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An Example of Non-Reducing Eigenspace of a Paranormal Operator

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Abstract

In this paper, we show an example of paranormal operator which has non-reducing eigenspace belonging to non-zero eigenvalue.

Introduction. A bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is called p-hyponormal, class A, p-quasihyponormal and paranormal iff $(T^*T)^p \geq (TT^*)^p$, $|T^2| \geq |T|^2$, $T^*\{(T^*T)^p-(TT^*)^p\}T \geq 0$ and $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for arbitrary $x \in \mathcal{H}$ respectively. It is well-known that the classes of all p-hyponormal, class A and p-quasihyponormal operators for $0 < p < 1$ are subclasses of paranormal operators and that any eigenspace belonging to non-zero eigenvalue of a p-hyponormal, class A or p-quasihyponormal operator T is always a reducing subspace of T. Also there is an example of quasihyponormal (hence it is class A), of which kernel is not a reducing subspace. So, it is interesting whether any eigenspace belonging to non-zero eigenvalue of a paranormal operator T is a reducing subspace of T or not.

In this paper, we show an example of paranormal operator which has a non-reducing eigenspace belonging to non-zero eigenvalue.

Definition. For arbitrary vectors x and y in a complex separable Hilbert space $\mathcal{H}$, we define the operator $x \otimes y$ on $\mathcal{H}$ by

$$(x \otimes y)z = \langle z, y\rangle x \quad \text{for all } z \in \mathcal{H}.$$

This operator $x \otimes y$ is called Schatten form and satisfies the followings.

1) $T(x \otimes y) = Tx \otimes y$, $(x \otimes y)T = x \otimes T^*y$ for $T \in \mathcal{B}(\mathcal{H})$.

2) $(x \otimes y)^* = y \otimes x$

The next theorem shows that for a paranormal operator T and eigenvalue λ of T, $\ker(T - \lambda)$ is not necessarily a reducing subspace of T even if $\lambda \neq 0$.

Theorem. Let $\{e_n\}_{n\geq 1}$ be the canonical base for $\ell^2(\mathbb{N})$, $\mathcal{H} = \mathbb{C}e_1 \oplus \ell^2(\mathbb{N})$, $\alpha > 0$ and U the unilateral shift on $\ell^2(\mathbb{N})$ defined by

$$Ue_n = e_{n+1} \quad \text{for } n \geq 1.$$

Then the operator T on $\mathcal{H}$ defined by

$$T = \begin{pmatrix} 1 & \alpha e_1 \otimes e_1 \\ 0 & U+1 \end{pmatrix}, \text{ on } \mathcal{H} = \mathbb{C}e_1 \oplus \ell^2(\mathbb{N}) \tag{1}$$

is paranormal for sufficiently small $\alpha > 0$. In fact, it is enough $\alpha \leq \frac{1}{4}$. Also $\ker(T-1) = \mathbb{C}e_1 \oplus \{0\}$ does not reduce T.

Proof. By Ando's characterization of paranormal operator [1], it suffices to show that

$$T^{*2}T^2 - 2kT^*T + k^2 \geq 0 \quad \text{for all } k > 0.$$

Since, $T^2 = \begin{pmatrix} 1 & 2\alpha e_1 \otimes e_1 \\ 0 & (U+1)^2 \end{pmatrix}$, we have

$$\begin{aligned}
& T^{*2}T^2 - 2kT^*T + k^2 \\
= & \begin{pmatrix} 1 & 2\alpha e_1 \otimes e_1 \\ 2\alpha e_1 \otimes e_1 & 4\alpha^2 e_1 \otimes e_1 + (U+1)^{*2}(U+1)^2 \end{pmatrix} \\
& \qquad -2k \begin{pmatrix} 1 & \alpha e_1 \otimes e_1 \\ \alpha e_1 \otimes e_1 & \alpha^2 e_1 \otimes e_1 + (U+1)^*(U+1) \end{pmatrix} + k^2 \\
= & \begin{pmatrix} (k-1)^2 & 2(1-k)\alpha e_1 \otimes e_1 \\ 2(1-k)\alpha e_1 \otimes e_1 & 4\alpha^2 e_1 \otimes e_1 - 2k\alpha^2 e_1 \otimes e_1 + (U+1)^{*2}(U+1)^2 - 2k(U+1)^*(U+1) + k^2 \end{pmatrix}.
\end{aligned}$$

If $k = 1$, then

$$T^{*2}T^2 - 2T^*T + 1 = 0 \oplus \left(2\alpha^2 e_1 \otimes e_1 + (U+1)^{*2}(U+1)^2 - 2(U+1)^*(U+1) + 1\right) \geq 0$$

because $U+1$ is hyponormal(hence it is paranormal) and $e_1 \otimes e_1$ is the orthogonal projection onto $\mathbb{C}e_1$.

For $k \neq 1$, we have only to prove that

$$(U+1)^{*2}(U+1)^2 - 2k(U+1)^*(U+1) + k^2 - 2k\alpha^2 e_1 \otimes e_1 \geq 0,$$

because if this holds, then

$$\begin{aligned}
T^{*2}T^2 - 2kT^*T + k^2 & \geq \begin{pmatrix} (k-1)^2 & 2(1-k)\alpha e_1 \otimes e_1 \\ 2(1-k)\alpha e_1 \otimes e_1 & 4\alpha^2 e_1 \otimes e_1 \end{pmatrix} \\
& = \begin{pmatrix} 1-k & 0 \\ 0 & 2\alpha e_1 \otimes e_1 \end{pmatrix} \begin{pmatrix} 1 & e_1 \otimes e_1 \\ e_1 \otimes e_1 & 1 \end{pmatrix} \begin{pmatrix} 1-k & 0 \\ 0 & 2\alpha e_1 \otimes e_1 \end{pmatrix} \\
& \geq 0.
\end{aligned}$$

Put $S = (U+1)^*(U+1)$. Then $0 \le S \le 4$,

$$(U+1)^{*2}(U+1)^2 = (U+1)^*\{(U+1)(U+1)^* + e_1 \otimes e_1\}(U+1) = S^2 + e_1 \otimes e_1,$$

and hence

$$(U+1)^{*2}(U+1)^2 - 2k(U+1)^*(U+1) + k^2 - 2k\alpha^2 e_1 \otimes e_1 = (S-k)^2 + (1-2k\alpha^2)e_1 \otimes e_1. \qquad (2)$$

Let $\alpha \le \frac{1}{4}$. Then $1 - 2k\alpha^2 \ge 0$ if $k \le 8$, hence (2) is positive. If $k \ge 8$, then $k - S \ge k - 4 > 0$ and $(k-S)^2 \ge (k-4)^2$, so we have

$$\begin{aligned}(2) &\ge (k-4)^2 - 2k\alpha^2 e_1 \otimes e_1 \\ &\ge (k-4)^2 - 2k = (k-2)(k-8) \ge 0 \quad \text{for } k \ge 8.\end{aligned}$$

References

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