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PINCHING THEOREMS OF PSEUDO-UMBILICAL SUBMANIFOLDS

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ABSTRACT. In this paper, we obtain some pinching theorems for pseudo-umbilical submanifolds.

§1. INTRODUCTION

Let M be an n-dimensional submanifold in $\overline{M}^{n+p}$. One of the interesting questions in the geometry of submanifolds of $\overline{M}^{n+p}$ is to obtain conditions under which they are totally geodesic. These conditions generally involve the pinching of sectional curvatures, Ricci curvatures, or the scalar curvature. For the submanifolds with parallel mean curvature in sphere, there are many results ([F][S]). Now, in this paper, we will give a pinching condition for the norm of the second fundamental form under which the submanifolds is totally geodesic.

Simon's formula ([Si]) is a basic and useful tool in the study of some problems of global rigidity for submanifolds immersed in kind Riemannian manifolds. This formula is related to a special sort of submanifolds, those that have parallel second fundamental form, and allow us to characterize some submanifolds of this family or totally geodesic submanifolds ([MRU][L]). Now, in this paper, we firstly obtain

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a modified version of Simons formula and then use it to deal with the pinching problem for the submanifolds in $\overline{M}^{n+p}$.

Throughout this paper, we use the similar notations and formulas as those used in [MRU]. Let M be an n-dimensional compact Riemannian manifold. We denote by UM the unit tangent bundle over M and by UM_p its fibre at $p \in M$. For any continuous funtion $f: UM \to R$, we have

$$\int_{UM} f dv = \int_M \int_{UM_p} f dv_p dp$$

where dp, dv_p and dv stand for the canonical measures on M, UM_p and UM respectively.

If T is a k-covariant tensor on M and ∇T is covariant derivative, then we have ([R1])

$$\int_{UM} \{\sum_{i=1}^{n} (\nabla T)(e_i, e_i, v, \cdots .v)\} dv = 0 \tag{1.1}$$

where $e_1, \cdots e_n$ is an orthonormal basis of $T_pM, p \in M$.

Suppose now that M is isometrically immersed in an (n+p)-dimensional Riemannian manifold $\overline{M}^{n+p}$. We denote by $\langle , \rangle$ the metric of $\overline{M}$ as well as that induced on M. Let σ be the second fundamental form of the isometrically immersion and A_ξ the Weingarten endomorphism for a normal vector ξ. If T_pM and $T_p^\perp M$ denote the tangent and normal spaces to M at p, one can define

$$L: T_pM \to T_pM \quad \text{and} \quad T: T_p^\perp M \times T_p^\perp M \to R$$

by the expresions

$$Lv = \sum_{i=1}^{n} A_{\sigma(v,e_i)} e_i \quad \text{and} \quad T(\xi, \eta) = trace A_\xi A_\eta$$

where $e_1, \cdots e_n$ is an orthonormal basis of T_pM. Then L is a self-adjoint linear map and T a symmetric bilinear map.

There are many submanifolds satisfying $T = k <,>$. Obviously, hypersurfaces represent a trivial case. In $CP^{n+p}(c)$, a Kaehler submanifold of order $\{k_1, k_2\}$

for some natural numbers k_1 and k_2 is one submanifold of this type ([R3]). In this paper, we have a pinching theorem for this kind of submanifolds in $\overline{M}^{n+p}$ as following:

Theorem 3.1. *For each compact isometric immersion $M^n \to \overline{M}^{n+p}$ with $T = k\langle,\rangle$, we have, for all positive constant x ,*

$$\begin{aligned}&(\frac{2(n+2)}{n}+\frac{n+8}{2x})\langle H,\xi\rangle\\ \leq&[x(n+2)+\frac{n(n^2+8n+8)}{16x}]\cdot\frac{H^2|\sigma|^2}{p}+\frac{n^2+12n+40}{8npx}|\sigma|^4.\end{aligned}$$

Theorem 3.3. *Let $M^n \to \overline{M}^{n+p}$ be a pseudo-umbilic immersion. Suppose that $T = k\langle,\rangle$. If*

$$|\sigma|^2 \leq \frac{1}{c_3(n)}(pc_1(n) - c_2(n))H^2,$$

then M is totally geodesic.

