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On p-quasihyponormal operators

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Abstract

For a p-quasihyponormal operator T with the polar decomposition $T = U|T|$, we show that $T_p = U|T|^p$ is quasihyponormal with spectrum $\sigma(T_p) = \{r^p e^{i\theta} : e^{i\theta} \in \sigma(T)\}$. From this, we obtain the following Putnam type inequality for a p-quasihyponormal operator T

$$\||T|^{2p} - |T^*|^{2p}\| \leq 2\|T\|^p (\frac{p}{\pi} \iint_{re^{i\theta}\in\sigma(T)} r^{2p-1}\, dr d\theta)^{\frac{1}{2}}.$$

These results are parallel with Xia, Aluthge and Chō-Itoh's results for p-hyponormal operators. Also we show that the Riesz idempotent E for T with respect to an isolated point λ of the spectrum $\sigma(T)$ satisfies $\text{ran}E = \ker(T-\lambda)$, moreover, if $\lambda \neq 0$ then E is self-adjoint and $\ker(T-\lambda) = \ker(T-\lambda)^*$.

1. Introductions

Studying p-hyponormal operators, i.e., operators T on a (separable) complex Hilbert space $\mathcal{H}$ such that $(T^*T)^p \geq (TT^*)^p$, for $0 < p < 1$ was first started by D. Xia [20], in that paper, he gave an example of semi-hyponormal operator but not hyponormal. Here we say that an operator T is hyponormal iff T is 1-hyponormal, semi-hyponormal iff T

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is $\frac{1}{2}$-hyponormal. After that, he have proved that many important results of hyponormal operators also hold for p-hyponormal operators for $p \geq \frac{1}{2}$. One of the famous results of them is to extend Putnam's inequality for hyponormal operators to the case of p-hyponormal operators for $p \geq \frac{1}{2}$ as follows:

Theorem (Xia [21]) Let T be a p-hyponormal operator for $p \geq \frac{1}{2}$ and $T = U|T|$ a polar decomposition of T. Then $U|T|^p$ is hyponormal with spectrum

$$\sigma(U|T|^p) = \{r^p e^{i\theta} : re^{i\theta} \in \sigma(T)\},$$

and hence

$$\||T|^{2p} - |T^*|^{2p}\| \leq \frac{1}{2\pi} \iint_{re^{i\theta} \in \sigma(T)} r^{2p-1}\, dr d\theta.$$

Above Xia's theorem was shown for all $p > 0$ by Aluthge. Also Xia and Aluthge extended Berger-Shaw's inequality to a class of operators which includes the class of p-hyponormal operators [2].

Aluthge's new method "Aluthge transform" is very important and necessarily to study p-hyponormal operators. The assertion is as follows:

Theorem(Aluthge[1]) For p-hyponormal operator $T = U|T|$, Aluthge transform $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is hyponormal if $p \geq \frac{1}{2}$, $(p + \frac{1}{2})$-hyponormal if $0 < p \leq \frac{1}{2}$.

By using previous Xia's lemma and above Aluthge's theorem, M. Chō-T. Huruya [5] and M. Chō-M. Itoh [6] also extended above Xia's theorem to the case $0 < p < \frac{1}{2}$.

Studying p-quasihyponormal operators, i.e., the operators T such that $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$ was first started by S. Arora-P. Arora [4]. By definition, if a p-quasihyponormal operator has dense range then it is p-hyponormal, so if we want to extend some results for p-hyponormal operators to the case of p-quasihyponormal operators we may assume p-quasihyponormal operator T does not have dense range, in particular, $0 \in \sigma(T)$. In this paper, we assume p-quasihyponormal operator T does not have dense range, because the results which we obtain have been already proven in the case of p-hyponormal operators. We remark some differences of properties between p-quasihyponormal operators and p-hyponormal operators. One is that p-hyponormality implies q-hyponormality for all $0 < q \leq p$, however, p-quasihyponormality does not imply q-quasihyponormality even if $0 < q < p$. In fact, there exists an example of p-quasihyponormal which is not

q-quasihyponormal for all $q > 0$ such that $q \neq p$. Also, it is not true that the Aluthge transform of p-quasihyponormal is q-quasihyponormal for some $q > 0$. There exists an example of p-quasihyponormal operator which Aluthge transform is no longer q-hyponormal for all $q > 0$. See [18]. That is, though Aluthge transform is very powerful to analyze p-hyponormal operators and also it is convenience because it does not change the spectrum of operators, but it does not work well to study p-quasihyponormal operators. However, we see that the deformed operator $T_p = U|T|^p$ of a p-quasihyponormal operator T is quasihyponormal(i.e., 1-quasihyponormal.) For quasihyponormal operators, there are many results have been obtained, e.g., Putnam type inequality, self-adjointness of Riesz idempotent with respect to non-zero isolated point of spectrum, Weyl's theorem(after we write definitions).

