently

$$||B^pA^pB^p||^{\frac{1}{p}} \leq \alpha \; ||BAB|| + \beta(m, M, p, \alpha)||B||^2 \qquad \text{for all } p > 1, \tag{19}$$

where $\beta(m, M, p, \alpha)$ is defined as (10).

Proof. For every unit vector $x \in H$, it follows that

$$\begin{aligned}
& ((BAB)^px, x) \\
\leq\ & (BABx, x)^p \quad \text{by Hölder-McCarthy inequality and } 0 < p < 1 \\
=\ & \left((A^p)^{\frac{1}{p}}\frac{Bx}{||Bx||}, \frac{Bx}{||Bx||}\right)^p ||Bx||^{2p} \\
\leq\ & \left(\alpha(A^p\frac{Bx}{||Bx||}, \frac{Bx}{||Bx||}) + \beta(m^p, M^p, \frac{1}{p}, \alpha)\right) ||Bx||^{2p} \quad \text{by Lemma 1 and } \tfrac{1}{p} > 1 \\
=\ & \alpha(A^pBx, Bx)||Bx||^{2p-2} + \beta(m^p, M^p, \frac{1}{p}, \alpha)||Bx||^{2p} \\
=\ & \alpha\left(B^pA^pB^p\frac{B^{1-p}x}{||B^{1-p}x||}, \frac{B^{1-p}x}{||B^{1-p}x||}\right)||Bx||^{2p-2}||B^{1-p}x||^2 + \beta(m^p, M^p, \frac{1}{p}, \alpha)||Bx||^{2p}
\end{aligned}$$

and

$$\begin{aligned}
||Bx||^{2p-2}||B^{1-p}x||^2 &= (B^2x, x)^{p-1}(B^{2-2p}x, x) \\
&\leq (B^2x, x)^{p-1}(B^2x, x)^{1-p} = 1 \quad \text{by } 0 < 1 - p < 1.
\end{aligned}$$

By combining two inequalities above, we have

$$\begin{aligned}
||BAB||^p &= ||(BAB)^p|| \\
&\leq \alpha||B^pA^pB^p|| + \beta(m^p, M^p, \frac{1}{p}, \alpha)||B||^{2p}
\end{aligned}$$

and hence we have the desired inequality (18).

Next, we show (18)$\Longrightarrow$(19). For $p > 1$, since $0 < \frac{1}{p} < 1$, it follows from (18) that

$$||BAB||^{\frac{1}{p}} \leq \alpha||B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}|| + \beta(m^{\frac{1}{p}}, M^{\frac{1}{p}}, p, \alpha)||B||^{\frac{2}{p}}.$$

By replacing A by A^p and B by B^p in the above inequality respectively, we have

$$||B^pA^pB^p||^{\frac{1}{p}} \leq \alpha||BAB|| + \beta(m, M, p, \alpha)||B^p||^{\frac{2}{p}},$$

and so we have the desired inequality (19). Similarly we can show (19)$\Longrightarrow$(18). Therefore (18) is equivalent to (19). □

Remark 7. *Bourin [3, 4] showed the following result: If A and Z are positive operators such that $0 < mI \le Z \le MI$ for some scalars $m < M$, then*

$$\|ZA\| \le \frac{M+m}{2\sqrt{Mm}} r(ZA),$$

where $r(\cdot)$ is the spectral radius. J.I.Fujii, M.Tominaga and one of the authors [6] exteneded the result above as follows: Under the same assumption, for each $\alpha > 0$

$$\text{(20)} \qquad \|(AZ^pA)^{\frac{1}{p}}\| \le \alpha\, r(ZA^{\frac{2}{p}}) + \beta(m, M, p, \alpha)\|A\|^{\frac{2}{p}} \qquad \textit{for all } p > 1,$$

where $\beta(m, M, p, \alpha)$ is defined by (10). Then it easily follows that Theorem 6 is equivalent to (20).

As a complementary result, we state the following theorem.

Theorem 8. *If A and B are positive operators on H such that $0 < mI \le A \le MI$ for some scalars $m < M$, then for each $\alpha > 0$*

$$\text{(21)} \qquad \|BAB\|^p \ge \alpha\|B^pA^pB^p\| + \overline{\beta}(m^p, M^p, \frac{1}{p}, \alpha)\|B^{-1}\|^{-2p} \qquad \textit{for all } p > 1,$$

or equivalently

$$\text{(22)} \qquad \|B^pA^pB^p\|^{\frac{1}{p}} \ge \alpha\|BAB\| + \overline{\beta}(m, M, p, \alpha)\|B^{-1}\|^{-2} \qquad \textit{for all } 0 < p < 1,$$

where $\overline{\beta}(m, M, p, \alpha)$ is defined as (12).