§2. Some Lemmas

Lemma 2.1. *Let M be an n-dimensional compact submanifold isometrically immersed in a Riemannian manifold $\overline{M}^{n+p}$. Then we have*

$$\begin{aligned}\int_{UM_p}|A_{\sigma(v,v)}v|^2 dv_p \geq& \frac{2}{n+2}\int_{UM_p}\langle Lv, A_{\sigma(v,v)}v\rangle dv_p\\ &+\frac{1}{n+2}\int_{UM_p}\langle A_{\sigma(e_i,e_i)}v, A_{\sigma(v,v)}v\rangle dv_p\ .\end{aligned} \tag{2.1}$$

Proof. Let $\triangle$ denote the Laplace operator on S^{n-1}. Then, for the function $f : UM_p \to T_pM$ defined by $f(v) = A_{\sigma(v,v)}v$, we have

$$(\triangle f)(v) = -3(n+1)A_{\sigma(v,v)}v + 4Lv + 2A_{\sigma(e_i,e_i)}v.$$

Since UM_p is a $(n-1)$-dimensional sphere, the first eigenvalue of $-\triangle = \nabla_{\nabla_{e_i}e_i} - \nabla_{e_i}\nabla_{e_i}$ is $n-1$. Then

$$-\int_{UM_p}\langle\triangle f, f\rangle dv_p \quad \geq \quad (n-1)\int_{UM_p}|f|^2 dv_p$$

and the lemma follows. $\square$

Let α be a 1-form on UM_p defined by

$$\alpha_v(e) = \langle A_{\sigma(v,v)}e, A_{\sigma(v,v)}v\rangle$$

where $v \in UM_p$, and $e \in T_vUM_p$. If $e_1, \cdots, e_{n-1}$ is an orthonormal basis of T_vUM_p, then the codifferential of α is

$$\begin{aligned}(\delta\alpha) &= \sum_{i=1}^{n} e_i \cdot \alpha_v(e_i)\\ &= -(n+4)\left|A_{\sigma(v,v)}v\right|^2 + 2\langle Lv, A_{\sigma(v,v)}v\rangle\\ &\quad + T(\sigma(v,v), \sigma(v,v)) + 2\sum_{i=1}^{n}\langle A_{\sigma(v,v)}e_i, A_{\sigma(v,e_i)}v\rangle,\end{aligned}$$

where $e_1, \cdots, e_{n-1}, e_n = v$ is an orthonormal basis of T_pM. Now integrating the above equality over UM_p and using divergence theorem, we have

$$\begin{aligned}&2\int_{UM_p}\{\sum_{i=1}^{n}\langle A_{\sigma(v,v)}e_i, A_{\sigma(v,e_i)}v\rangle\}dv_p\\ =&(n+4)\int_{UM_p}\left|A_{\sigma(v,v)}v\right|^2 dv_p - 2\int_{UM_p}\langle Lv, A_{\sigma(v,v)}v\rangle dv_p\\ &-\int_{UM_p} T(\sigma(v,v), \sigma(v,v))dv_p \qquad (2.2)\end{aligned}$$

Lemma 2.2. *For any Riemannian immersion $M^n \to \overline{M}^{n+p}$, we have*

$$\begin{aligned}2\int_{UM_p}\sum_{i=1}^{n}\left|A_{\sigma(v,e_i)}v\right|^2 dv_p = &\int_{UM_p}\{\frac{n+4}{2}\left|f(v)\right|^2\\ &-\langle A_{nH}v, f(v)\rangle + \frac{1}{2}T(\sigma(v,v), \sigma(v,v))\}dv_p\ . \qquad (2.3)\end{aligned}$$

Proof. For the 1-form α defined by

$$\alpha_v(e) = \langle A_{\sigma(v,e)}v, A_{\sigma(v,v)}v\rangle,$$

we have

$$(\delta\alpha)(v) = \sum_{i=1}^{n}\{2\left|A_{\sigma(v,e_i)}v\right|^2 + \langle A_{\sigma(v,e_i)}v, A_{\sigma(v,v)}e_i\rangle$$
$$+\langle A_{\sigma(e_i,e_i)}v, A_{\sigma(v,v)}v\rangle\} - (n+4)\left|f(v)\right|^2 + \langle Lv, f(v)\rangle.$$