In this paper, we show the following results 1), 2), 3) and 4).

For a p-quasihyponormal operator T,

1) $T_p = U|T|^p$ is quasihyponormal with the spectrum

$$\{r^p e^{i\theta} : re^{i\theta} \in \sigma(T)\},$$

2) $\||T|^{2p} - |T^*|^{2p}\| \leq 2\|T\|^p(\frac{p}{\pi}\iint_{re^{i\theta}\in\sigma(T)} r^{2p-1}\,drd\theta)^{\frac{1}{2}}$,

3) for each isolated point of $\sigma(T)$, the Riesz idempotent E for T with respect to λ defined by

$$E = \frac{1}{2\pi i}\int_\gamma (z-T)^{-1}\,dz,$$

satisfies that $\mathrm{ran}E = \ker T$ if $\lambda = 0$, $\mathrm{ran}E = \ker(T-\lambda) = \ker(T-\lambda)^*$ and $E = E^*$ if $\lambda \neq 0$, where γ is a circle with center λ and small enough radius ϵ such that $\{z : |z-\lambda| \leq \epsilon\} \cap \sigma(T) = \{\lambda\}$,

4) Weyl's theorem holds for T, i.e.,

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

Here we denote the Weyl spectrum of T and the set of all eigenvalues of T with finite multiplicities such that each of which is an isolated point in $\sigma(T)$ by $w(T)$ and $\pi_{00}(T)$ respectively. For definitions of $w(T)$ and $\pi_{00}(T)$, we mention them in the next section.

2. Preliminaries

Definitions and Notations An operator T on $\mathcal{H}$ is called Fredholm if it has closed range and both $\ker T$ and Coker$T = \mathcal{H}/\text{ran}T$ are finite dimension, also is called semi-Fredholm if it has closed range and either $\ker T$ or Coker$T$ is finite dimension. For a semi-Fredholm operator T there corresponds an index ind(T) which called Fredholm index defined by

$$\text{ind}(T) = \dim\ker T - \dim \text{Coker}T = \dim\ker T - \dim\ker T^*.$$

It is well-known that ind$(\cdot)$ is a continuous mapping from the set of all semi-Fredholm operators to the discrete space $\mathbb{Z} \cup \{\pm\infty\}$. We denote the set of all Fredholm operators on $\mathcal{H}$ with Fredholm index 0 by $\mathcal{F}_0$.

$$\begin{aligned}
\sigma(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\} \quad : \text{spectrum of } T \\
\sigma_p(T) &= \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\} \quad : \text{point spectrum of } T \\
\sigma_a(T) &= \{\lambda \in \mathbb{C} : \exists\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}, \|x_n\| = 1 \text{ and } \|(T-\lambda)x_n\| \to 0 \text{ as } n \to \infty\} \\
&\quad : \text{approximate point spectrum of } T \\
w(T) &= \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{F}_0\} \quad : \text{Weyl spectrum} \\
\pi_{00}(T) &= \{\lambda \in \sigma_p(T) : \dim\ker(T-\lambda) < \infty \text{ and } \lambda \text{ is isolated in } \sigma(T)\}.
\end{aligned}$$

We say that $\lambda \in \mathbb{C}$ is a normal approximate eigenvalue of T if $\lambda \in \sigma_a(T)$(i.e., λ is an approximate eigenvalue of T) and for each sequence of unit vectors in $\mathcal{H}$ such that $\|(T-\lambda)x_n\| \to 0$ as $n \to \infty$,

$$\|(T-\lambda)^* x_n\| \to 0 \quad \text{as } n \to \infty.$$

Denote the set of all normal approximate eigenvalues of T by $\sigma_{na}(T)$.

Proposition 1.(Hölder-McCarcy inequality) For $A \geq 0$ and $x \in \mathcal{H}$,

$$\langle Ax, x\rangle \leq \|x\|^{2(1-\frac{1}{p})}\langle A^p x, x\rangle^{\frac{1}{p}} \quad \text{if } p \geq 1 \tag{1}$$

$$\langle Ax, x\rangle \geq \|x\|^{2(1-\frac{1}{p})}\langle A^p x, x\rangle^{\frac{1}{p}} \quad \text{if } 0 < p \leq 1. \tag{2}$$

For p-hyponormal operator T, $\sigma_a(T) = \sigma_{na}(T)$ was proven by D. Xia [20] and M. Chō-T. Huruya [5]. Though there exists p-quasihyponormal operator T such that $\ker T$

does not reduce T, see [13], we see that every eigenspace of a p-quasihyponormal operator T with respect to non-zero eigenvalue always reduces T. The next lemma contains this assertion.

