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A Characterization of Stability of Discrete Hopfield Neural Networks

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Abstract

We discuss the stability of discrete Hopfield neural networks (Shortly, DHNNs) in synchronous mode. To do this, we introduce an equivalent relation in the set $M_{m,n}(\mathbb{R})$ of all m by n matrices over the real field $\mathbb{R}$ and then obtain a classification of matrices. Thanks to this classification we establish a classification of all discrete Hopfield neural networks with n neurons in such a way that two DHNNs belong to the same class if and only if they have the same dynamic property. Lastly, a characterization of the stability of a DHNN with two neurons in synchronous mode is obtained.

1. Introduction

Discrete Hopfield neural networks(DHNNs) were proposed mainly as an associative memory model by Hopfield in [1]. A discrete Hopfield neural network(DHNN) can be viewed as a single layer consisting of n neurons which are connected each other. Each neuron in the networks is in one of two possible states, either 1 or -1. The state of neuron i at time t is denoted by $v_i(t)$. The state of the network at time t is denoted by the vector $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$.

In recent years, for the purpose of associative memory and combinatorial optimization, several schemes have been established in light of generalized Hopfield neural networks [2-4]. A network can operate in different modes. If the computation is performed at all neurons at the same time, we say that the network is operating in synchronous mode. If the computation is performed only at a single neuron at each time, we say that the network is operating in asynchronous mode. The general form of a DHNN with n neurons in synchronous mode can be described as follows.

$$v_i(t+1) = \operatorname{sgn}^*\left(\sum_{j=1}^{n} w_{ij}v_j(t) - \theta_i\right) \; (i = 1, 2, \ldots, n) \tag{1.1}$$

where

$$(v_1(t), v_2(t), \ldots, v_n(t)) \in \{-1, 1\}^n \equiv V^n,$$

$$[w_{ij}] \in M_n(\mathbb{R}), (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n,$$

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and

$$\text{sgn}^* s = \begin{cases} 1 & \text{if } s \geq 0, \\ -1 & \text{if } s < 0. \end{cases}$$

The vector $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$ is called the state at t of the neurons, $W = [w_{ij}]$ and $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$ are called the weight-matrix and the threshold of the system (1.1), respectively.

Put

$$W(\Theta) = [W \quad \Theta^T], v(t) = (v_1(t), v_2(t), \ldots, v_n(t), -1), \tag{1.2}$$

$$\text{sgn}^*(x_1, x_2, \ldots, x_n)^T = (\text{sgn}^* x_1, \text{sgn}^* x_2, \ldots, \text{sgn}^* x_n)^T, \tag{1.3}$$

then the system (1.1) can be rewritten as

$$v(t+1)^T = \text{sgn}^* W(\Theta) v(t)^T, \tag{1.4}$$

which is denoted simply by (W, Θ).

Definition 1.1. A network (W, Θ) is called to be stable if for every initial state $v(0) \in V^n$, there exists a integer $t_0 \geq 0$ such that $v(t+1) = v(t)$ for all integer $t \geq t_0$. If (W, Θ) is not stable, then we call it to be unstable.

Remark 1.1. From Definition 1.1, we know that a network (W, Θ) is stable if and only if there is an integer $t_0 \geq 0$, such that $v(t+1) = v(t)$ for all integers $t \geq t_0$ since V^n is finite.

The stability of a DHNN is a very important property and has been developed in the literatures [5-8]. In fact, these scheme for the purpose of associative memory and combinatorial optimization are all based on the stability of the network. In recent years, many authors had studied the stability of the network by introducing a special energy function [1,2,5-8]. But the results obtained are only sufficient conditions. In present paper, we introduce an equivalent relation in the set $M_{m,n}(\mathbf{R})$ of all m by n matrices over the real field $\mathbf{R}$ and then obtain a classification of matrices. With this classification we establish a classification of all discrete Hopfield neural networks with n neurons in such a way that two DHNNs belong to the same class if and only if they have the same dynamic property. Lastly, a sufficient and necessary condition about the stability of DHNN with two neurons in synchronous mode is obtained.