Integrating this and using (2.2), we get (2.3). □

Since

$$\begin{aligned} &2\sum_{i=1}^{n}\langle A_{\sigma(v,e_i)}v, A_{\sigma(v,v)}e_i\rangle \\ &\le b\sum_{i=1}^{n}\left|A_{\sigma(v,e_i)}v\right|^2 + \frac{1}{b}\sum_{i=1}^{n}\left|A_{\sigma(v,v)}e_i\right|^2 \\ &= b\sum_{i=1}^{n}\left|A_{\sigma(v,e_i)}v\right|^2 + \frac{1}{b}T(\sigma(v,v),\sigma(v,v)), \end{aligned} \tag{2.4}$$

where $b(>0)$ is a constant. By (2.2),(2.3) and (2.4), we have , for $\forall b > 0$,

$$\begin{aligned} &\int_{UM_p}\{(n+4-\frac{b(n+4)}{4})\left|f(v)\right|^2 - 2\langle Lv, f(v)\rangle \\ &-(1+\frac{b}{4}+\frac{1}{b})T(\sigma(v,v),\sigma(v,v))\}dv \le 0. \end{aligned} \tag{2.5}$$

Lemma 2.3. *Let $M^n \to \overline{M}^{n+p}$ be a compact Riemannian immersion. Then we have*

(1)

$$\begin{aligned} &\int_{UM_p}(n+2)\langle A_H v, f(v)\rangle dv_p \\ &= \int_{UM_p}\{2\sum_{i=1}^{n}\langle A_H e_i, A_{\sigma(v,e_i)}v\rangle + T(H,\sigma(v,v))\}dv_p\,. \end{aligned}$$

(2) $\int_{UM_p}\langle A_H v, Lv\rangle dv_p = \int_{UM_p}\sum_{i=1}^{n}\langle A_H e_i, A_{\sigma(v,e_i)}v\rangle dv_p$.

(3)

$$\begin{aligned} \int_{UM_p}\langle A_H v, Lv\rangle dv_p &= \frac{1}{n}\int_{UM_p}\sum_{i=1}^{n}\langle A_H e_i, Le_i\rangle dv_p \\ &= \frac{1}{n}\int_{UM_p}\langle H, \xi\rangle dv_p\,, \end{aligned}$$

where $\xi = \sum_{i=1}^{n} \sigma(e_i, Le_i)$.

(4) $\int_{UM_p} T(H, \sigma(v,v))dv_p = \int_{UM_p} T(H,H)dv_p$.

(5)

$$\int_{UM_p} (n+2)T(\sigma(v,v),\sigma(v,v))dv_p = \int_{UM_p} \{nT(H,\sigma(v,v)) + 2\sum_{i=1}^{n} T(\sigma(v,e_i),\sigma(v,e_i))\}dv_p .$$

(6)

$$\int_{UM_p} \sum_{i=1}^{n} T(\sigma(v,e_i),\sigma(v,e_i))dv_p = \frac{1}{n}\int_{UM_p} \sum_{i,j=1}^{n} T(\sigma(e_i,e_j),\sigma(e_i,e_j))dv_p ,$$

(7) $\int_{UM_p} \langle A_H v, f(v)\rangle dv_p = \int_{UM_p} \{\frac{1}{n+2}T(H,H) + \frac{2}{n(n+2)}\langle H,\xi\rangle\}dv_p$,

(8)

$$\int_{UM_p} T(\sigma(v,v),\sigma(v,v))dv_p = \int_{UM_p} \{\frac{n}{n+2}T(H,H) + \frac{2}{n(n+2)}\sum_{i,j=1}^{n} T(\sigma(e_i,e_j),\sigma(e_i,e_j))\}dv_p ,$$

(9)

$$\int_{UM_p} (2 - \frac{b(n+4)}{4})|f(v)|^2 dv_p \leq \int_{UM_p} \{(1+\frac{b}{4}+\frac{1}{b})T(\sigma(v,v),\sigma(v,v)) - (1+\frac{b}{2})n\langle A_H v, f(v)\rangle\}dv_p ,$$

for each b.

Proof. By taking some proper 1-form on UM_p respectively, we can obtain (1) $\sim$ (6) and then (7) and (8) as their corollaries. Using lemma 2.1, (2.5) implies (9). □

Remark. When $b(> 0)$ is small, (9) gives a estimation of the upper bound of $|f(v)|^2$.