Lemma 1. Let T be a p-quasihyponormal operator for some $p > 0$. Then $\sigma_a(T)\backslash\{0\} \subset \sigma_{na}(T)$. Hence the following (i) and (ii) hold.

(i) If $\|(T-\lambda)x_n\| \to 0$ with $\|x_n\| = 1$ and $\lambda = |\lambda|e^{i\theta} \neq 0$, then

$$\|(T-\lambda)^* x_n\| \to 0,$$

hence, $\|(|T| - |\lambda|)x_n\| \to 0$, $\|(U - e^{i\theta})x_n\| \to 0$ and $\|(U - e^{i\theta})^* x_n\| \to 0$, where $T = U|T|$ is the polar decomposition of T.

(ii) If $\lambda \neq 0$ is an eigenvalue of T, then $\ker(T-\lambda) \subset \ker(T-\lambda)^*$, hence $\ker(T-\lambda)$ reduces T.

Proof. First, we show the case $0 < p \leq 1$. In this case T is of the form

$$T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\mathrm{ran} T} \oplus \ker T^*,$$

such that $(A^*A)^p \geq (AA^* + SS^*)^p \geq (AA^*)^p$. In particular, A is p-hyponormal. See Uchiyama [16]. Recall that for any p-hyponormal operator B satisfies $\sigma_a(B) = \sigma_{na}(B)$.

If $\lambda \in \sigma_a(T) \setminus \{0\}$ and $\{x_n\}_{n=1}^{\infty}$ is an arbitrary sequence of unit vectors in $\mathcal{H}$ which satisfies

$$\|(T-\lambda)x_n\| \to 0 \quad \text{as } n \to \infty,$$

then $x_n = y_n \oplus z_n \in \mathcal{H} = \overline{\mathrm{ran} T} \oplus \ker T^*$ satisfies

$$z_n \to 0, \; \|y_n\| \to 1, \quad \text{and} \quad \|(A-\lambda)y_n\| \to 0,$$

as $n \to \infty$ Hence, $\|(A-\lambda)^* y_n\| \to 0$. From this, for each $t > 0$, we have

$$\|(|A|^t - |\lambda|^t)y_n\| \to 0 \quad \text{and} \quad \|(|A^*|^t - |\lambda|^t)y_n\| \to 0,$$

as $n \to \infty$. Since $(A^*A)^p \geq (AA^* + SS^*)^p \geq (AA^*)^p$ we also have $\|((AA^* + SS^*)^p - |\lambda|^{2p})y_n\| \to 0$ and therefore $\|((AA^* + SS^*) - |\lambda|^2)y_n\| \to 0$. This implies $\|S^* y_n\| \to 0$. Hence we have

$$\begin{aligned} \|(T-\lambda)^* x_n\| &= \|(A-\lambda)^* y_n \oplus (S^* y_n - \overline{\lambda} z_n)\| \\ &\leq \|(A-\lambda)^* y_n\| + \|S^* y_n\| + |\lambda|\|z_n\| \to 0. \end{aligned}$$

Next we show the case $p > 1$. Let $\{x_n\}$ be an arbitrary sequence of unit vectors in $\mathcal{H}$ such that

$$\|(T-\lambda)x_n\| \to 0 \quad \text{as } n \to \infty.$$

Then

$$\|T^*x_n\| \geq |\langle T^*x_n, x_n\rangle| = \langle x_n, Tx_n\rangle \to |\lambda|,$$

so we have

$$\liminf_{n\to\infty} \|T^*x_n\| \geq |\lambda|.$$

We shall show that $\limsup_{n\to\infty} \|T^*x_n\| \leq |\lambda|$. Without loss of generality, we may assume that $\|T^*x_n\| \geq \frac{|\lambda|}{2}$ for all n.