Proof. By a similar way in Theorem 6, for every unit vector $x \in H$ we have

$$\begin{aligned} \|Bx\|^{2p-2}\|B^{1-p}x\|^2 &= (B^2x, x)^{p-1}(B^{2-2p}x, x) \\ &\ge (B^2x, x)^{p-1}(B^2x, x)^{1-p} = 1 \quad \text{by } 1 - p < 0 \end{aligned}$$

and

$$\begin{aligned} &((BAB)^px, x) \\ &\ge \alpha\left(B^pA^pB^p\frac{B^{1-p}x}{\|B^{1-p}x\|}, \frac{B^{1-p}x}{\|B^{1-p}x\|}\right)\|Bx\|^{2p-2}\|B^{1-p}x\|^2 + \overline{\beta}(m^p, M^p, \frac{1}{p}, \alpha)\|Bx\|^{2p} \\ &\ge \alpha\left(B^pA^pB^p\frac{B^{1-p}x}{\|B^{1-p}x\|}, \frac{B^{1-p}x}{\|B^{1-p}x\|}\right) + \overline{\beta}(m^p, M^p, \frac{1}{p}, \alpha)\|Bx\|^{2p}. \end{aligned}$$

By a suitable unit vector $x \in H$, it follows that

$$\begin{aligned} \|BAB\|^p &= \|(BAB)^p\| \\ &\ge \alpha\|B^pA^pB^p\| + \overline{\beta}(m^p, M^p, \frac{1}{p}, \alpha)\|Bx\|^{2p}. \end{aligned}$$

Since $(B^2x, x) \ge \|B^{-2}\|^{-1}(x, x)$ and $p > 1$, we have

$$\|Bx\|^{2p} \ge \|B^{-2}\|^{-p}$$

and hence we have the desired inequality (21). We can show (21) $\Longleftrightarrow$ (22) by a similar proof as in Theorem 6. □

If we choose α such that $\beta = 0$ (resp. $\overline{\beta} = 0$) in Theorem 6 (resp. Theorem 8), then we have the following ratio type reverse inequality to Araki's inequality.

Corollary 9. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$\|B^pA^pB^p\| \leq K(m,M,p)\ \|BAB\|^p \qquad \text{for all } p > 1 \tag{23}$$

and

$$K(m,M,p)\|BAB\|^p \leq \|B^pA^pB^p\| \qquad \text{for all } 0 < p < 1, \tag{24}$$

where $K(m,M,p)$ is defined as (5) in §1.

In particular,

$$\|B^2A^2B^2\| \leq \frac{(M+m)^2}{4Mm}\|BAB\|^2 \tag{25}$$

and

$$\frac{2\sqrt[4]{Mm}}{\sqrt{M}+\sqrt{m}}\|BAB\|^{\frac{1}{2}} \leq \|B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\|. \tag{26}$$

Proof. For $p > 1$, if we choose α such that $\beta(m,M,p,\alpha) = 0$ in Theorem 6, then it follows that $\alpha^p = K(m,M,p)$ and so we have the desired inequality (23). Similarly we have the inequality (24) by Theorem 8. We have (24) (resp. (25)) if we put $p = 2$ in (23) (resp. $p = 1/2$ in (24). □

If we put $\alpha = 1$ in Theorem 6 and Theorem 8, then we have the following difference type reverse inequality to Araki's inequality.

Corollary 10. *If A and B are positive operators on H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then*

$$\|B^pA^pB^p\| - \|BAB\|^p \leq C(m,M,p)\|B^{-1}\|^{-2p} \qquad \text{for all } p > 1, \tag{27}$$

and

$$\|BAB\|^p - \|B^pA^pB^p\| \leq -C(m,M,p)\|B\|^{2p} \qquad \text{for all } 0 < p < 1, \tag{28}$$

where $C(m,M,p)$ is defined as (17).

In particular,

$$\|B^2A^2B^2\| - \|BAB\|^2 \leq \frac{(M-m)^2}{4}\|B^{-1}\|^{-4} \tag{29}$$

and

$$\|BAB\|^{\frac{1}{2}} - \|B^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq \frac{(\sqrt{M}-\sqrt{m})^2}{4(\sqrt{M}+\sqrt{m})}\|B\|. \tag{30}$$

Proof. For $p > 1$, if we put $\alpha = 1$ in Theorem 8, then it follows that $\beta(m^p, M^p, \frac{1}{p}, 1) = -C(m,M,p)$ and so we have the desired inequality (27). Similarly we have the inequality (28) by Theorem 6. We have (29) (resp. (30)) if we put $p = 2$ in (27) (resp. $p = 1/2$ in (28). □

Moreover, we obtain the following reverse inequality to the Cordes inequality by Corollary 9.