§3. Main theorems and their Proofs

From Lemma 2.3, we can prove

Theorem 3.1. *For each compact isometric immersion $M^n \to \overline{M}^{n+p}$ with $T = k\langle,\rangle$, we have, for all positive constant x ,*

$$(\frac{2(n+2)}{n} + \frac{n+8}{2x})\langle H, \xi\rangle \le [x(n+2) + \frac{n(n^2+8n+8)}{16x}] \cdot \frac{H^2|\sigma|^2}{p} + \frac{n^2+12n+40}{8npx}|\sigma|^4. \tag{3.1}$$

Proof. For $\forall x > 0$, by (2) and (3) of Lemma 2.3, we have

$$\begin{aligned}\int_{UM_p} \frac{2}{n}\langle H, \xi\rangle dv_p &= \int_{UM_p} 2\sum_{i=1}^{n}\langle A_H e_i, A_{\sigma(v,e_i)}v\rangle dv_p \\ &\le \int_{UM_p}\{x\sum_{i=1}^{n}|A_H e_i|^2 + \frac{1}{x}\sum_{i=1}^{n}|A_{\sigma(v,e_i)}v|^2\}dv_p \\ &= \int_{UM_p}\{x\sum_{i=1}^{n}|A_H e_i|^2 + \frac{1}{x}[\frac{n+4}{4}|f(v)|^2 \\ &\quad - \frac{1}{2}\langle A_{nH}v, f(v)\rangle + \frac{1}{4}T(\sigma(v,v),\sigma(v,v))]\}dv_p\,.\end{aligned} \tag{3.2}$$

Substituting Lemma 2.3(9) with $b = \frac{4}{n+4}$ into (3.2), we have

$$\begin{aligned}\int_{UM_p} \frac{2}{n}\langle H, \xi\rangle dv_p \le \int_{UM_p} &\{xT(H,H) + \frac{1}{x}[(1+\frac{b}{4}+\frac{1}{b})\cdot\frac{n+4}{4} + \frac{1}{4}] \\ &\cdot T(\sigma(v,v),\sigma(v,v)) - \frac{1}{x}[\frac{n+4}{4}(1+\frac{b}{2})n + \frac{1}{2}n]\langle A_H v, f(v)\rangle\}dv_p\,.\end{aligned} \tag{3.3}$$

Since $T = k\langle,\rangle = \frac{|\sigma|^2}{p}\langle,\rangle$, we get

$$T(H,H) = \frac{|\sigma|^2}{p}H^2, \quad \sum_{i,j=1}^{n} T(\sigma(e_i,e_j),\sigma(e_i,e_j)) = \frac{|\sigma|^4}{p}. \tag{3.4}$$

Substituting (7) and (8) of Lemma 2.3 and (3.4) into (3.3), we obtain

$$\int_{UM_p} (\frac{2(n+2)}{n} + \frac{n+8}{2x})\langle H, \xi\rangle dv_p$$
$$\leq \int_{UM_p} \{[x(n+2) + \frac{n(n^2+8n+8)}{16x}]\frac{H^2|\sigma|^2}{p}$$
$$+ \frac{n^2+12n+40}{8npx}|\sigma|^4\}dv_p. \tag{3.5}$$

Because, at point $p \in M$, H, ξ and $|\sigma|^2$ in (3.5) are constants, we have (3.1) and the proof is finished. □

Remark. In the proof of this Theorem, we haven't used the modified Simons formula ([LC]) as in [MRU]. Here $\xi = \sum_{i=1}^{n} \sigma(e_i, Le_i)$ is also called *the third mean curvature vector* or B.Y.Chen's vector.

Corollary 3.2. *Let $M^n \to \overline{M}^{n+p}(p > \frac{n^2}{2})$ be a compact Riemannian immersion. Suppose that M is pseudo-umbilical and $T = k\langle,\rangle$. If*

$$|\sigma|^2 \leq \frac{p(3n^2+24n-8)}{n^2+12n+40}H^2, \tag{3.6}$$

then M is totally geodesic.

Proof. Since $\langle H, \xi\rangle = H^2|\sigma|^2$, by (3.1) with $x = n$, we have

$$(\frac{4(n+2)+(n+8)}{2n} - \frac{17n^2+40n+8}{16p})H^2 \leq \frac{n^2+12n+8}{8n^2p}|\sigma|^2. \tag{3.7}$$

If $p > \frac{n^2}{2}$, then

$$\text{left hand side of (3.7)} \geq \frac{3n^2+24n-8}{8n^2}H^2.$$

From these, the proof is finished. □

Remark. Corollary 3.2 removes the condition that M is Einstein and have parallel mean curvature vector, but unfortunately, it requirs that p is large enough.