Since T is p-quasihyponormal,

$$0 \leq T^*\{(T^*T)^p - (TT^*)^p\}T.$$

Hence

$$\begin{aligned}
0 &\leq \langle T^*\{(T^*T)^p - (TT^*)^p\}Tx_n, x_n\rangle \\
&= \langle(|\lambda|^4(TT^*)^{p-1} - |\lambda|^2(TT^*)^p)x_n, x_n\rangle + O(\|(T-\lambda)x_n\|) \\
&= |\lambda|^4\langle((TT^*)^p)^{\frac{p-1}{p}}x_n, x_n\rangle - |\lambda|^2\langle(TT^*)^p x_n, x_n\rangle + O(\|(T-\lambda)x_n\|) \\
&\leq |\lambda|^4\langle(TT^*)^p x_n, x_n\rangle^{\frac{p-1}{p}} - |\lambda|^2\langle(TT^*)^p x_n, x_n\rangle + O(\|(T-\lambda)x_n\|) \quad \text{by (2)} \\
&= |\lambda|^2\||T^*|^p x_n\|^{\frac{2(p-1)}{p}}(|\lambda|^2 - \langle(TT^*)^p x_n, x_n\rangle^{\frac{1}{p}}) + O(\|(T-\lambda)x_n\|) \\
&\leq |\lambda|^2\||T^*|^p x_n\|^{\frac{2(p-1)}{p}}(|\lambda|^2 - \|T^*x_n\|^2) + O(\|(T-\lambda)x_n\|) \quad \text{by (1)}.
\end{aligned}$$

Since $\||T^*|^p x_n\|^{\frac{2(p-1)}{p}} \geq \|T^*x_n\|^{2(p-1)} \geq (\frac{|\lambda|}{2})^{2(p-1)}$, above inequalities shows that

$$0 \leq |\lambda|^2 - \|T^*x_n\|^2 + O(\|(T-\lambda)x_n\|).$$

Hence we have

$$\limsup_{n\to\infty} \|T^*x_n\| \leq |\lambda|^2,$$

and $\lim \|T^*x_n\| = |\lambda|$. The assertion of this lemma is immediately from this since

$$\begin{aligned}
\|(T-\lambda)^*x_n\|^2 &= \|T^*x_n\|^2 + |\lambda|^2 - \lambda\langle T^*x_n, x_n\rangle - \overline{\lambda}\langle x_n, T^*x_n\rangle \\
&= \|T^*x_n\|^2 + |\lambda|^2 - \lambda\langle x_n, Tx_n\rangle - \overline{\lambda}\langle Tx_n, x_n\rangle \\
&\to |\lambda|^2 + |\lambda|^2 - |\lambda|^2 - |\lambda|^2 = 0 \quad \text{as } n \to \infty.
\end{aligned}$$

The assertions of (i) and (ii) are immediate from the fact $\sigma_a(T) \setminus \{0\} \subset \sigma_{na}$.

Since we only consider the operator valued function $T(t) = U|T|^{1+t(q-1)}$ $(q > 0)$ for p-quasihyponormal operator T, we prepare the following an elementary lemma to be understood easily.

Lemma 2. Let $T : [0,1] \to \mathcal{B}(\mathcal{H})$ be a norm continuous mapping, i.e., $s_n, s \in [0,1]$ and $s_n \to s$ implies $\|T(s_n) - T(s)\| \to 0$. If $T(0)$ and $T(1)$ are semi-Fredholm operator such as $\mathrm{ind}(T(0)) \neq \mathrm{ind}(T(1))$. Then there exists an $s_0 \in (0,1)$ such that $T(s_0)$ is not semi-Fredholm, in particular $0 \in \sigma_a(T(s_0))$.

Proof. Assume that there is no such point s_0. Then $\{T(s) : s \in [0,1]\}$ is a connected subset of semi-Fredholm operators. Hence each of them has same Fredholm index because Fredholm index $\mathrm{ind}(\cdot)$ is a continuous function from {semi-Fredholm operators } to $\mathbb{Z} \cup \{\pm\infty\}$. This is a contradiction. Thus there exists an $s_0 \in (0,1)$ such that $T(s_0)$ is not semi-Fredholm.

If $0 \notin \sigma_a(T(s_0))$, then $T(s_0)$ is bounded below and hence it is a semi-Fredholm operator with $\mathrm{ind}(T(s_0)) \leq -1$, this contradicts the fact that $T(s_0)$ is not semi-Fredholm.

Now, we can prove 1), more generally, the following.

Lemma 3. Let $T = U|T|$ be p-quasihyponormal for some $p > 0$ and $q > 0$ be arbitrary. Then $T_q = U|T|^q$ is $\frac{p}{q}$-quasihyponormal with spectrum

$$\sigma(T_q) = \{r^q e^{i\theta} : re^{i\theta} \in \sigma(T)\}.$$

Proof. It is easy to see that an operator S is p-quasihyponormal if and only if

$$P\{(S^*S)^p - (SS^*)^p\}P \geq 0,$$

where P is the orthogonal projection onto $[\mathrm{ran}S](= \overline{\mathrm{ran}S})$.

Since $[\mathrm{ran}T] = [\mathrm{ran}T_q]$, $(T^*T)^p = (T_q{}^*T_q)^{\frac{p}{q}}$ and $(TT^*)^p = (T_qT_q{}^*)^{\frac{p}{q}}$, the operator T_q is $\frac{p}{q}$-quasihyponormal.

