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Two classes of Lorentzian stationary surfaces in semi-Riemannian space forms

Makoto SAKAKI

Abstract. We give certain two classes of 2-dimensional Lorentzian metrics which can be realized as induced metrics of Lorentzian stationary surfaces in semi-Riemannian space forms.

1. Introduction

Let $N_\nu^n(c)$ denote the n-dimensional simply connected semi-Riemannian space form of constant curvature c and index ν. A surface in $N_\nu^n(c)$ is called Lorentzian if its induced metric is Lorentzian. We say that a Lorentzian surface in $N_\nu^n(c)$ is stationary if its mean curvature vector vanishes identically. We are interested in the following question: Which 2-dimensional Lorentzian metrics can be realized as induced metrics of Lorentzian stationary surfaces in $N_\nu^n(c)$?

There are several related results for minimal surfaces in Riemannian space forms (cf. [4], [5], [2], [3]). In the previous paper [7], refering to [3], we gave two classes of 2-dimensional Riemannian metrics which can be realized as spacelike stationary surfaces in $N_\nu^n(c)$. In this paper, we will give two classes of 2-dimensional Lorentzian metrics which can be realized as Lorentzian stationary surfaces in $N_\nu^n(c)$.

Let M be a 2-dimensional Lorentzian manifold with Gaussian curvature K and Laplacian Δ. For each real number c, set

$$F_1^c = 2(K-c), \quad F_{p+1}^c = F_p^c + 2(p+1)K - \sum_{q=1}^{p} \Delta \log(F_q^c) \ \text{ if } \ F_p^c > 0.$$

Our results are stated as follows.

Theorem 1. *Let M be a 2-dimensional simply connected Lorentzian manifold. Suppose that $F_p^c > 0$ for $p < m$, and $F_m^c = 0$ identically. Then there* *exists an isometric stationary immersion of M into $N_m^{2m}(c)$.*

Theorem 2. *Let M be a 2-dimensional simply connected Lorentzian manifold with metric ds^2. Suppose that $F_p^c > 0$ for $p \leq m$, and the metric $d\hat{s}^2 = (\prod_{p=1}^m F_p^c)^{1/(m+1)} ds^2$ is flat. Then there exists a one-parameter family of isometric stationary immersions of M into $N_m^{2m+1}(c)$.*

Remark. The conditions of the theorems may be seen as generalized Ricci conditions (cf. [4], [3], [7]).

Using Theorems 1 and 2, we may obtain Lorentzian stationary surfaces with constant curvature in pseudo-hyperbolic spaces.

Corollary 1. *For every positive integer m, there exists an isometric stationary immersion of $N_1^2(-2/m(m+1))$ into $N_m^{2m}(-1)$.*

Corollary 2. *For every positive integer m, there exists a one-parameter family of isometric stationary immersions of the Minkowski plane R_1^2 into $N_m^{2m+1}(-1)$.*

Remarks. (i) See [1] for minimal surfaces with constant curvature in Riemannian space forms.

(ii) The author does not know the explicit representations of the surfaces in the corollaries.

In Section 5, we show the existence of 2-dimensional Lorentzian metrics with nonconstant curvature which satisfy the conditions of the theorems.

2. Preliminaries

Unless otherwise stated, we use the following conventions on the ranges of indices:

$$1 \leq i, j, \cdots \leq 2, \quad 3 \leq \alpha, \beta, \cdots \leq n, \quad 1 \leq A, B, \cdots \leq n.$$

Let $\{e_A\}$ be a local orthonormal frame field in $N_\nu^n(c)$, and $\{\omega^A\}$ be the dual coframe. Here the metric of $N^n_\nu(c)$ is given by

$$ds^2 = (\omega^1)^2 - (\omega^2)^2 + \sum_{A=3}^{n-\nu+1} (\omega^A)^2 - \sum_{A=n-\nu+2}^{n} (\omega^A)^2.$$