2. Equi-Activity of Matrices

In the sequel, we use $M_{m,n}(\mathbf{R})$ to denote the set of all m by n matrices over $\mathbf{R}$ and identify $M_{1,n}(\mathbf{R})$ with $\mathbf{R}^n$.

Definition 2.1. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_{m,n}(\mathbb{R})$ are called equi-active and written $A \approx B$ if

$$\mathrm{sgn}^*(Av^T) = \mathrm{sgn}^*(Bv^T), \quad \forall v \in V^n. \tag{2.1}$$

It is easy to check that the relation $\approx$ is an equivalent relation in $M_{m,n}(\mathbb{R})$. That is, the following statements hold:

(i) $A \approx A$;

(ii) $A \approx B \Rightarrow B \approx A$;

(iii) $A \approx B$ and $B \approx C \Rightarrow A \approx C$;

for all A, B, C in $M_{m,n}(\mathbb{R})$.

Thus, with the relation we obtain a classification $M_{m,n}(\mathbb{R})/\approx$ of $M_{m,n}(\mathbb{R})$. For $A \in M_{m,n}(\mathbb{R})$, denote by $E(A)$ the equivalent class of A, that is,

$$E(A) = \{B \in M_{m,n}(\mathbb{R}) | B \approx A\}.$$

We also use the following symbol.

$$E(M_{m,n}(\mathbb{R})) = \{E(A) : \ A \in M_{m,n}(\mathbb{R})\}.$$

Remark 2.1. For matrices $[W_1, \Theta_1]$ and $[W_2, \Theta_2]$ in $M_{n,n+1}(\mathbb{R})$, if $[W_1, \Theta_1] \approx [W_2, \Theta_2]$, then the DHNNs (W_1, Θ_1) and (W_2, Θ_2) completely have the same dynamic property and therefore have the same stabilities.

From [5] we know that if $W \in M_n(\mathbb{R})$ is positive definite, then for each $\Theta \in \mathbb{R}^n$, the network (W, Θ) is stable. By Remark 2.1 we can get the following result.

Corollary 2.1. If $W \in M_n(\mathbb{R})$ is equi-active to a positive definite matrix, then the system $(W, 0)$ is stable in synchronous mode.

For example, let

$$A = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

It is easy to check that A, B and C are equi-active and that A is positive definite, B is positive semidefinite and C is indefinite. From Corollary 2.1 we know that they are all stable.

Lemma 2.1. Let w, v be in $\mathbb{R}^n$ with $\mathrm{sgn}^* w = \mathrm{sgn}^* v$, then

$$\mathrm{sgn}^*(w + v) = \mathrm{sgn}^* w. \tag{2.2}$$

Proposition 2.1. For each A in $M_{m,n}(\mathbb{R}), E(A)$ is a cone.

Proof. Let $X, Y \in E(A)$ and $s > 0$, then for each $v = (v_1, v_2, \ldots, v_n) \in V^n$, we have, from Lemma 2.1, that

$$\mathrm{sgn}^*((sX)v^T) = \mathrm{sgn}^*(Xv^T) = \mathrm{sgn}^*(Av^T),$$

$$\mathrm{sgn}^*((X+Y)v^T) = \mathrm{sgn}^*(Xv^T + Yv^T) = \mathrm{sgn}^*(Xv^T) = \mathrm{sgn}^*(Av^T).$$

Thus sX and $X + Y$ are in $E(A)$. This completes the proof.

Let $\langle x, y\rangle$ denote the inner product of vectors x and y in $\mathbb{R}^n$. For a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define

$$l(x) = \min\{(\sum_{i=1}^{n} x_i v_i)^2 | \ (v_1, v_2, \ldots, v_n) \in V^n\}, \tag{2.3}$$

$$\|x\|_2 = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} \tag{2.4}$$

and

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}. \tag{2.5}$$

For a matrix $A = [a_{ij}] \in M_{m,n}(\mathbb{R})$, set

$$A_i = (a_{i1}, a_{i2}, \ldots, a_{in})(i = 1, 2, \ldots, m) \tag{2.6}$$

and

$$\|A\|_2 = (\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}^2)^{\frac{1}{2}}. \tag{2.7}$$