We show the latter. It suffices to show that

$$\{r^q e^{i\theta} : re^{i\theta} \in \sigma(T)\} \subset \sigma(T_q),$$

because if this holds, by using symmetric argument, we also have

$$\{s^{\frac{1}{q}} e^{i\tau} : se^{i\tau} \in \sigma(T_q)\} \subset \sigma(T),$$

and we have the conclusion. Since $0 \in \sigma(T)$ if and only if $0 \in \sigma(T_q)$, we have only to prove if $\lambda = re^{i\theta} \in \sigma(T) \setminus \{0\}$, then $\lambda_q = r^q e^{i\theta} \in \sigma(T_q)$.

First we consider the case 1') $\lambda \in \sigma_a(T)$. In this case, λ is a normal approximate eigenvalue of T by Lemma 1, hence there exists a sequence of unit vectors $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that

$$\|(|T| - r)x_n\| \to 0 \quad \text{and} \quad \|(U - e^{i\theta})x_n\| \to 0 \quad \text{as } n \to \infty.$$

From this we have

$$\|(T_q - \lambda_q)x_n\| \le \|U(|T|^q - r^q)x_n\| + r^q\|(U - e^{i\theta})x_n\| \to 0.$$

Hence $\lambda_q \in \sigma_a(T_q) \subset \sigma(T_q)$.

Next, we consider the case 2') $\lambda \in \sigma(T) \setminus \sigma_a(T)$. Suppose that λ_q does not belong to $\sigma(T_q)$. Then operator valued mapping $S(\cdot) : [0,1] \to \mathcal{B}(\mathcal{H})$ defined by

$$S(t) = U|T|^{1+t(q-1)} - r^{1+t(q-1)}e^{i\theta} = T_{1+t(q-1)} - \lambda_{1+t(q-1)}$$

is continuous and satisfies

$$\begin{aligned} S(0) &= T - \lambda \text{ is semi-Fredholm with } \mathrm{ind}(S(0)) \le -1, \\ S(1) &= T_q - \lambda_q \text{ is invertible.} \end{aligned}$$

Hence by Lemma 2, there exists an $s \in (0,1)$ such that $S(s)$ is not semi-Fredholm, i.e., $\lambda_{1+s(q-1)} = r^{1+s(q-1)}e^{i\theta} \in \sigma_a(T_{1+s(q-1)})$. Since $T_{1+s(q-1)}$ is $\frac{p}{1+s(q-1)}$-quasihyponormal we have $\lambda_{1+s(q-1)} \in \sigma_{na}(T_{1+s(q-1)})$ by Lemma 1. By using the same argument as case 1'), we have that $\lambda \in \sigma_a(T)$ and $\lambda_q \in \sigma_a(T_q)$. This is a contradiction. Hence $\lambda_q = r^q e^{i\theta} \in \sigma(T_q)$. This completes the proof.

Remark 1. (i) If we choose $R = \mathbb{C} \setminus \{0\}$ or $R = \sigma(T) \setminus \{0\}$, $T(t) = U|T|^{1+t(q-1)}$ and $\tau_t(re^{i\theta}) = r^{1+t(q-1)}e^{i\theta}$ for $re^{i\theta} \in R$, the above lemma is also shown directly by Xia's lemma and Lemma 1.

(ii) Since $re^{i\theta} \mapsto r^q e^{i\theta}$ is a homeomorphism on $\mathbb{C}$, this maps each isolated point of $\sigma(T)$ to an isolated point of $\sigma(T_q)$. That also maps $\sigma_a(T)$ onto $\sigma_a(T_q)$, hence it maps $\sigma(T) \setminus \sigma_a(T)$ onto $\sigma(T_q) \setminus \sigma_a(T_q)$.

3. Main results

In [16], the author obtained an extension of Putnam's inequality for quasihyponormal operators as follows.

Proposition 2. If T is a quasihyponormal operator, then

$$\|T^*T - TT^*\| \leq 2\|T\|(\frac{1}{\pi}m(\sigma(T)))^{\frac{1}{2}}, \tag{3}$$

where $m(\cdot)$ is the planar Lebesgue measure on $\mathbb{C}$.

Remark 2. One may think that the above inequality is quite different from original Putnam inequality for hyponormal operators, and it should be improved at least as follows:

$$\|T^*T - TT^*\| \leq Cm(\sigma(T)),$$

where C is a uniformly constant which is independent on quasihyponormal operator T.

However, we remark that there does not exist such constant C. We show an example of a sequences $\{T_n\}$ of quasihyponormal operators such that $m(\sigma(T_n)) = 1$ for all n and $\|{T_n}^*T_n - T_n{T_n}^*\| \to \infty$ as $n \to \infty$.

