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NON-SMOOTH GALOIS POINT ON A QUINTIC CURVE WITH ONE SINGULAR POINT

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ABSTRACT. Let C be an irreducible plane quintic curve with only one singular point P, which is a double point. Then, we consider a projection of C from P. This projection induces an extension of rational function fields $k(C)/k(\mathbb{P}^1)$. In this paper, we give the defining equation of the curve C when the extension is Galois.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero, which we fix as the ground field of our discussion. Let C be an irreducible (possibly singular) curve of degree d in the projective plane $\mathbb{P}^2 = \mathbb{P}^2(k)$ and $K = k(C)$ the rational function field of C. For each point $P \in C$, let $\pi_P : C \cdots \rightarrow l$ be a projection from C to a line l with the center P. This rational map induces the extension of fields $K/k(l)$. The structure of this extension does not depend on the choice of l, but on P, so that we write K_P instead of $k(l)$.

Definition 1. *A point $P \in C$ is called a Galois point if the extension K/K_P is Galois. In particular, a Galois point is called a non-smooth Galois point [resp. a smooth Galois point] if it is singular. [resp. nonsingular.]*

In the papers [5], [6] and [8], Yoshihara raised the following questions:

(1) When is the extension K/K_P Galois? Namely, when is the point P Galois?
(2) How many Galois points do there exist on C (or $\mathbb{P}^2\backslash C$)?
(3) Let L_P be the Galois closure of K/K_P. What can we say about L_P?
(4) What is the Galois group $\mathrm{Gal}(L_P/K_P)$?
(5) Determine intermediate fields between K_P and L_P.

These were treated in detail for nonsingular plane curves in papers [5], [6], [8] and Miura's paper [2]. Miura also studied these questions for singular plane quartic curves in [1] and [3].

Let $(X : Y : Z)$ be homogeneous coordinates on $\mathbb{P}^2$ and (x, y) affine coordinates such that $x = X/Z$ and $y = Y/Z$. For a nonsingular plane curve, we have an answer to Question (1) as follows.

Proposition 1 ([8], Proposition 5). *Let C be a nonsingular plane curve of degree d $(d \geq 4)$. Then, the point $P \in C$ is Galois if and only if the defining equation*

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of C can be expressed as a standard form $y + h(x,y)$ by taking a suitable projective transformation which moves P to $(0,0)$, where $h(x,y)$ is a form of degree d with distinct factors.

If P is a Galois point of C, then an element σ of the Galois group $\mathrm{Gal}(K/K_P)$ induces a birational map $C \cdots \rightarrow C$. In this paper, we use the same symbol $\sigma \in \mathrm{Gal}(K/K_P)$ to denote this birational map, when there is no fear of confusion. Moreover, if an element $\sigma \in \mathrm{Gal}(K/K_P)$ is the restriction of a projective transformation of $\mathbb{P}^2$, then we say that σ belongs to $PGL(3,k)$, and denote by $\sigma \in PGL(3,k)$.

For singular plane curves, we have no good answer to Question (1). The reason is that the following well-known assertion does not hold true for a singular plane curve:

> An automorphism of a nonsingular plane curve of degree d ($d \geq 4$) is the restriction of some projective transformation of $\mathbb{P}^2$.

So, the question seems difficult. However, we have the following.

Proposition 2 ([4], Proposition 2). *Let C be a plane curve of degree d and P be a singular point of C with multiplicity m_P. Suppose that P is a Galois point. Then, the Galois group $\mathrm{Gal}(K/K_P)$ is contained in $PGL(3,k)$ if and only if C is projectively equivalent to the curve given by $f_{m_P}(x,y) + f_d(x,y) = 0$, where $f_i(x,y)$ is a homogeneous polynomial of x and y of degree i ($i = m_P$ or d).*

There was no study on non-smooth Galois points. The purpose of this paper is to show when the point P is Galois under the following assumption: the plane quintic curve C has only one singular point P, which is a double point. This case is the most simple one of Question (1) for non-smooth Galois points.