In the case where $A_i \neq 0$ for $i = 1, 2, \ldots, m$, we define

$$\alpha(A) = \min\{\frac{1}{n\|A_1\|_2} l(A_1), \ldots, \frac{1}{n\|A_m\|_2} l(A_m)\}. \tag{2.8}$$

Proposition 2.2. Let $A, B \in M_{m,n}(\mathbb{R})$ such that $\alpha(A) > 0$, then
(a) If $\|B\|_2 < \alpha(A)$, *then* $B + A \in E(A)$;
(b) If $\|B - A\|_2 < \alpha(A)$, *then* $B \in E(A)$;
(c) $E(A)$ is an open set in $(M_{m,n}(\mathbb{R}), \|\cdot\|_2)$.

Proof. (a) For every v in V^n, we have

$$\mathrm{sgn}^*(B+A)v^T = (\mathrm{sgn}^*(B_1 + A_1)v^T, \mathrm{sgn}^*(B_2 + A_2)v^T, \ldots, \mathrm{sgn}^*(B_m + A_m)v^T),$$

$$\mathrm{sgn}^* Av^T = (\mathrm{sgn}^* A_1 v^T, \mathrm{sgn}^* A_2 v^T, \ldots, \mathrm{sgn}^* A_m v^T).$$

Let $\|B\|_2 < \alpha(A)$. Then

$$\|B_i\|_2 \leq \|B\|_2 < \alpha(A) \leq \frac{1}{n\|A_i\|_2} l(A_i).$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}(B_i + A_i)v^T \cdot A_i v^T &= (\langle A_i, v\rangle)^2 + \langle B_i, v\rangle \cdot \langle A_i, v\rangle \\ &\geq l(A_i) - n\|B_i\|_2\|A_i\|_2 \\ &> 0.\end{aligned}$$

This shows that $\mathrm{sgn}^*(B_i+A_i)v^T = \mathrm{sgn}^* A_i v^T (i = 1, 2, \ldots, m)$ and so $\mathrm{sgn}^*(B+A)v^T = \mathrm{sgn}^* Av^T$ for all v in V^n. This shows that $B + A \in E(A)$.

(b) Let $\|B - A\|_2 < \alpha(A)$, then by (a) $B = (B - A) + A \in E(A)$.

(c) This is proved from (b). This completes the proof.

Proposition 2.3. Let $A, B \in M_{m,n}(\mathbb{R})$, then

$$A \approx B \text{ in } M_{m,n}(\mathbb{R}) \Longleftrightarrow A_i \approx B_i \text{ in } M_{1,n}(\mathbb{R}) \equiv \mathbb{R}^n \ (i = 1, 2, \ldots, m).$$

3. Equi-Activity of Vectors

From Proposition 2.3, we can see that equi-activity of matrices is equivalent to that of vectors. Therefore, it suffices to discuss equi-activity of vectors in $\mathbb{R}^n$. To do this, some notations will be needed.

Let $\{e_1, e_2, \ldots, e_n\}$ be the canonical basis for Hilbert space $\mathbb{R}^n$. For each vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define $P_i(x) = x_i \ (i = 1, 2, \ldots, n)$ and

$$V_+^n(x) = \{v \in V^n : \langle x, v\rangle \geq 0\}, \tag{3.1}$$

$$P_{i,j}(x) = P_j(x)e_i + P_i(x)e_j + \sum_{k \neq i,j} P_k(x)e_k, \tag{3.2}$$

$$M_i(x) = -P_i(x)e_i + \sum_{k \neq i} P_k(x)e_k. \tag{3.3}$$

Then we obtain an element $V_+^n(x)$ of the power set $\mathcal{P}(V^n)$ of V^n, which is the set of all subsets of V^n. It is easy to show that all $P_{i,j}$ and M_i are Hermitian matrices on $\mathbb{R}$ and have the following properties:

(i) $(P_{i,j})^2 = (M_k)^2 = I, \ P_{i,j} = P_{j,i}$;

(ii) $P_{i,j}M_k = M_k P_{i,j} (k \neq i, j), P_{i,j}M_i = M_j P_{i,j}, P_{i,j}M_j = M_i P_{i,j}$;

(iii) $P_{i,j}(V^n) = M_k(V^n) = V^n$, for all $i, j, k \in I_n = \{1, 2, \ldots, n\}$.