Let $\{\epsilon_n\}$ be the canonical orthogonal basis for $\ell^2(\mathbb{N})$, U the unilateral shift on $\ell^2(\mathbb{N})$ defined by $U\epsilon_n = \epsilon_{n+1}$ for $n \in \mathbb{N}$ and P_n the orthogonal projection onto the one-dimensional subspace $\mathbb{C}\epsilon_n$. Let $\mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Define operators T_n on $\mathcal{H}$ by

$$\begin{pmatrix} U+n & P_1 \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}).$$

Then T_n is quasihyponormal but not hyponormal with $\sigma(T_n) = \{0\} \cup \{z : |z-n| \leq 1\}$, i.e., $m(\sigma(T_n)) = 1$ for all n, and $\|T_n\| = n+1$. Since,

$$T_n{}^*T_n - T_nT_n{}^* = \begin{pmatrix} 0 & nP_1 \\ nP_1 & P_1 \end{pmatrix},$$

we have $\|T_n{}^*T_n - T_nT_n{}^*\| = \frac{1+\sqrt{4n^2+1}}{2} (= O(\|T_n\|)) \to \infty$ as $n \to \infty$. Hence there does not exist such constant C, and this example also shows that $\|T\|$ on the right hand side of the inequality in Proposition 2 is necessary.

By Proposition 2 and Lemma 3, we have the following Putnam type inequality for p-quasihyponormal operators. It shows that a p-quasihyponormal operator with Lebesgue null set spectrum is always a normal operator. In particular, every p-quasihyponormal operator on a finite dimensional Hilbert space is a normal operator and also compact, p-quasihyponormal operator is normal.

Theorem 1. If T is p-quasihyponormal for some $p > 0$, then

$$\|(T^*T)^p - (TT^*)^p\| \leq 2\|T\|^p (\frac{p}{\pi} \iint_{re^{i\theta} \in \sigma(T)} r^{2p-1}\, dr d\theta)^{\frac{1}{2}}.$$

Proof. Since $T_p = U|T|^p$ is quasihyponormal with spectrum $\sigma(T_p) = \{r^p e^{i\theta} : re^{i\theta} \in \sigma(T)\}$, by lemma 3,

$$\begin{aligned} \|(T^*T)^p - (TT^*)^p\| &= \|T_p{}^*T_p - T_pT_p{}^*\| \\ &\leq 2\|T_p\|(\frac{1}{\pi} m(\sigma(T_p)))^{\frac{1}{2}} \\ &= 2\|T\|^p (\frac{1}{\pi} \iint_{re^{i\theta} \in \sigma(T_p)} r\, dr d\theta)^{\frac{1}{2}} \\ &= 2\|T\|^p (\frac{p}{\pi} \iint_{re^{i\theta} \in \sigma(T)} r^{2p-1}\, dr d\theta)^{\frac{1}{2}}. \end{aligned}$$

The following results have been already shown the case of p-quasihyponormal operators for $0 < p \leq 1$. See [13]. In here, we shall show those for all $p > 0$ by using deformed operator T_p, Lemma 3 and the results for quasihyponormal case.

Lemma 4. If T is p-quasihyponormal, then it is isoloid, i.e., every isolated point of $\sigma(T)$ is an eigenvalue of T.

Proof. If $\lambda = re^{i\theta}$ is isolated in $\sigma(T)$, then $\lambda_p = r^p e^{i\theta}$ is isolated point of spectrum $\sigma(T_p)$ of quasihyponormal operator $T_p = U|T|^p$ by Lemma 3 and Remark 1 (ii). By Tanahashi-Uchiyama [13], the Riesz idempotent E_p with respect to λ_p defined by

$$E_p = \frac{1}{2\pi i}\int_{\gamma}(z - T_p)^{-1}dz \tag{4}$$

satisfies $\text{ran}E_p = \ker T_p$ if $\lambda = 0$, $\text{ran}E_p = \ker(T_p - \lambda_p) = \ker(T_p - \lambda_p)^*$ and E_p is self-adjoint if $\lambda \neq 0$, where γ is a circle with center λ_p and small enough radius ϵ such that $\{z : |z - \lambda_p| \leq \epsilon\} \cap \sigma(T_p) = \{\lambda_p\}$. Since $\ker T_p = \ker T$, and $\ker(T_p - \lambda_p) = \ker(T - \lambda)$ if $\lambda \neq 0$ by Lemma 1, we have the conclusion.

One may wonder whether above E_p coincides with the Riesz idempotent E for T with respect to λ or not. The answer is yes, i.e., $E = E_p$. We prove this in the proof of the next theorem.