2. Statement of results

We use the same notation as is used in Section 1 and restrict ourselves to the case where C is an irreducible quintic curve with only one singular point P, which is a double point. We denote by $g(C)$ the genus of a nonsingular model of a curve C. Note that from the genus formula, $g(C) = 0, 1, 2, 3, 4$ or 5. Our main theorem is stated as follows.

Theorem. *Let C be an irreducible plane quintic curve. Suppose that C has only one singular point P, which is a double point. Then we have the following.*

(1) *If $g(C) = 0$ or 3, then P cannot be a Galois point.*
(2) *If $g(C) = 1$, then P is a Galois point if and only if C is projectively equivalent to the curve given by the equation*

$$y^2 - 6xy\,(x + 2y) + 3x\left(3x^3 + 12x^2y + 10xy^2 - 3y^3\right) + 3xy\left(6x^3 + 21x^2y + 19xy^2 + y^3\right) = 0. \quad \text{(C1)}$$

(3) *If $g(C)=2$, then P is a Galois point if and only if C is projectively equivalent to the curve given by the equation*

$$y^2 - 54c^4(1+c)xy(x+y) + 243c^6(1+c)^2 x(x+y)\left((1+c)y^2 + 3c^2x(x+y)\right) - 729c^8(1+c)^4 xy(x+y)\left(-(1+c)y^2 + 9c^2x(x+y)\right) = 0, \quad \text{(C2)}$$

where $c \in k$ and $c \neq 0, -1$.

(4) *If $g(C)=4$, then P is a Galois point if and only if C is projectively equivalent to the curve given by the equation*

$$y^2 + h_5(x,y) = 0 \qquad \textit{or} \qquad \text{(C3)}$$

$$y^2 + 3x^2y + 3x^4 + h_5(x,y) = 0, \qquad \text{(C4)}$$

where $h_5(x,y)$ is a form of degree five.

(5) *If $g(C)=5$, then P is a Galois point if and only if C is projectively equivalent to the curve given by the equation*

$$xy + h_5(x,y) = 0, \qquad \text{(C5)}$$

where $h_5(x,y)$ is a form of degree five.

Remark 1. *Let $\rho : \tilde{C} \to C$ be the resolution of the singularity of C. Then, the number of points $\rho^{-1}(P)$ is equal to one when the curve C is given by Equation* (C3)*, on the other hand, the number is two when the curve C is given by Equation* (C1), (C2), (C4) *or* (C5).

As a corollary of Theorem, we also see when the Galois group $\mathrm{Gal}(K/K_P)$ is contained in $PGL(3,k)$.

Corollary 1. *With the same assumptions as in Theorem, suppose that P is a Galois point. Then we have the following.*

(1) *If either*
  (a) $g(C) = 1, 2$ *or*
  (b) $g(C) = 4$ *and C is projectively equivalent to the curve given by Equation* (C4),

then $\mathrm{Gal}(K/K_P) \not\subset PGL(3,k)$.

(2) *If either*
  (a) $g(C) = 4$ *and C is projectively equivalent to the curve given by Equation* (C3) *or*
  (b) $g(C) = 5$,

then $\mathrm{Gal}(K/K_P) \subset PGL(3,k)$.

Let $F = F(X,Y,Z) = 0$ be the homogeneous defining equation of C and $f = f(x,y) = F(x,y,1) = 0$ its dehomogenized equation. Moreover, we put $f(x,y) = \Sigma f_i(x,y)$, where $f_i = f_i(x,y)$ is the homogeneous part of f of degree i. When $g(C) = 4$ or 5, we have the easy criterion for the point P to be Galois, which is similar to [8, Lemma 11] as follows.

Corollary 2. *With the same assumptions as in Theorem, suppose that $g(C) = 4$ or 5. Let the coordinates of P be $(0:0:1)$ by taking a suitable projective transformation. Then P is a Galois point if and only if ${f_3}^2 = 3f_2f_4$.*

3. Proofs

We use the following notations.

Notation 1.