Theorem 11. *If A and B are positive operators on H such that $0 < mI \le A \le MI$ for some scalars $m < M$, then*

$$\|A^pB^p\| \le K(m^2, M^2, p)^{\frac{1}{2}}\ \|AB\|^p \quad \textit{for all } p > 1 \tag{31}$$

or equivalently

$$K(m^2, M^2, p)^{\frac{1}{2}}\|AB\|^p \le \|A^pB^p\| \qquad \textit{for all } 0 < p < 1. \tag{32}$$

In particular,

$$\|A^2B^2\| \le \frac{M^2+m^2}{2Mm}\|AB\|^2 \tag{33}$$

and

$$\sqrt{\frac{2\sqrt{Mm}}{M+m}}\|AB\|^{\frac{1}{2}} \le \|A^{\frac{1}{2}}B^{\frac{1}{2}}\|. \tag{34}$$

Proof. For a given $p > 1$, it follows from Corollary 9 that

$$\|B^pA^pB^p\| \le K(m, M, p)\|BAB\|^p$$

and hence

$$\|A^{\frac{p}{2}}B^p\|^2 \le K(m, M, p)\|A^{\frac{1}{2}}B\|^{2p}.$$

If we replace A by A^2, then we have

$$\|A^pB^p\|^2 \le K(m^2, M^2, p)\|AB\|^{2p}$$

as desired. □

The equivalence among the reverse inequalities of Araki, Cordes and Löwner-Heinz inequalities is now given as follows.

Theorem 12. *For a given $p > 1$, the following are mutually equivalent: For all positive operators A, B such that $0 < mI \le A \le MI$ for some scalars $m < M$*

(A) $\quad A \ge B \ge 0$ *implies* $K(m, M, p)A^p \ge B^p$.

(B) $\quad \|A^pB^p\| \le K(m^2, M^2, p)^{1/2}\|AB\|^p$.

(C) $\quad \|B^pA^pB^p\| \le K(m, M, p)\|BAB\|^p$.

(B') $\quad K(m^2, M^2, 1/p)^{1/2}\|AB\|^{\frac{1}{p}} \le \|A^{\frac{1}{p}}B^{\frac{1}{p}}\|$.

(C') $\quad K(m, M, 1/p)\|BAB\|^{\frac{1}{p}} \le \|B^{\frac{1}{p}}A^{\frac{1}{p}}B^{\frac{1}{p}}\|$.

Proof. The proof is divided into three parts, namely the equivalence $(A) \Longrightarrow (B) \Longrightarrow (C) \Longrightarrow (A)$, $(B) \Longleftrightarrow (B')$ and $(C) \Longleftrightarrow (C')$.

$(A) \Longrightarrow (B)$. It follows that

$$\begin{aligned}(A) &\Longleftrightarrow \|A^{-\frac{1}{2}}B^{\frac{1}{2}}\| \le 1 \to \|A^{-\frac{p}{2}}B^{\frac{p}{2}}\|^2 \le K(m, M, p)\\ &\Longleftrightarrow \|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \le 1 \to \|A^{\frac{p}{2}}B^{\frac{p}{2}}\|^2 \le K(M^{-1}, m^{-1}, p) = K(m, M, p)\\ &\Longleftrightarrow \|AB\| \le 1 \to \|A^pB^p\| \le K(m^2, M^2, p).\end{aligned}$$

If we put $B_1 = B/\|AB\|$, then it follows from $\|AB_1\| = 1$ that

$$\|A^pB_1^p\| \le K(m^2, M^2, p)^{\frac{1}{2}} \Longleftrightarrow \|A^pB^p\| \le K(m^2, M^2, p)^{\frac{1}{2}}\|AB\|^p.$$

$(B) \Longrightarrow (C)$. If we replace A by $A^{\frac{1}{2}}$ in (B), then it follows that

$$\|A^{\frac{p}{2}}B^p\| \le K(m,M,p)^{\frac{1}{2}}\|A^{\frac{1}{2}}B\|^p.$$

Squaring both sides, we have

$$\|B^pA^pB^p\| \le K(m,M,p)\|BAB\|^p.$$

$(C) \Longrightarrow (A)$. If we replace B by $B^{\frac{1}{2}}$ and A by A^{-1} in (C), then it follows that

$$\|B^{\frac{p}{2}}A^{-p}B^{\frac{p}{2}}\| \le K(M^{-1},m^{-1},p)\|B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\|^p.$$

By rearranging it, we have

$$\|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \le K(m,M,p)\|A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\|^p.$$

Since $A \ge B \ge 0$, it follows from $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le 1$ that

$$\|A^{-\frac{p}{2}}B^pA^{-\frac{p}{2}}\| \le K(m,M,p)$$

and hence

$$B^p \le K(m,M,p)A^p.$$

$(B) \Longleftrightarrow (B')$: If we replace A and B by $A^{\frac{1}{p}}$ and $B^{\frac{1}{p}}$ in (B) respectively, then it follows that

$$\begin{aligned}(B) &\Longleftrightarrow \|AB\| \le K(m^{\frac{2}{p}},M^{\frac{2}{p}},p)^{\frac{1}{2}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\|^p \\ &\Longleftrightarrow \|AB\|^{\frac{1}{p}} \le K(m^{\frac{2}{p}},M^{\frac{2}{p}},p)^{\frac{1}{2p}}\|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \\ &\Longleftrightarrow K(m^2,M^2,1/p)^{\frac{1}{2}}\|AB\|^{\frac{1}{p}} \le \|A^{\frac{1}{p}}B^{\frac{1}{p}}\| \quad \text{by (v) in Lemma 5} \\ &\Longleftrightarrow (B')\end{aligned}$$

Similarly we have $(C) \Longleftrightarrow (C')$ and so the proof is complete. □

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