Now, we define a map $g^1 : UM_p \to T_pM$ by

$$g^1(v) = A_{\sigma(v,v)}v - Lv.$$

By a direct computation, we have

$$(-\triangle g^1)(v) = 3(n+1)f(v) - (n+3)Lv - 2nA_Hv.$$

Here $\triangle$ is the Laplacian of UM_p. Since $\int_{UM_p} g^1(v)dv_p = 0$, we get

$$\int_{UM_p} \langle(-\triangle g^1)(v), g^1(v)\rangle \quad \geq \quad (n-1)\int_{UM_p} |g^1(v)|^2.$$

Then the above relation gives

$$\int_{UM_p} \{(2n+4)|f(v)|^2 - (2n+8)\langle Lv, f(v)\rangle$$
$$- 2n\langle f(v), A_Hv\rangle + 4|Lv|^2 + 2n\langle Lv, A_Hv\rangle\}dv_p \geq 0. \tag{3.8}$$

In a similar way, for the 1-form $g^2(v) = f(v) + Lv$, we have

$$\int_{UM_p} \{(2n+4)|f(v)|^2 - 2n\langle Lv, f(v)\rangle$$
$$- 2n\langle f(v), A_Hv\rangle - 4|Lv|^2 - 2n\langle Lv, A_Hv\rangle\}dv_p \geq 0. \tag{3.9}$$

By (3.8) and (3.9) we get

$$\int_{UM_p} \{(2n+4)|f(v)|^2 - (2kn+4k+4)\langle Lv, f(v)\rangle$$
$$- 2n\langle f(v), A_Hv\rangle + 4k|Lv|^2 - 2nk\langle Lv, A_Hv\rangle\}dv_p \geq 0. \tag{3.10}$$

Choosing $k = -\frac{2}{n+4}$ and using $\langle 2Lv + f(v), 2Lv + f(v)\rangle \;\geq\; 0$, (3.10) gives

$$\int_{UM_p} (2n+4+\frac{2}{n+4})|f(v)|^2 \geq \int_{UM_p} \{2n\langle f(v), A_Hv\rangle + \frac{4n}{n+4}\langle Lv, A_Hv\rangle\}dv_p.$$

On the other hand, by Lemma 2.3(9) with $b = \frac{4}{n+4}$, we have

$$\int_{UM_p} |f(v)|^2 dv_p \leq \int_{UM_p} \{(1 + \frac{1}{n+4} + \frac{n+4}{4})T(\sigma(v,v), \sigma(v,v))$$
$$- (1 + \frac{2}{n+4})n\langle A_Hv, f(v)\rangle\}dv_p.$$

So, from these, we get

$$\int_{UM_p}(1+\frac{1}{n+4}+\frac{n+4}{4})T(\sigma(v,v),\sigma(v,v))dv_p$$
$$\geq \int_{UM_p}\{[\frac{2n(n+4)}{(2n+4)(n+4)+2}+(1+\frac{2}{n+4})n]\langle f(v),A_Hv\rangle$$
$$+\frac{4n}{(n+4)(2n+4)+2}\langle A_Hv,f(v)\rangle\}dv_p.$$

From (3),(7) and (8) of Lemma 2.3, we have

$$c_1(n)\langle H,\xi\rangle \quad \leq \quad c_2(n)\frac{H^2|\sigma|^2}{p}+c_3(n)\frac{|\sigma|^4}{p}, \tag{3.11}$$

where

$$c_1(n)=(\frac{2(n+4)}{(2n+4)(n+4)+2}+(1+\frac{2}{n+4}))\cdot\frac{2}{n+2}+\frac{4}{(n+4)(2n+4)+2}$$
$$c_2(n)=(1+\frac{1}{n+4}+\frac{n+4}{4})\cdot\frac{n}{n+2}$$
$$-(\frac{2n(n+4)}{(2n+4)(n+4)+2}+(1+\frac{2}{n+4})n)\cdot\frac{1}{n+2}$$
$$c_3(n)=(1+\frac{1}{n+4}+\frac{n+4}{4})\cdot\frac{n}{n+2} \tag{3.12}$$

It is easy to see that $c_1(n)\sim\frac{2}{n}, c_2(n)\sim\frac{n}{4}, c_3(n)\sim\frac{1}{2n}$. If the immersion is pseudo-umbilic, we have

$$|\sigma|^2\geq\frac{1}{c_3(n)}(pc_1(n)-c_2(n))H^2.$$

So, by (3.10) and Lemma 2.1, we get

Theorem 3.3. *Let $M^n\to\overline{M}^{n+p}$ be a pseudo-umbilic immersion. Suppose that $T=k\langle,\rangle$. If*

$$|\sigma|^2\leq\frac{1}{c_3(n)}(pc_1(n)-c_2(n))H^2,$$

then M is totally geodesic.

Remark. When n is large enough, the pinching constant is about $(4p-\frac{n^2}{2})H^2$. In this case, the result of Theorem 3.3 is better than that of corollary 3.2 .

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