- $\omega := (-1+\sqrt{-3})/2$
- $\sim$: *the linearly equivalence of divisors*
- $|D|$: *the complete linear system associated with a divisor D*
- $\boldsymbol{L}_C(D) := \{\, \phi \in k(C) \mid \phi = 0$ or $\operatorname{div}(\phi) + D \geq 0 \,\}$
- $l(D)$: *the dimension of $\boldsymbol{L}_C(D)$ as a k-vector space*
- Φ_L: *the rational map corresponding to a linear system L*
- $\boldsymbol{V}_C(L, D) := \{\, \phi \in k(C) \mid \phi = 0$ or $\operatorname{div}(\phi) + D \in L \,\}$, *where L is a sub-linear system of $|D|$*
- $\langle \phi_0, \cdots, \phi_n \rangle$: *the k-vector space generated by elements $\phi_0, \cdots, \phi_n$*

Notation 2. *Under the assumptions that $g(C) \geq 1$ and P is a Galois point, we use the following notation. Let $\rho : \tilde{C} \to C$ be the resolution of the singularity of C, and we put $\{\, P_1, P_2 \,\} := \rho^{-1}(P)$, where points P_1 and P_2 may be the same. Let Q be a ramification point of $\pi_P \circ \rho : \tilde{C} \to l$ such that $Q \neq P_1, P_2$. We denote by L and M the linear systems corresponding to the morphisms ρ and $\pi_P \circ \rho$, respectively. Namely, we may write that $\rho = \Phi_L$ and $\pi_P \circ \rho = \Phi_M$. Here, we note that $L \subset |3Q + P_1 + P_2|$ and $M \subset L \cap |3Q|$. Let τ be the number $\min\{\, n \in \mathbb{N} \mid l(\tau Q) = 3 \,\}$ and C_0 the image of $\Phi_{|\tau Q|} : \tilde{C} \to \mathbb{P}^2$. Then we note that the degree of the map $\Phi_{|\tau Q|} : \tilde{C} \to C_0$ is equal to one. Indeed, from $M \subset |\tau Q|$ and $\deg \Phi_M = 3$, if $\deg \Phi_{|\tau Q|} = 3$ then $\deg C_0 = 1$, this contradicts that $l(\tau Q) = 3$. Let $\xi : \tilde{C_0} \to C_0$ be the resolution of singularities of C_0. We denote by N the linear system corresponding to the morphism $\Phi_L \circ \Phi_{|\tau Q|}^{-1} \circ \xi : \tilde{C_0} \to C$. Noting that $\xi^{-1} \circ \Phi_{|\tau Q|} : \tilde{C} \to \tilde{C_0}$ is an isomorphism, we put $D := \xi^{-1} \circ \Phi_{|\tau Q|}(3Q + P_1 + P_2)$. Let $\iota : \tilde{C_0} \to \mathbb{P}^2$ be the composition of ξ and the inclusion map $C_0 \hookrightarrow \mathbb{P}^2$, and $\iota^*(x)$ and $\iota^*(y)$ the rational functions $x \circ \iota$ and $y \circ \iota$, respectively. Let σ be a generator of $\operatorname{Gal}(K/K_P)$, which is isomorphic to the cyclic group of order three. If $g(C) \leq 4$, then we denote by T_PC the tangent line to C at P, and let $(C, T_PC)_P$ be the intersection number of C and T_PC at P.*

Now, we note the following, which is clear.

Remark 2. *The canonical divisor $K_{\tilde{C}}$ of $\tilde{C}$ is linearly equivalent to $6Q + (g(C) - 4)(P_1 + P_2)$.*

Let us prove Theorem examining the cases that $g(C) = 0, 1, 2, 3, 4$ and 5 separately.

(1). The case $g(C) = 0$.