Theorem 2. If T is a p-quasihyponormal and λ is an isolated point of $\sigma(T)$, then the Riesz idempotent E with respect to λ defined by

$$E = \frac{1}{2\pi i}\int_{\gamma'}(z - T)^{-1}dz, \tag{5}$$

satisfies $\text{ran}E = \ker T$ if $\lambda = 0$, $\text{ran}E = \ker(T - \lambda) = \ker(T - \lambda)^*$ and $E = E^*$ if $\lambda \neq 0$, where γ' is a circle with center λ and small enough radius δ such that $\{z : |z - \lambda| \leq \delta\} \cap \sigma(T) = \{\lambda\}$.

Proof Let E_p be same as in the proof of Lemma 4. Then

$$\text{ran}E_p = \ker(T_p - \lambda_p) = \ker(T - \lambda) \subset \text{ran}E.$$

First we show in the case $\lambda \neq 0$. Since, E_p is the orthogonal projection onto $\ker(T-\lambda)$ and $\ker(T-\lambda)$ reduces T by Lemma 1, $E_pT = \lambda E_p = TE_p$ hence E_p also commutes with E. Assume that $E \neq E_p$. Then $\operatorname{ran}(E-E_p)$ is a non-zero T-invariant closed subspace which contained in $\operatorname{ran}(1-E_p)$, because $E - E_p = (1-E_p)E$ is idempotent. Since $E = E(E_p \oplus 1 - E_p) = E_p \oplus (1-E_p)E = E_p \oplus E - E_p$, $\{\lambda\} = \sigma(T|\operatorname{ran}_E) = \sigma(T|\operatorname{ran}_{E_p}) \cup \sigma(T|\operatorname{ran}_{(E-E_p)}) = \{\lambda\} \cup \sigma(T|\operatorname{ran}_{(E-E_p)})$, and we have $\sigma(T|\operatorname{ran}_{(E-E_p)}) = \{\lambda\}$. Hence λ is an approximate eigenvalue of $T|\operatorname{ran}_{(E-E_p)}$. Thus there exists a sequence $\{x_n\}$ of unit vectors in $\operatorname{ran}(E-E_p)$ such that

$$\|(T-\lambda)x_n\| \to 0.$$

By Lemma 1, $\lambda \in \sigma_{na}(T)$, so we have $\|(T_p - \lambda_p)x_n\| \to 0$ as $n \to 0$. This contradicts the fact that $\sigma(T_P|\operatorname{ran}_{(1-E_P)})$ does not contain λ_p. Hence, we have $E = E_p$ and the assertion is immediate from this.

Next, we show the case $\lambda = 0$. Let $T = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}$ on $\mathcal{H} = \operatorname{ran}E_p \oplus (\operatorname{ran}E_p)^\perp = \ker T \oplus \overline{\operatorname{ran}T^*}$ be 2×2 matrix representation and $T = U|T|$ be the polar decomposition of T. Then U is of the form

$$U = \begin{pmatrix} 0 & U_1 \\ 0 & U_2 \end{pmatrix},$$

because $\ker U = \ker T$. Hence,

$$T = \begin{pmatrix} 0 & U_1(A^*A+B^*B)^{\frac{1}{2}} \\ 0 & U_2(A^*A+B^*B)^{\frac{1}{2}} \end{pmatrix},$$

and

$$T_p = \begin{pmatrix} 0 & U_1(A^*A+B^*B)^{\frac{p}{2}} \\ 0 & U_2(A^*A+B^*B)^{\frac{p}{2}} \end{pmatrix}.$$

Put $C = U_1(A^*A+B^*B)^{\frac{p}{2}}$ and $D = U_2(A^*A+B^*B)^{\frac{p}{2}}$. We shall show that D is invertible. If D is not invertible, then by assumption, 0 is an isolated point of $\sigma(D)$. Let F be the Riesz idempotent for D with respect to 0. Then $F \neq 0$ and

$$\begin{aligned} E_p &= \frac{1}{2\pi i}\int_\gamma (z-T_p)^{-1}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \begin{pmatrix} z^{-1} & z^{-1}C(z-D)^{-1} \\ 0 & (z-D)^{-1} \end{pmatrix} dz \\ &= \begin{pmatrix} \frac{1}{2\pi i}\int_\gamma z^{-1}\,dz & \frac{1}{2\pi i}\int_\gamma z^{-1}C(z-D)^{-1}\,dz \\ 0 & \frac{1}{2\pi i}\int_\gamma (z-D)^{-1}\,dz \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 & \frac{1}{2\pi i}\int_\gamma z^{-1}C(z-D)^{-1}\,dz \\ 0 & F \end{pmatrix},$$

