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ON THE GEOMETRY OF QUASI-KÄHLER MANIFOLDS WITH NORDEN METRIC

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ABSTRACT. The basic class of the non-integrable almost complex manifolds with Norden metric is considered. Its curvature properties are studied. The isotropic Kähler type of investigated manifolds is introduced and characterized geometrically.

The generalized B-manifolds are introduced in [1]. They are also known as almost complex manifolds with Norden metric in [2] and as almost complex manifolds with B-metric in [3]. In the present paper these manifolds are called almost complex manifolds with Norden metric.

The aim of the present work is to further study of the geometry of one of the basic classes of almost complex manifolds with Norden metric. This is the class of the quasi-Kähler manifolds with Norden metric, which is the only basic class with non-integrable almost complex structure.

In §1 we recall the notions of the almost complex manifolds with Norden metric, we give some of their curvature properties and introduce isotropic Kähler type of the considered manifolds.

In §2 we specialize some curvature properties for the quasi-Kähler manifolds with Norden metric and the corresponding invariants.

1. ALMOST COMPLEX MANIFOLDS WITH NORDEN METRIC

Let (M, J, g) be a $2n$-dimensional almost complex manifold with Norden metric, i.e. J is an almost complex structure and g is a metric on M such that

$$J^2X = -X, \qquad g(JX, JY) = -g(X,Y) \tag{1.1}$$

for all differentiable vector fields X, Y on M, i.e. $X, Y \in \mathfrak{X}(M)$.

The associated metric $\tilde{g}$ of g on M given by $\tilde{g}(X,Y) = g(X, JY)$ for all $X, Y \in \mathfrak{X}(M)$ is a Norden metric, too. Both metrics are necessarily of signature (n, n). The manifold $(M, J, \tilde{g})$ is an almost complex manifold with Norden metric, too.

Further, X, Y, Z, U (x, y, z, u, respectively) will stand for arbitrary differentiable vector fields on M (vectors in T_pM, $p \in M$, respectively).

The Levi-Civita connection of g is denoted by ∇. The tensor filed F of type $(0,3)$ on M is defined by

$$F(X,Y,Z) = g\big((\nabla_X J)Y, Z\big). \tag{1.2}$$

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It has the following symmetries

$$F(X,Y,Z)=F(X,Z,Y)=F(X,JY,JZ). \tag{1.3}$$

Let $\{e_i\}$ $(i=1,2,\dots,2n)$ be an arbitrary basis of T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$.

The Lie form θ associated with F is defined by

$$\theta(z)=g^{ij}F(e_i,e_j,z). \tag{1.4}$$

A classification of the considered manifolds with respect to F is given in [2]. Eight classes of almost complex manifolds with Norden metric are characterized there by conditions for F. The three basic classes are given as follows

$$\begin{aligned}
\mathcal{W}_1 &: F(x,y,z)=\tfrac{1}{4n}\{g(x,y)\theta(z)+g(x,z)\theta(y)\\
&\qquad\qquad +g(x,Jy)\theta(Jz)+g(x,Jz)\theta(Jy)\};\\
\mathcal{W}_2 &: \underset{x,y,z}{\mathfrak{S}} F(x,y,Jz)=0, \quad \theta=0;\\
\mathcal{W}_3 &: \underset{x,y,z}{\mathfrak{S}} F(x,y,z)=0,
\end{aligned} \tag{1.5}$$

where $\mathfrak{S}$ is the cyclic sum by three arguments.

The special class $\mathcal{W}_0$ of the Kähler manifolds with Norden metric belonging to any other class is determined by the condition $F=0$.

Let R be the curvature tensor field of ∇ defined by

$$R(X,Y)Z=\nabla_X(\nabla_Y Z)-\nabla_Y(\nabla_X Z)-\nabla_{[X,Y]}Z. \tag{1.6}$$

The corresponding tensor field of type $(0,4)$ is determined as follows

$$R(X,Y,Z,U)=g(R(X,Y)Z,U). \tag{1.7}$$

Theorem 1.1. *Let (M,J,g) be an almost complex manifold with Norden metric. Then the following identities are valid*

(i) $R(X,Y,JZ,U)-R(X,Y,Z,JU)=(\nabla_X F)(Y,Z,U)-(\nabla_Y F)(X,Z,U)$;

(ii) $(\nabla_X F)(Y,JZ,U)+(\nabla_X F)(Y,Z,JU)$
$=-g\big((\nabla_X J)Z,(\nabla_Y J)U\big)-g\big((\nabla_X J)U,(\nabla_Y J)Z\big)$;

(iii) $(\nabla_X F)(Y,Z,U)=(\nabla_X F)(Y,U,Z)$;

(iv) $g^{ij}(\nabla_{e_i}F)(e_j,Jz,u)+g^{ij}(\nabla_{e_i}F)(e_j,z,Ju)$
$=-2g^{ij}g\big((\nabla_{e_i}J)z,(\nabla_{e_j}J)u\big)$.