From [7, Proposition 3], we may assume that $P = (0 : 0 : 1)$ and C is given by the equation

$$(y - x^2)(y - x^2 + \alpha y^2 - \alpha x^2 y + 2xy^2) + y^5 = 0,$$

where $\alpha \in k$. Putting $t = x/y$, we have that $K_P = k(t)$ and $K = K_P(x)$. Thus, we obtain the minimal polynomial of x over K_P as follows:

$$x^3 + \frac{2t^3 - 2\alpha t^2 + 1}{t\,(t^4 + -2t + \alpha)} x^2 + \frac{(\alpha t^2 - 2)}{t^4 - 2t + \alpha} x + \frac{t}{t^4 + -2t + \alpha}$$

So, we have that the discriminant of this polynomial is

$$\psi_\alpha(t) := \frac{t^6\left((4\alpha^3 + 27)\,t^4 - 36\alpha t^3 + 8\alpha^2 t^2 - 4t + 4\alpha\right)}{(t^4 - 2t + \alpha)^4}.$$

From the extension degree of K/K_P is equal to three, we infer that the extension K/K_P is Galois if and only if $\sqrt{\psi_\alpha(t)} \in K_P = k(t)$. However, we obtain easily that $\sqrt{\psi_\alpha(t)} \notin k(t)$ for any $\alpha \in k$. Therefore, P cannot be a Galois point.

(2). The case $g(C) = 1$.

First, we can check easily that if C is given by Equation (C1), then the point $P = (0 : 0 : 1)$ is Galois. Indeed, we have $\sqrt{\psi} \in K_P$, where ψ is the discriminant of the minimal polynomial of $x \in K = K_P(x)$ over K_P.

Next, suppose that $P = (0 : 0 : 1)$ is a Galois point. Then, we note that $\tau = 3$, $\Phi_{|3Q|}$ is an isomorphism, and C_0 is a nonsingular cubic curve. The generator $\sigma \in \mathrm{Gal}(K/K_P) \subset \mathrm{Aut}(\tilde{C})$ induces an automorphism of C_0, i.e., there is an injection $\mathrm{Gal}(K/K_P) \hookrightarrow \mathrm{Aut}(C_0)$. (We use the same symbol $\sigma \in \mathrm{Gal}(K/K_P)$ to denote its image.) Hence, we may assume that C_0 is given by the equation $y^2 = x^3 - 1$ and $\Phi_{|3Q|}(Q) = (0 : 1 : 0)$. Moreover, we see that $\mathrm{Gal}(K/K_P) \subset PGL(3, k)$ and may assume that

$$\sigma = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Claim 1. *We have that $P_1 \neq P_2$.*

Proof. Suppose the contrary. Then, from there are five infinitely near singular points over P, we infer that $(C, T_P C)_P \neq 3$. Moreover, since Φ_M is a Galois cover, we have that $(C, T_P C)_P \neq 4$. So, we conclude that $(C, T_P C)_P = 5$. Hence, putting $P' := P_1 = P_2$, we have that $\sigma(\Phi_{|3Q|}(P')) = \Phi_{|3Q|}(P')$. Thus, we obtain that $\Phi_{|3Q|}(P') = (0 : 1 : \sqrt{-1})$ or $(0 : 1 : -\sqrt{-1})$, so we may assume that $\Phi_{|3Q|}(P') = (0 : 1 : \sqrt{-1})$. Then, we have that $N \subset |D|$ and

$$\boldsymbol{L}_{C_0}(D) = \langle 1, \iota^*(y), \iota^*(x), \frac{\iota^*(y) + \sqrt{-1}}{\iota^*(x)}, \frac{(\iota^*(y) + \sqrt{-1})^2}{\iota^*(x)^2} \rangle.$$

Note that $\pi_P \circ \Phi_N$ is given by the linear system corresponding to the k-vector space $\langle 1, \iota^*(y) \rangle$, we may put

$$\boldsymbol{V}_{C_0}(N, D) = \langle 1, \iota^*(y), A\iota^*(x) + B\frac{\iota^*(y) + \sqrt{-1}}{\iota^*(x)} + \frac{(\iota^*(y) + \sqrt{-1})^2}{\iota^*(x)^2} \rangle,$$

where $A, B \in k$. Therefore, the defining equation of C (i.e., the image of Φ_N) is computed as follows (see Remark 3).