Let $\mathcal{G}_n(\mathbb{R})$ denote the multiplicative group generated by

$$\{I\} \cup \{P_{i,j} : i, j \in I_n\} \cup \{M_k : k \in I_n\}.$$

Then, from the properties (i), (ii) and (iii) above, it can be seen that the general element of $\mathcal{G}_n(\mathbb{R})$ has the form of

$$\left(\prod_{i\in I} M_i\right)\left(\prod_{(i,j)\in J} P_{i,j}\right), I \subset I_n, J \subset I_n \times I_n,$$

and that $\mathcal{G}_n(\mathbb{R})$ is a subgroup of the unitary group $\mathcal{U}(M_n(\mathbb{R}))$ of matrix algebra $M_n(\mathbb{R})$.

Theorem 3.1. Let x, y be two vectors in $\mathbb{R}^n$ and $T \in M_n(\mathbb{R})$ be a matrix with $T^*(V^n) \subset V^n$, then

(a) $x \approx y \Longleftrightarrow V_+^n(x) = V_+^n(y)$.

(b) $x \approx y \Longrightarrow T(x) \approx T(y)$.

(c) $T(E(x)) \subset E(T(x))$.

If, in addition, $T^*T = I$ and $T(V^n) \subset V^n$, then

(d) $x \approx y \Longleftrightarrow T(x) \approx T(y)$.

(e) $T(E(x)) = E(T(x))$.

Proof. (a) It is clear from the definition.

(b) For each v in V^n, we have

$$\langle Tx, v\rangle = \langle x, T^*v\rangle,\ \langle Ty, v\rangle = \langle y, T^*v\rangle.$$

Thus,

$$v \in V_+^n(Tx) \Longleftrightarrow T^*v \in V_+^n(x),$$
$$v \in V_+^n(Ty) \Longleftrightarrow T^*v \in V_+^n(y).$$

Let $x \approx y$, then by (a) we obtain $V_+^n(x) = V_+^n(y)$ and so $V_+^n(Tx) = V_+^n(Tx)$, that is, $T(x) \approx T(y)$.

(c) Let $y \in E(x)$, then $x \approx y$ and thus from (b) $T(x) \approx T(y)$. Hence $T(y) \in E(T(x))$. This shows that $T(E(x)) \subset E(T(x))$.

(d) Under the additional condition, by (b) we get

$$x \approx y \Longrightarrow T(x) \approx T(y) \Longrightarrow x = T^*T(x) \approx T^*T(y) = y.$$

(e) It is given by (d). This completes the proof.

Corollary 3.1. Let x, y be two vectors in $\mathbb{R}^n$, then for every $i, j, k \in I_n$, we have

(a) $x \approx y \Longleftrightarrow P_{i,j}(x) \approx P_{i,j}(y)$.

(b) $x \approx y \Longleftrightarrow M_k(x) \approx M_k(y)$.

(c) $P_{i,j}(E(x)) = E(P_{i,j}(x))$.

(d) $M_k(E(x)) = E(M_k(x))$.

Corollary 3.2. For every $T \in \mathcal{G}_n(\mathbb{R})$, we have
(a) $x \approx y \Longleftrightarrow T(x) \approx T(y)$.
(b) $T(E(x)) = E(T(x))$.

We recall that a matrix $A = [a_{ij}]$ is called to be *strictly diagonal dominant* if $a_{ii} > \sum_{j \neq i} |a_{ij}| (i = 1, 2, \ldots, m)$.

Theorem 3.2. A matrix $A = [a_{i,j}]$ in $M_n(\mathbb{R})$ is equi-active to the unit matrix I if and only if it is strictly diagonal dominant.