this contradicts the fact that $\operatorname{ran} E_p = \ker T$. Hence D is invertible, therefore A^*A+B^*B, U_2 and $B = U_2(A^*A+B^*B)^{\frac{1}{2}}$ are also invertible because

$$0 < D^*D \le (A^*A+B^*B)^p$$

implies the invertibility of $(A^*A+B^*B)^p$, so is A^*A+B^*B and $U_2 = D(A^*A+B^*B)^{-p}$. So, we have

$$\begin{aligned} E_p &= \begin{pmatrix} 1 & \frac{1}{2\pi i}\int_\gamma z^{-1}C(z-D)^{-1}\,dz \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -CD^{-1}\frac{1}{2\pi i}\int_\gamma \{z^{-1}-(z-D)^{-1}\}\,dz \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -CD^{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -U_1(A^*A+B^*B)^{\frac{p}{2}}(A^*A+B^*B)^{\frac{-p}{2}}U_2^{-1} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -U_1U_2^{-1} \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and similarly,

$$\begin{aligned} E &= \frac{1}{2\pi i}\int_{\gamma'} \begin{pmatrix} z^{-1} & z^{-1}A(z-B)^{-1} \\ 0 & (z-B)^{-1} \end{pmatrix} dz \\ &= \begin{pmatrix} 1 & -AB^{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -U_1U_2^{-1} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, $E_p = E$. Thus the assertion also holds.

Remark 3. By the proof of Theorem 2, we easily see that if T is p-quasihyponormal for some $p > 0$ and $\lambda = re^{i\theta}$ is an isolated point of $\sigma(T)$, then for every $s > 0$, the Riesz idempotent E_s for $T_s = U|T|^s$ with respect to $\lambda_s = r^s e^{i\theta}$ coincides with E_1 because T_s is $\frac{p}{s}$-quasihyponormal.

Theorem 3. Weyl's theorem holds for p-quasihyponormal operator T, i.e., $\sigma(T) \setminus w(T) = \pi_{00}(T)$.

Proof. Let $\lambda \in \sigma(T) \setminus w(T)$. Then $T - \lambda \in \mathcal{F}_0$ and $0 < \dim\ker(T - \lambda) < \infty$. We shall show that λ is isolated in $\sigma(T)$.

First, we consider the case $\lambda = 0$. Since $T \in \mathcal{F}_0$, $|T|^s \in \mathcal{F}_0$ for all $s > 0$ and $U \in \mathcal{F}_0$ because $\mathrm{ran}U = \mathrm{ran}T$ is closed and $\ker U = \ker T$, that is $\mathrm{ind}(U) = \mathrm{ind}(T)$. Thus we have $T_p = U|T|^p \in \mathcal{F}_0$. Since T_p is quasihyponormal hence Weyl's theorem holds for T by [17] or [19], i.e.,

$$0 \in \sigma(T_p) \setminus w(T_p) = \pi_{00}(T_p).$$

Hence 0 is isolated in $\sigma(T_p)$, therefore 0 is isolated in $\sigma(T)$ by Lemma 3 and Remark 1 (ii).

Next, we consider the case $\lambda \neq 0$. Since $\ker(T - \lambda)$ reduces T by Lemma 1, T is of the form

$$T = \lambda \oplus T_1 \quad \text{on} \quad \mathcal{H} = \ker(T - \lambda) \oplus [\mathrm{ran}(T - \lambda)^*],$$

where T_1 is p-quasihyponormal and $\ker(T_1 - \lambda) = \{0\}$. Since $\ker(T - \lambda)$ is finite dimensional, $T_1 - \lambda$ is Fredholm with index 0, hence it is invertible. This implies that λ is isolated in $\sigma(T) = \{\lambda\} \cup \sigma(T_1)$.

Conversely, if $\lambda \in \pi_{00}(T)$, then the Riesz idempotent E with respect to λ defined by (5) satisfies $\mathrm{ran}E = \ker(T - \lambda)$ by Theorem 2 and $\sigma(T|_{\mathrm{ran}(1-E)})$ does not contain λ. Hence,

$$\begin{aligned}\mathrm{ran}(T - \lambda) &= \mathrm{ran}(T - \lambda)E + \mathrm{ran}(T - \lambda)(1 - E) \\ &= \mathrm{ran}(1 - E),\end{aligned}$$

(i.e., $T - \lambda$ is semi-Fredholm) and since $\dim(\mathcal{H}/\mathrm{ran}(T - \lambda)) = \dim(\mathcal{H}/\mathrm{ran}(1 - E)) = \dim \mathrm{ran}E = \dim\ker(T - \lambda)$, $T - \lambda \in \mathcal{F}_0$, so we have $\lambda \in \sigma(T) \setminus w(T)$.

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