$$\begin{aligned} x^2 &+ 2\sqrt{-1}\,xy - y^2 - 3\sqrt{-1}\,(A-1)\,Bx^4 + 3\,(A+1)\,Bx^3y \\ &- 3\sqrt{-1}\,(A-1)\,Bx^2y^2 + 3\,(A+1)\,Bxy^3 + \left(-(A-1)^3 + \sqrt{-1}\,B^3\right)x^5 \\ &+ \left(-2\sqrt{-1}\,(A-1)^2\,(A+2) + B^3)\right)x^4y + \left(-6 + 6A^2 + \sqrt{-1}\,B^3\right)x^3y^2 \\ &\qquad + \left(-2\sqrt{-1}\,(A-2)\,(A+1)^2 + B^3\right)x^2y^3 + (1+A)^3xy^4 = 0 \end{aligned}$$

Here, we check that the number of infinitely near singular points over $P = (0:0:1)$ of this curve. Then, it is equal to two. However, since the quintic curve C has only one singular point P with multiplicity two and $g(C) = 1$, the number of infinitely near singular points over P must be equal to five. This is a contradiction. □

Noting that $P_1 \neq P_2$ and $\Phi_M(P_1) = \Phi_M(P_2)$, let us put that $P_2 = \sigma(P_1)$, $P_3 := \sigma(P_2)$ and $P_1 = \sigma(P_3)$, and let (a, b) be the affine coordinates of $\Phi_{|3Q|}(P_3)$. Then, we obtain that

$$\boldsymbol{L}_{C_0}(D) = \langle 1, \iota^*(y), \iota^*(x), \frac{\iota^*(y) + b}{\iota^*(x) - \omega a}, \frac{\iota^*(y) + b}{\iota^*(x) - \omega^2 a} \rangle.$$

Hence, we may put

$$\boldsymbol{V}_{C_0}(N, D) = \langle 1, \iota^*(y), A\iota^*(x) + B\frac{\iota^*(y) + b}{\iota^*(x) - \omega a} + \frac{\iota^*(y) + b}{\iota^*(x) - \omega^2 a} \rangle,$$

where $A, B \in k$. Therefore, the defining equation of C is computed (see Remark 3) as

$$\begin{aligned} b^2x^2 &- 2bxy + y^2 - 3a^2bBx^3 + 3a^2(B + \omega - \omega B)x^2y + 3\omega a^2b(1-B)x^3 \\ &- 3A(1+B)x^3y - 3A(1+B)xy^3 - 3\,(1+B)\,(-a - Ab - \omega a + \omega aB)\,x^2y^2 \\ &\qquad + 3\left(Ab - ab^2 + 3aB + AbB + 2ab^2B - \omega ab^2 + \omega ab^2B^2\right)x^4 \\ &+ \big(- 3abA^2 - b^2A^3 + b^3 - 9a^2AB - 3bB - 6bB^2 - 3b^3B^2 + b^3B^3 - 3\omega baA^2 \\ &\qquad + 3\omega bB + 3\omega abA^2B + 3\omega b^3B - 3\omega bB^2 - 3\omega b^3B^2\big)x^5 \\ &\quad + \big(3aA^2 + 2A^3 + b^2 - 3B - 6B^2 - 3b^2B^2 + b^2B^3 + 3\omega aA^2 + 3\omega B \\ &\qquad - 3\omega aA^2B + 3\omega b^2B - 3\omega B^2 - 3\omega b^2B^2\big)x^4y \\ &+ \big(- A^3 - b - 3abA^2 - b^2A^3 - 9a^2AB - 3bB - 3bB^2 - bB^3 - 3\omega abA^2 + 3\omega abA^2B\big)x^3y^2 \\ &+ \big(- 1 + 3aA^2 + 2bA^3 - 3B - 3B^2 - B^3 + 3\omega aA^2 - 3\omega aA^2B\big)x^2y^3 - A^3xy^4 = 0. \end{aligned}$$

Here, considering the blowing-ups at five infinitely near singular points over P, we conclude that $a^3 = 4$, $b^2 = 3$, $A = -2\omega b/3a^2$ and $B = \omega^2$. So, we may assume that

$a = \sqrt[3]{4}$ and $b = \sqrt{3}$. By taking the inverse image of the projective transformation

$$\begin{pmatrix} \sqrt{3}/12 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 4/\left(3(1+\sqrt{-3})\sqrt[3]{2}^2\right) & 0 & -1/\left(3(1+\sqrt{-3})\sqrt[3]{2}^2\right) \end{pmatrix},$$

we obtain Equation (C1).