Proof. The equality (i) follows from the Ricci identity for J

$$(\nabla_X(\nabla_Y J))Z-(\nabla_Y(\nabla_X J))Z-(\nabla_{[X,Y]}J)Z=R(X,Y)JZ-JR(X,Y)Z$$

and the property of covariant constancy of g, i.e. $\nabla g=0$.

The property (1.3) of F and the definition of the covariant derivative of F imply the equations (ii) and (iii).

The equation (iv) is a corollary of (ii) by the action of contraction of $X=e_i$ and $Y=e_j$ for an arbitrary basis $\{e_i\}(i=1,2,\dots,2n)$ of T_pM. □

The square norm $\|\nabla J\|^2$ of ∇J is defined by

$$\|\nabla J\|^2 = g^{ij}g^{kl}g\big((\nabla_{e_i}J)\,e_k, (\nabla_{e_j}J)\,e_l\big). \tag{1.8}$$

A manifold (M,J,g) belongs to the class $\mathcal{W}_0$ if and only if $\nabla J = 0$. It is clear that if $(M,J,g)\in\mathcal{W}_0$, then $\|\nabla J\|^2$ vanishes, too, but the inverse proposition is not always true. That is, in general, the vanishing of the square norm $\|\nabla J\|^2$ does not always imply the Kähler condition $\nabla J = 0$.

An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2$ to be zero is called an *isotropic Kähler manifold with Norden metric.*

A special subclass of the investigated manifolds consisting of isotropic Kähler but non-Kähler manifold with Norden metric is considered in [5]. In the next section we will focus on this case.

2. The Quasi-Kähler Manifolds with Norden Metric

Let (M,J,g) be a quasi-Kähler manifold with Norden metric (in short a $\mathcal{W}_3$-manifold), i.e. it belongs to the class $\mathcal{W}_3$.

Proposition 2.1. *The following properties are valid for an arbitrary $\mathcal{W}_3$-manifold.*

(i) $(\nabla_X J)JY + (\nabla_Y J)JX + (\nabla_{JX}J)Y + (\nabla_{JY}J)X = 0;$

(ii) $\underset{X,Y,Z}{\mathfrak{S}} F(JX,Y,Z) = 0;$

(iii) $\underset{Y,Z,U}{\mathfrak{S}} (\nabla_X F)(Y,Z,U) = 0;$

(iv) $g^{ij}(\nabla_X F)(e_i,e_j,z) = g^{ij}(\nabla_X F)(z,e_i,e_j) = 0,$

where $\mathfrak{S}$ is the cyclic sum by three arguments.

Proof. The equalities (i) and (ii) are equivalent to the characteristic condition (1.5) for the class $\mathcal{W}_3$. The defining equation of the covariant derivative of F, the condition $\nabla g = 0$ and the definition (1.5) of $\mathcal{W}_3$ imply the equalities (iii) and (iv). □

In [5] it is proved that on every $\mathcal{W}_3$-manifold the curvature tensor R satisfies the following identity

$$\begin{aligned}
&R(X,JZ,Y,JU) + R(X,JY,U,JZ) + R(X,JY,Z,JU)\\
&+R(X,JZ,U,JY) + R(X,JU,Y,JZ) + R(X,JU,Z,JY)\\
&+R(JX,Z,JY,U) + R(JX,Y,JU,Z) + R(JX,Y,JZ,U)\\
&+R(JX,Z,JU,Y) + R(JX,U,JY,Z) + R(JX,U,JZ,Y)\\
&= -\underset{X,Y,Z}{\mathfrak{S}} g\Big((\nabla_X J)Y + (\nabla_Y J)X, (\nabla_Z J)U + (\nabla_U J)Z\Big).
\end{aligned} \tag{2.1}$$

Let us consider an associated tensor of the Ricci tensor ρ defined by the equation $\rho^*(y,z) = g^{ij}R(e_i,y,z,Je_j)$ on an almost complex manifold with Norden metric. The tensor ρ^* is symmetric because of the first Bianchi identity.

By virtue of the identity (2.1) we get immediately the following

Lemma 2.2. *For a $\mathcal{W}_3$-manifold (M,J,g) with the Ricci tensor ρ of ∇ and its associated tensor $\rho^*(y,z) = g^{ij}R(e_i,y,z,Je_j)$ we have*

$$\text{(i)}\quad \begin{aligned} &\rho^*(Jy,z)+\rho^*(y,Jz)+\rho(y,z)-\rho(Jy,Jz)\\ &= -g^{ij}g\Big((\nabla_{e_i}J)y+(\nabla_yJ)e_i,\,(\nabla_zJ)e_j+(\nabla_{e_j}J)z\Big); \end{aligned}$$

(ii) $\|\nabla J\|^2 = -2g^{ij}g^{kl}g\Big((\nabla_{e_i}J)e_k,\,(\nabla_{e_l}J)e_j\Big)$.