Set $I_1 = \{1, 3, 4, \cdots, n-\nu+1\}$ and $I_2 = \{2, n-\nu+2, n-\nu+3, \cdots, n\}$. We can define the connection forms $\{\omega^A_B\}$ by

$$de_B = \sum_A \omega^A_B e_A.$$

Then $\omega^A_B + \omega^B_A = 0$ if $A, B \in I_1$, or $A, B \in I_2$. And $\omega^A_B = \omega^B_A$ if $A \in I_1, B \in I_2$. The structure equations are given by

$$d\omega^A = -\sum_B \omega^A_B \wedge \omega^B,$$

$$d\omega^A_B = -\sum_C \omega^A_C \wedge \omega^C_B + \frac{1}{2} \sum_{C,D} R^A{}_{BCD} \omega^C \wedge \omega^D,$$

$$R^A{}_{BCD} = c\varepsilon_B (\delta^A{}_C \delta_{BD} - \delta^A{}_D \delta_{BC}),$$

where $\varepsilon_B = 1$ for $B \in I_1$, and $\varepsilon_B = -1$ for $B \in I_2$.

Let M be a Lorentzian surface in $N^n_\nu(c)$. We choose the frame $\{e_A\}$ so that $\{e_i\}$ are tangent to M. Then $\omega^\alpha = 0$ on M. In the following, our argument will be restricted to M. Then we have

$$0 = d\omega^\alpha = -\sum_i \omega^\alpha_i \wedge \omega^i.$$

So there is a symmetric tensor h^α_{ij} such that

$$\omega^\alpha_i = \sum_j h^\alpha_{ij} \omega^j,$$

where h^α_{ij} are the components of the second fundamental form h of M. The Gaussian curvature K of M is given by

$$d\omega^1_2 = -K\omega^1 \wedge \omega^2.$$

The mean curvature vector H of M is given by

$$H = \frac{1}{2} \sum_\alpha (h^\alpha_{11} - h^\alpha_{22}) e_\alpha.$$

We say that M is stationary if $H = 0$ on M.

3. Proof of Theorem 1

Proof of Theorem 1. We choose an orthonormal frame field $\{e_1, e_2\}$ on M with dual coframe $\{\omega^1, \omega^2\}$. Here the metric on M is given by

$$ds^2 = (\omega^1)^2 - (\omega^2)^2.$$

Let $\omega_2^1 = \omega_1^2$ be the connection form satisfying

$$d\omega^1 = -\omega_2^1 \wedge \omega^2, \quad d\omega^2 = -\omega_1^2 \wedge \omega^1.$$

Unless otherwise stated, we use the following conventions on the ranges of indices:

$$1 \le i, j, \cdots \le 2, \quad 3 \le \alpha, \beta, \cdots \le 2m, \quad 1 \le A, B, \cdots \le 2m.$$

Let E be a vector bundle of rank $2m-2$ over M with orthonormal sections $\{e_\alpha\}$ such that $\langle e_\alpha, e_\beta \rangle = \tilde{\varepsilon}_\alpha \delta_{\alpha\beta}$, where $\tilde{\varepsilon}_\alpha = 1$ if α is odd, and $\tilde{\varepsilon}_\alpha = -1$ if α is even. Let h be a symmetric section of $\mathrm{Hom}(TM \times TM, E)$ such that

$$(h_{ij}^3) = \begin{pmatrix} 0 & \sqrt{F_1^c}/2 \\ \sqrt{F_1^c}/2 & 0 \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} \sqrt{F_1^c}/2 & 0 \\ 0 & \sqrt{F_1^c}/2 \end{pmatrix},$$

$$(h_{ij}^5) = \cdots = (h_{ij}^{2m}) = (0).$$

For $1 \le p \le m-1$, set

$$\omega_{2p+1}^{2p-1} = -\omega_{2p-1}^{2p+1} = -(\sqrt{F_p^c}/2)\omega^2, \quad \omega_{2p+1}^{2p} = \omega_{2p}^{2p+1} = (\sqrt{F_p^c}/2)\omega^1,$$