(3). The case $g(C) = 2$.

We can check easily that if C is given by Equation (C2), then the point $P = (0 : 0 : 1)$ is Galois.

Suppose that $P = (0 : 0 : 1)$ is a Galois point.

Claim 2. *We have that $P_1 \neq P_2$.*

Proof. Suppose the contrary. Then, by an argument similar to that in the proof of Claim 1, we see that $(C, T_P C)_P = 5$. So, putting $P' := P_1 = P_2$, we infer that $3Q \sim 3P'$. Hence, we have that $K_{\tilde{C}} \sim 2P'$, so $l(2P') = l(K_{\tilde{C}}) = 2$. From the Riemann-Roch theorem, we infer that $l(3P') = 2$. Therefore we have that $|3P'| = |2P'|$. However, we see that $M = |3Q| = |3P'|$ and $\deg \Phi_M = 3$, this contradicts that $\deg \Phi_{|3P'|} = \deg \Phi_{|2P'|} = 2$. □

Since $\Phi_M(P_1) = \Phi_M(P_2)$, we may put that $P_2 = \sigma(P_1)$, $P_3 := \sigma(P_2)$ and $P_1 = \sigma(P_3)$. Noting that $\tau = 4$, from $\sigma^*|4Q| = |4Q|$, we infer that the birational map $\Phi_{|4Q|} \circ \sigma \circ \Phi_{|4Q|}^{-1} : C_0 \cdots \rightarrow C_0$ belongs to $PGL(3, k)$. From Proposition 1, we may assume that $\Phi_{|4Q|}(Q) = (0 : 0 : 1)$ and C_0 is given by the equation $y + f_4(x, y) = 0$, where $f_4(x, y)$ is a form of degree four. Then, because $g(C) = g(C_0) = 2$, C_0 has one double point. Hence, by taking a suitable projective transformation, we may assume that C_0 is given by the equation $y + x^2(x + y)(x + ay) = 0$, where $a \in k$. Here, we claim that P_3 is a Weierstrass point, so P_1 and P_2 are also Weierstrass points. Indeed, noting that $3Q \sim P_1 + P_2 + P_3$ and $l(K_{\tilde{C}} - 2P_3) = l(6Q - 2P_1 - 2P_2 - 2P_3) = 1$, from the Riemann-Roch theorem, we infer that $l(2P_3) = 2$. Because of this, we may put $\Phi_{|4Q|}(P_3) = (\sqrt{a} : 1 : \alpha\sqrt{a})$, where $\alpha \in k$ such that $\alpha^3 = -(\sqrt{a} + 1)^2$. Then, we obtain that

$$\boldsymbol{L}_{\tilde{C}_0}(D) = \langle 1, \frac{\iota^*(x)}{\iota^*(y)}, \frac{\iota^*(x)(\iota^*(x) - \sqrt{a}\,\iota^*(y))}{\iota^*(y)(\omega\alpha\iota^*(x) - 1)}, \frac{\iota^*(x)(\iota^*(x) - \sqrt{a}\,\iota^*(y))}{\iota^*(y)(\omega^2\alpha\iota^*(x) - 1)} \rangle.$$