The last lemma and the identity (2.1) imply the next

Theorem 2.3. *Let (M,J,g) be a $\mathcal{W}_3$-manifold. Then*

$$\|\nabla J\|^2 = -2(\tau+\tau^{**}),$$

where τ is the scalar curvature of ∇ and $\tau^{**} = g^{ij}g^{kl}R(e_i,e_k,Je_l,Je_j)$.

Hence, the last theorem implies the following

Corollary 2.4. *If (M,J,g) is an isotropic Kähler $\mathcal{W}_3$-manifold then $\tau^{**}=-\tau$.*

According to [5], if (M,J,g), $\dim M\geq 4$, is a $\mathcal{W}_3$-manifold which has the Kähler property of R: $R(X,Y,JZ,JU)=-R(X,Y,Z,U)$, then the norm $\|(\nabla_xJ)x\|^2$ vanishes for every vector $x\in T_pM$.

Since $\|(\nabla_xJ)x\|^2 = g\Big((\nabla_xJ)x,\,(\nabla_xJ)x\Big)=0$ holds, then applying the substitutions $x\to x+y$, primary, and $x\to x+z$, $y\to y+u$, secondary, we receive the following condition

$$\underset{x,y,z}{\mathfrak{S}}\, g\Big((\nabla_xJ)y+(\nabla_yJ)x,\,(\nabla_zJ)u+(\nabla_uJ)z\Big)=0.$$

Therefore, for the traces we have

$$g^{ij}g^{kl}g\Big((\nabla_{e_i}J)e_k,\,(\nabla_{e_j}J)e_l\Big)=0.$$

Having in mind (1.8) we obtain the following

Proposition 2.5. *If (M,J,g), $\dim M\geq 4$, is a $\mathcal{W}_3$-manifold with Kähler curvature tensor, then it is isotropic Kählerian.*

Let α_1 and α_2 be holomorphic 2-planes determined by the basis (x,Jx) and (y,Jy), respectively. The holomorphic bisectional curvature $h(x,y)$ of the pair of holomorphic 2-planes α_1 and α_2 is introduced in [4] by the following way

$$h(x,y) = -\frac{R(x,Jx,y,Jy)}{\sqrt{\{g(x,x)\}^2+\{g(x,Jx)\}^2}\sqrt{\{g(y,y)\}^2+\{g(y,Jy)\}^2}}, \tag{2.2}$$

where x, y do not lie along the totally isotropic directions, i.e. the both of the couples $\big(g(x,x),g(x,Jx)\big)$ and $\big(g(y,y),g(y,Jy)\big)$ are different from the couple $(0,0)$. The holomorphic bisectional curvature is invariant with respect to the basis of the 2-planes α_1 and α_2. In particular, if $\alpha_1=\alpha_2$, then the holomorphic bisectional curvature coincides with the holomorphic sectional curvature of the 2-plane $\alpha_1=\alpha_2$.

Let us note that $R(x,Jx,y,Jy)$ is the main component of the curvature tensor on the 4-dimensional holomorphic space spanned by the frame $\{x,y,Jx,Jy\}$, which is contained in T_pM. Therefore, an important problem is the vanishing of $R(x,Jx,y,Jy)$.

Proposition 2.6. *Let (M, J, g), $\dim M \geq 4$, be a $\mathcal{W}_3$-manifold and let x, y be arbitrary vectors which do not lie along the totally isotropic directions in T_pM such that $\{x, y, Jx, Jy\}$ is a basis of a 4-dimensional holomorphic space in T_pM. Then the vanishing of $R(x, Jx, y, Jy)$ is equivalent to the vanishing of the bisectional curvature $h(x, y)$ of the pair of the holomorphic 2-planes $\{x, Jx\}$ and $\{y, Jy\}$.*

Theorem 2.7. *Let (M, J, g), $\dim M \geq 4$, be a $\mathcal{W}_3$-manifold and $R(x, Jx, y, Jy) = 0$ for all $x, y \in T_pM$. Then (M, J, g) is an isotropic Kähler $\mathcal{W}_3$-manifold.*

Proof. Let the condition $R(x, Jx, y, Jy) = 0$ be valid. At first we substitute $x + z$ and $y + u$ for x and y, respectively. According to the properties of R, we get

$$R(x, Jy, z, Ju) + R(Jx, y, Jz, u) - R(x, Jy, Jz, u) - R(Jx, y, z, Ju) = 0 \tag{2.3}$$

and two similar equations which imply the vanishing of the left side of (2.1). Therefore

$$\underset{x,y,z}{\mathfrak{S}} g\Big((\nabla_x J)y + (\nabla_y J)x, (\nabla_z J)u + (\nabla_u J)z\Big) = 0.$$

By contracting the last equation, having in mind (1.8), we receive the condition $\|\nabla J\|^2 = 0$. □

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