$$\omega_{2p+2}^{2p-1} = \omega_{2p-1}^{2p+2} = (\sqrt{F_p^c}/2)\omega^1, \quad \omega_{2p+2}^{2p} = -\omega_{2p}^{2p+2} = -(\sqrt{F_p^c}/2)\omega^2,$$

$$\omega_{2p+2}^{2p+1} = \omega_{2p+1}^{2p+2} = (p+1)\omega_2^1 + \frac{1}{2}\sum_{q=1}^{p} *d\log(F_q^c),$$

and

$$\omega_B^A = 0 \quad \text{otherwise},$$

where $*$ is the Hodge star operator given by $*\omega^1 = \omega^2$ and $*\omega^2 = \omega^1$. We note that

$$d * df = (\Delta f)\omega^1 \wedge \omega^2$$

for a smooth function f on M. We define a compatible connection ${}^\perp\nabla$ of E so that

$${}^\perp\nabla e_\beta = \sum_\alpha \omega_\beta^\alpha e_\alpha.$$

By a computation using the condition of Theorem 1, we can show that $\{\omega_B^A\}$ satisfy the structure equations:

$$d\omega_2^1 = -\sum_\alpha \omega_\alpha^1 \wedge \omega_2^\alpha - c\omega^1 \wedge \omega^2,$$

$$d\omega_\alpha^1 = -\omega_2^1 \wedge \omega_\alpha^2 - \sum_\beta \omega_\beta^1 \wedge \omega_\alpha^\beta, \quad d\omega_\alpha^2 = -\omega_1^2 \wedge \omega_\alpha^1 - \sum_\beta \omega_\beta^2 \wedge \omega_\alpha^\beta,$$

$$d\omega_\beta^\alpha = -\sum_i \omega_i^\alpha \wedge \omega_\beta^i - \sum_\gamma \omega_\gamma^\alpha \wedge \omega_\beta^\gamma,$$

which are the integrability conditions. Therefore, by the fundamental theorem, there exists an isometric immersion of M into $N_m^{2m}(c)$, with second fundamental form h and normal connection ${}^\perp\nabla$. So it is stationary, and we get the conclusion.

Remark. To show the structure equations above, it is convenient to separate the cases for

$$d\omega_2^1, \quad d\omega_{2p+2}^{2p+1}\ (1 \le p \le m-2), \quad d\omega_{2m}^{2m-1},$$

$$d\omega_{2p+1}^{2p-1}, \ d\omega_{2p+2}^{2p-1}, \ d\omega_{2p+3}^{2p-1}, \ d\omega_{2p+4}^{2p-1}, \ d\omega_{2p+1}^{2p}, \ d\omega_{2p+2}^{2p}, \ d\omega_{2p+3}^{2p}, \ d\omega_{2p+4}^{2p}$$

$(1 \le p \le m-1)$, and other trivial ones.

Proof of Corollary 1. For $N_1^2(-2/m(m+1))$, we have

$$F_p^{-1} = 2 - \frac{2p(p+1)}{m(m+1)}$$

for $1 \le p \le m$. Hence by Theorem 1, there exists an isometric stationary immersion of $N_1^2(-2/m(m+1))$ into $N_m^{2m}(-1)$.

Through the natural anti-isometries (cf. [6, p.110]), Corollary 1 is equivalent to the following:

Corollary 3. *For every positive integer m, there exists an isometric stationary immersion of $N_1^2(2/m(m+1))$ into $N_m^{2m}(1)$.*

4. Proof of Theorem 2

Proof of Theorem 2. We choose an orthonormal coframe field $\{\omega^1, \omega^2\}$ on M, with connection form $\omega_2^1 = \omega_1^2$. Here the metric on M is given by

$$ds^2 = (\omega^1)^2 - (\omega^2)^2.$$

Unless otherwise stated, we use the following conventions on the ranges of indices:

$$1 \le i, j, \cdots \le 2, \quad 3 \le \alpha, \beta, \cdots \le 2m+1, \quad 1 \le A, B, \cdots \le 2m+1.$$