So, noting that $N \subset |D|$, we may put

$$\boldsymbol{V}_{\tilde{C}_0}(N, D) = \langle 1, \frac{\iota^*(x)}{\iota^*(y)}, A\frac{\iota^*(x)(\iota^*(x) - \sqrt{a}\,\iota^*(y))}{\iota^*(y)(\omega\alpha\iota^*(x) - 1)} + \frac{\iota^*(x)(\iota^*(x) - \sqrt{a}\,\iota^*(y))}{\iota^*(y)(\omega^2\alpha\iota^*(x) - 1)} \rangle,$$

where $A \in k$. Therefore, the defining equation of C is computed (see Remark 3) as

$$
\begin{aligned}
&(\sqrt{-1}+b^3)^2x^2+2(-1+\sqrt{-1}b^3)xy-y^2+3b^4(-1+\sqrt{-1}b^3)(A-\omega+\omega A)x^2y\\
&\quad -3b^4(A-\omega+\omega A)xy^2+3b^2(\sqrt{-1}+b^3)^2(1+A)(\omega(A-1)-1)x^3y\\
&\quad -3b^2(A(-2+2\sqrt{-1}b^3+3b^6)+2(-1\sqrt{-1}b^3)(1+\omega)+2(\omega-\sqrt{-1}b^3\omega)A^2)x^2y^2\\
&\quad +3b^2(1+\omega+A-\omega A^2)xy^3-(1+A)^3(-1+\sqrt{-1}b^3)^3x^4y\\
&\quad -3(1-\sqrt{-1}b^3)(-1+\sqrt{-1}b^3+A(-3+3\sqrt{-1}b^3+(-1+\sqrt{-1}b^3)A^2\\
&\qquad -(-1+\omega)b^6+A(-3+3\sqrt{-1}b^3+(2+\omega)b^6)))x^3y^2\\
&\quad +3(1-\sqrt{-1}b^3+(1-\sqrt{-1}b^3)A^3+A(3-3\sqrt{-1}b^3+(-1+\omega)b^6)\\
&\qquad -A^2(-3+3\sqrt{-1}b^3+(2+\omega)b^6))x^2y^3+(1+A)^3xy^4=0,
\end{aligned}
$$

where $b \in k$ such that $b^2 = \alpha$ and $b^3 = -\sqrt{-1}(\sqrt{a}+1)$. Considering the blowing-ups of this curve at four infinitely near singular points over P, we conclude that $A = \omega^2$. Letting $c = -\sqrt{-1}\,b^3$ and taking the inverse image of the projective transformation

$$
\begin{pmatrix}
1/(\sqrt{-1}\,(1+c)) & 0 & 0\\
\sqrt{-1} & \sqrt{-1} & 0\\
0 & 0 & -2/(9(-\sqrt{-1}+\sqrt{3})(\sqrt{-1}\,c)^{2/3}c^2(1+c)^2)
\end{pmatrix},
$$

we obtain Equation (C2).

(4). The case $g(C) = 3$.

Then first, we infer that $L = |3Q + P_1 + P_2|$ from $l(3Q + P_1 + P_2) = 3$ and $L \subset |3Q + P_1 + P_2|$. On the other hand, we note that $l(P_1 + P_2) = 1$. Indeed, if $l(P_1 + P_2) = 2$ then we infer that $\Phi_{|P_1+P_2|} = \pi_R \circ \Phi_{|3Q+P_1+P_2|}$, where π_R is a projection of C from some point $R \in \mathbb{P}^2$. However, we have that $\deg \Phi_{|3Q+P_1+P_2|} = 1$ and $\deg \Phi_{|P_1+P_2|} = 2$, this contradicts that $\deg \pi_R \geq 3$. Next, we see that $P_1 + P_2 \sim \sigma^*(P_1 + P_2)$, because we have that $6Q - P_1 - P_2 \sim K_{\tilde{C}} \sim \sigma^* K_{\tilde{C}} \sim 6Q - \sigma^*(P_1 + P_2)$. Hence, we obtain that $P_1 + P_2 = \sigma^*(P_1 + P_2)$. Thus, we conclude that $L = \sigma^* L$ and the birational map $\Phi_L \circ \sigma \circ \Phi_L^{-1} : C \cdots \to C$ belongs to $PGL(3, k)$. Therefore, from Proposition 2, we may assume that C is given by the equation $y^2 + f_5(x, y) = 0$, where $f_5(x, y)$ is a form of degree five. However, the genus of a nonsingular model of this curve is equal to four. This contradicts that $g(C) = 3$.

