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THE FIRST EIGENVALUE $\lambda_{1,p}$ OF THE p-LAPLACE OPERATOR

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ABSTRACT. In this paper, we give an estimate of the first eigenvalue $\lambda_{1,p}$ of the p-Laplace operator associated to a Riemannian manifold M^m. Precisely, we show that for $p \geq 2$

$$\lambda_{1,p} \geq \left(\frac{(m-1)k}{p-1-\frac{1}{(p-2+\sqrt{m})^2}} \right)^{p/2}$$

provided that the Ricci curvature of M is no less than $(m-1)k$ where k is a positive constant. The estimate improves a recent result by A.M.Matei and is equal to the optimal result when $p=2$.

1. INTRODUCTION AND THE STATE OF THE RESULT

Let (M,g) be an m-dimensional connected compact Riemannian manifold without boundary. The first eigenvalue of the Laplace-Beltrami operator on M has been extensively studied in mathematical literature. Many connections between this invariant and other geometrical quantities have been pointed out. Recently, there has been an increasing interest for the $p-$Laplacian operator Δ_p defined by

$$\Delta_p f := -\mathrm{div}(|df|^{p-2} df), \qquad p > 1.$$

See [1]-[8],[10]-[12]. An eigenfunction of Δ_p is a nonzero function f such that there exists a real number λ satisfying

$$\Delta_p f = \lambda |f|^{p-2} f.$$

The real number λ is then called an eigenvalue of Δ_p on M. Obviously, 0 is an eigenvalue associated with the constant eigenfunctions. The set $\sigma_p(M)$ of the remaining eigenvalues is a nonempty, unbounded subset of $(0,\infty)$ [5]. Its infimum $\lambda_{1,p}(M) = inf \sigma_p(M)$ itself is a positive eigenvalue and we have the following variational characterization [14]

$$\lambda_{1,p}(M) = inf \left\{ \frac{\int |df|^p}{\int |f|^p}; \quad 0 \not\equiv f \in W^{1,p}(M), \quad \int |f|^{p-2} f = 0 \right\}, \tag{1.1}$$

where, and throughout this paper, the integration is over M with the standard volume element induced by the Riemannian metric. So finding first nonzero eigenvalue is related to the problem of finding the best constant in the inequality

$$|f|_{L^p} \leq C |df|_{L^p}$$

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obtained by the continuous embedding $W^{1,p}(M) \to L^p(M)$. For $p = 2$, we have the following well-known theorem of Lichnerowicz-Obata [9][13].

Theorem 1.1. *Let M be a m-dimensional connected compact Riemannian manifold. Suppose there exists positive constant k such that $Ric^M \geq (m-1)k$. Then*

$$\lambda_{1,2}(M) \geq \lambda_{1,2}(S_k^m) = mk. \tag{1.2}$$

Equality holds if and only if M is isometric to S_k^m.

For $p \geq 2$, a low bound of $\lambda_{1,p}$ was obtained in [11] (c.f. Theorem 3.2) as follows

Theorem 1.2. *With the same notation and assumptions as in the above theorem, then*

$$\lambda_{1,p}(M) \geq \left(\frac{(m-1)k}{p-1}\right)^{p/2}, \qquad p \geq 2. \tag{1.3}$$

The estimation (1.3) is clearly not optimal if we compare (1.3) with (1.2) for $p = 2$. The purpose of this article is to improve the estimation (1.3) of the first eigenvalue $\lambda_{1,p}(M)$ and we have

Theorem 1.3. *Let M be a m-dimensional compact Riemannian manifold without boundary. Suppose $Ric^M \geq (m-1)k > 0$. Then*

$$\lambda_{1,p}(M) \geq \left(\frac{(m-1)k}{p-1-\frac{1}{(p-2+\sqrt{m})^2}}\right)^{p/2}, \qquad p \geq 2. \tag{1.4}$$

Remark 1.4. For $p > 2$, one does not even know the exact value $\lambda_{1,p}(S^m)$ for the standard sphere. Our estimation (1.4) is reduced to (1.2) for the usual Laplacian.

2. The Proof of the Theorem 1.3

We start the proof with a lemma.

Lemma 2.1. *Let M^m be a compact Riemannian manifold. For $f \in C^\infty(M)$, we have*

$$|Hessf||df|^{p-2} \geq \frac{1}{p-2+\sqrt{m}}|\Delta_p f|. \tag{2.1}$$

Proof. The inequality (2.1) is well-known for $p = 2$. We only consider the case $p > 2$. For $f \in C^\infty(M)$, we have

$$\Delta_p f = |df|^{p-2}\Delta f - (p-2)|df|^{p-4}(\mathrm{Hess}f)(\nabla f, \nabla f). \tag{2.2}$$

For any constant $r \in R$, set $s = (p-2)^{\frac{1}{2}}m^{-1/4}$ and $t = (p-2)r/s$, we have

$$\begin{aligned} &|Hessf + r|df|^{p-2}g|^2 \\ &= |Hessf|^2 + 2r|df|^{p-2}\Delta f + r^2 m|df|^{2p-4}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &|s \cdot Hessf - t|df|^{p-4}df \otimes df|^2 \\ &= s^2|Hessf|^2 - 2st|df|^{p-4}Hessf(\nabla f, \nabla f) + t^2|df|^{2p-4}. \end{aligned} \tag{2.4}$$

Summing up (2.3) and (2.4), using (2.2), we get for any $r \in R$

$$(1+s^2)|Hessf|^2 + 2r\Delta_p f + r^2\left(m + \frac{(p-2)^2}{s^2}\right)|df|^{2p-4} \geq 0.$$

So the discriminant of the left hand side is non positive, which implies the lemma. $\square$

Following the arguments as in [11], we are now in the position to prove the theorem 1.3.