Let E be a vector bundle of rank $2m-1$ over M with orthonormal sections $\{e_\alpha\}$ such that $\langle e_\alpha, e_\beta \rangle = \tilde{\varepsilon}_\alpha \delta_{\alpha\beta}$, where $\tilde{\varepsilon}_\alpha = 1$ if α is odd, and $\tilde{\varepsilon}_\alpha = -1$ if α is even. Let h be a symmetric section of $\mathrm{Hom}(TM \times TM, E)$ such that

$$(h_{ij}^3) = \begin{pmatrix} 0 & \sqrt{F_1^c}/2 \\ \sqrt{F_1^c}/2 & 0 \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} \sqrt{F_1^c}/2 & 0 \\ 0 & \sqrt{F_1^c}/2 \end{pmatrix},$$

$$(h_{ij}^5) = \cdots = (h_{ij}^{2m+1}) = (0).$$

Let $\{\omega_B^A\}$ $(1 \le A, B \le 2m)$ be defined as in the proof of Theorem 1.

The flatness of the metric $d\hat{s}^2$ is equivalent to

$$\sum_{p=1}^{m} \Delta \log(F_p^c) = 2(m+1)K.$$

So the equation

$$dt = -(m+1)\omega_2^1 - \frac{1}{2}\sum_{p=1}^{m} *d\log(F_p^c)$$

is integrable. Let t be a solution of this equation. For each real number θ, set

$$\omega_{2m+1}^{2m-1} = -\omega_{2m-1}^{2m+1} = \sqrt{F_m^c/2}\{\sinh(t+\theta)\omega^1 - \cosh(t+\theta)\omega^2\},$$

$$\omega^{2m}_{2m+1} = \omega^{2m+1}_{2m} = \sqrt{F^c_m/2}\{\cosh(t+\theta)\omega^1 - \sinh(t+\theta)\omega^2\},$$
$$\omega^A_{2m+1} = \omega^{2m+1}_A = 0 \quad \text{for} \quad 1 \leq A \leq 2m-2.$$

We define a compatible connection ${}^\perp\nabla$ of E so that

$${}^\perp\nabla e_\beta = \sum_\alpha \omega^\alpha_\beta e_\alpha.$$

By a computation, we can show that $\{\omega^A_B\}$ satisfy the structure equations. Hence, there exists a one-parameter family of isometric stationary immersions of M into $N^{2m+1}_m(c)$.

Remark. To show the structure equations above, we should consider the cases for $d\omega^{2m-1}_{2m}$ and $d\omega^A_{2m+1}$. Other cases are the same as in the proof of Theorem 1.

Proof of Corollary 2. It is immediate from Theorem 2.

5. A remark

Here we show the existence of 2-dimensional Lorentzian metrics with nonconstant curvature which satisfy the conditions of the theorems. Let

$$ds^2 = e^{2u}(dx^2 - dy^2)$$

be a 2-dimensional Lorentzian metric, where $u = u(x)$ is a smooth function depending only on x. Then we have

$$\Delta = e^{-2u}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right), \qquad K = -e^{-2u}u'', \qquad K' = 2e^{-2u}u'u'' - e^{-2u}u''',$$

and F^c_p is represented by $u^{(k)}$ $(0 \leq k \leq 2p)$.

The condition $F^c_m = 0$ of Theorem 1 becomes an ordinary differential equation for u of $2m$-th order. For $m \geq 2$, choosing a suitable initial condition at $x = 0$ so that $F^c_p(0) > 0$ $(1 \leq p \leq m-1)$ and $K'(0) \neq 0$, we can show the existence of a solution u of $F^c_m = 0$. This u gives a 2-dimensional Lorentzian metric with nonconstant curvature satisfying the condition of Theorem 1.

The condition of Theorem 2 becomes an ordinary differential equation for u of $(2m+2)$-th order. For $m \geq 1$, choosing a suitable initial condition as above, we can show the existence of 2-dimensional Lorentzian metrics with nonconstant curvature satisfying the condition of Theorem 2.

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Department of Mathematical System Science
Faculty of Science and Technology
Hirosaki University
Hirosaki 036-8561
Japan
E-mail: sakaki@cc.hirosaki-u.ac.jp

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