(5). The case $g(C) = 4$.

We can check easily that if C is given by Equation (C3) or (C4), then the point $P = (0 : 0 : 1)$ is Galois.

Next, suppose that the point P is Galois. Then, we infer that $L = |3Q + P_1 + P_2|$ from $L \subset |3Q + P_1 + P_2|$ and $l(3Q + P_1 + P_2) = 3$. Now, we assume that $P_1 = P_2$. Then, by an argument similar to that in the proof of Claim 1, we conclude that $(C, T_P C)_P = 5$, and $P_1 = P_2 = \sigma(P_1) = \sigma(P_2)$. Thus, we see that $\sigma^* L = L$, and therefore we conclude that $\mathrm{Gal}(K/K_P) \subset PGL(3, k)$. From Proposition 2, by taking a suitable projective transformation, we obtain Equation (C3). Next, let us assume that $P_1 \neq P_2$. Then, since $\Phi_M(P_1) = \Phi_M(P_2)$, we may put that $P_2 = \sigma(P_1)$,

$P_3 := \sigma(P_2)$ and $P_1 = \sigma(P_3)$. Noting that $\tau = 5$, since $\sigma^*|5Q| = |5Q|$, the birational map $\Phi_{|5Q|} \circ \sigma \circ \Phi_{|5Q|}^{-1} : C_0 \cdots \to C_0$ belongs to $PGL(3,k)$. From Proposition 2, we may assume that $\Phi_{|5Q|}(Q) = (0:0:1)$ and C_0 is given by the equation $x^2 + f_5(x,y) = 0$, where $f_5(x,y)$ is a form of degree five. Moreover, by taking a suitable projective transformation, we may assume that $\Phi_{|5Q|}(P_3) = (1:0:1)$. Then, we have that $\Phi_{|5Q|}(P_1) = (\omega : 0 : 1)$ and $\Phi_{|5Q|}(P_2) = (\omega^2 : 0 : 1)$, hence, we conclude that

$$\boldsymbol{V}_{\tilde{C}_0}(N,D) = \langle 1, \frac{\iota^*(y)}{\iota^*(x)}, \frac{\iota^*(x)-1}{\iota^*(y)} \rangle.$$

Therefore, we obtain Equation (C4) (see Remark 3).

(6). The case $g(C) = 5$.

We can check easily that if C is given by Equation (C5), then the point $P = (0:0:1)$ is Galois.

Suppose that the point P is Galois. By an argument similar to that in (4) the case $g(C) = 3$, we conclude that $\sigma^* L = L$ and the birational map $\Phi_L \circ \sigma \circ \Phi_L^{-1} : C \cdots \to C$ belongs to $PGL(3,k)$. Therefore, from Proposition 2, by taking a suitable projective transformation, C is given by Equation (C5). Now we complete the proof of Theorem.

Remark 3. *In the previous proof, we can compute the defining equation of C from $\boldsymbol{V}_{\tilde{C}_0}(N,D)$ as follows. Let us assume that*

$$\boldsymbol{V}_{\tilde{C}_0}(N,D) = \langle 1, \phi_1(\iota^*(x), \iota^*(y)), \phi_2(\iota^*(x), \iota^*(y)) \rangle,$$

and $C_0 \subset \mathbb{P}^2$ is given by the equation $g(x,y) = 0$. Then, we put that $(Y/X) = \phi_1(\iota^*(x), \iota^*(y))$ and $(Z/X) = \phi_2(\iota^*(x), \iota^*(y))$, and we have that $g(\iota^*(x), \iota^*(y)) = 0$. Here, we eliminate $\iota^*(x)$ and $\iota^*(y)$ from these equations by elimination theory [9, Chapter *XI*]. *Thus, we obtain the defining equation of C.*

Corollary 1 and 2 is obvious from Theorem.

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