We need only consider the cases for $p > 2$. Obviously, the infimum (1.1) does not change when we replace $W^{1,p}(M)$ by $C^\infty(M)$. For any $f \in C^\infty(M)$, since $\delta = -\mathrm{div}$ is conjugate to the exterior differential operator d, we have

$$\begin{aligned}\int \Delta_p f \Delta f &= \int \delta(|df|^{p-2}df)\Delta f = \int |df|^{p-2}(df, d\Delta f)\\ &= \int |df|^{p-2}(df, \Delta df).\end{aligned}$$

By the Bochner's formula

$$\langle df, \Delta df\rangle = |Hessf|^2 + \frac{1}{2}\Delta(|df|^2) + Ric^M(df, df).$$

We have

$$\begin{aligned}\int \Delta_p f \Delta f &= \int |df|^{p-2}|Hessf|^2\\ &+ \frac{1}{2}\int |df|^{p-2}\Delta(|df|^2) + \int |df|^{p-2}Ric^M(df, df).\end{aligned} \tag{2.5}$$

Now,

$$\begin{aligned}\int |df|^{p-2}\Delta(|df|^2) &= \int \langle d(|df|^{p-2}), d(|df|^2)\rangle\\ &= 2(p-2)\int |df|^{p-2}|d(|df|)|^2 \geq 0.\end{aligned} \tag{2.6}$$

From the Young inequality, we have for $\forall \varepsilon > 0$,

$$|f|^2|df|^{p-2} \leq \frac{2}{p}\varepsilon^{4-2p}|f|^p + \frac{p-2}{p}\varepsilon^4|df|^p. \tag{2.7}$$

We have

$$|Hessf|^2|df|^{p-2} \geq 2\eta|Hessf||df|^{p-2}|f| - \eta^2|df|^{p-2}|f|^2. \tag{2.8}$$

By (2.7),and (2.8) and the lemma 2.1, we get

$$\begin{aligned}&|Hessf|^2|df|^{p-2}\\ &\geq \frac{2\eta}{p-2+\sqrt{m}}|\Delta_p f||f| - \frac{2\eta^2\varepsilon^{4-2p}}{p}|f|^p - \frac{\eta^2(p-2)\varepsilon^4}{p}|df|^p.\end{aligned} \tag{2.9}$$

Applying (2.6),(2.9) and the Ricci curvature assumption to the equality (2.5), we have

$$\begin{aligned}\int \Delta_p f \Delta f &\geq \frac{2\eta}{p-2+\sqrt{m}}\int |\Delta_p f||f| - \frac{2\eta^2\varepsilon^{4-2p}}{p}\int |f|^p\\ &+ \left((m-1)k - \frac{\eta^2(p-2)\varepsilon^4}{p}\right)\int |df|^p.\end{aligned} \tag{2.10}$$

It is clear that inequality (2.10) also holds for an eigenfunction f corresponding to the first eigenvalue $\lambda_{1,p}$ of $\triangle_p$. On the other hand, when f is such an eigenfunction, we have

$$\begin{aligned}\lambda_{1,p}\int |f|^p &= \int f\triangle_p f = \int f\delta(|df|^{p-2}df)\\ &= \int |df|^{p-2}\langle df, df\rangle = \int |df|^p,\end{aligned} \tag{2.11}$$

and

$$\int |\triangle_p f||f| = \lambda_{1,p}\int |f|^p = \int |df|^p. \tag{2.12}$$

Also, we have

$$\begin{aligned}\int \triangle_p f\triangle f &= \lambda_{1,p}\int |f|^{p-2}f\triangle f = \lambda_{1,p}\int \langle d(|f|^{p-2}f), df\rangle\\ &= \lambda_{1,p}\int (p-2)|df|^{p-3}\langle d|f|, \frac{1}{2}d|f|^2\rangle + |f|^{p-2}|df|^2\\ &= (p-1)\lambda_{1,p}\int |f|^{p-2}|df|^2.\end{aligned}$$

So the Hölder inequality implies

$$\int \triangle_p f\triangle f \le (p-1)\lambda_{1,p}\left(\int |f|^p\right)^{1-\frac{2}{p}}\left(\int |df|^p\right)^{\frac{2}{p}}. \tag{2.13}$$

Using (2.13),(2.12) (2.11), we have by (2.10),

$$(p-1)\lambda_{1,p}^{\frac{2}{p}} \ge (m-1)k + \frac{2\eta}{p-2+\sqrt{m}} - \eta^2\left(\frac{2\varepsilon^{4-2p}}{p\lambda_{1,p}} + \frac{(p-2)\varepsilon^4}{p}\right).$$

Now the theorem follows from the above inequality if we set $\varepsilon = \lambda_{1,p}^{-\frac{1}{2p}}$ and $\eta = \frac{1}{p-2+\sqrt{m}}\lambda_{1,p}^{\frac{2}{p}}$.

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