Proof. Sufficiency. Let $A = [a_{ij}]$ be strictly diagonal dominant, then

$$a_{ii} > \sum_{j \neq i} |a_{ij}|, \ i = 1, 2, \ldots, n.$$

Thus, for every $v = (v_1, v_2, \ldots, n) \in V^n$, we have

$$\langle e_i, v \rangle \langle A_i, v \rangle = a_{ii} + \sum_{j \neq i} a_{ij} v_i v_j \geq a_{ii} - \sum_{j \neq i} |a_{ij}| > 0.$$

This proves that $e_i \approx A_i$ for all $i \in I_n$. Therefore, $I \approx A$.

Necessity. Let $I \approx A$. Then $e_i \approx A_i$ for all $i \in I_n$. For each fixed $i \in I_n$, define $v_i = -1$, $v_j = \text{sgn}^* a_{ij}$ if $j \neq i$ and $a_{ij} \neq 0$, $v_j = 1$ if $a_{ij} = 0$. Then $v = (v_1, v_2, \ldots, v_n) \in V^n$ such that $\langle e_i, v \rangle = -1 < 0$ and thus

$$0 > \langle A_i, v \rangle = -a_{ii} + \sum_{j \neq i} |a_{ij}|.$$

This shows that $A = [a_{ij}]$ is strictly diagonal dominant. This completes the proof.

4. The Stability of DHNNs with Two Neurons

This section is devoted to a sufficient and necessary condition for a DHNN with two neurons to be stable in synchronous mode. To do this, we will accurately describe the classification of 2-dimension vectors as follows.

Proposition 4.1. $E((0, \ldots, 0)) = \{(0, \ldots, 0)\}$.

Proof. Let $v_i = -\text{sgn}^* x_i$ and $v = (v_1, v_2, \ldots, v_n)$, then $xv^T = -(|x_1| + |x_2| + \ldots + |x_n|) = 0$ since $x \approx 0$. Thus $x = 0$. This completes the proof.

Proposition 4.2. If $(a, b) \in \mathbb{R}^2$, then
(A1) $E((1, 0)) = \{(a, b) | \ a > |b|\}$;
(A2) $E((0, 1)) = \{(a, b) | \ b > |a|\}$;
(A3) $E((-1, 0)) = \{(a, b) | \ -a > |b|\}$;

(A4) $E((0,-1)) = \{(a,b)|\ -b > |a|\}$;

Proof. If $(a,b) \in E((1,0)), v_1 = (-1,1), v_2 = (-1,-1)$, then

$$\text{sgn}^*(a,b)v_1^T = \text{sgn}^*(a,b)v_2^T = -1.$$

This implies that $-a+b<0; -a-b<0$. Therefore, $a > |b|$. Conversely, if $a > |b|$, then for $\forall v = (v_1, v_2) \in V^2$, we have $\text{sgn}^*(1,0)v^T = \text{sgn}^* v_1 = \text{sgn}^*(a,b)v^T$. Thus $(a,b) \in E((1,0))$. This shows that (A1) holds. The others can be given by (A1) and Corollary 3.2. This completes the proof.

Proposition 4.3. If $(a,b) \in \mathbb{R}^2$, then
(B1) $E((1,1)) = \{(a,b)|\ a = b > 0\}$;
(B2) $E((-1,-1)) = \{(a,b)|\ a = b < 0\}$;
(B3) $E((1,-1)) = \{(a,b)|\ a = -b > 0\}$;
(B4) $E((-1,1)) = \{(a,b)|\ -a = b > 0\}$.

Proof. Let $(a,b) \in E((1,1)), v_1 = (1,-1), v_2 = (-1,1), v_3 = (1,1)$, then

$$\text{sgn}^*(a,b)v_1^T = \text{sgn}^*(a,b)v_2^T = \text{sgn}^*(a,b)v_3^T = 1.$$

This shows that $a-b \geq 0, b-a \geq 0$ and $a+b \geq 0$. By Proposition 4.1, $a = b > 0$. Conversely, if $a = b > 0$, then it is obvious that $(a,b) \in E((1,1))$. This shows that (B1) holds. The others can be given by (B1) and Corollary 3.2. This completes the proof.

Propositions 4.2 and 4.3 can be rewritten shortly as follows.

Proposition 4.2′. If $(a,b) \in \mathbb{R}^2$, then

$$E(T(1,0)) = \{T(a,b)|\ a > |b|\}, \quad \forall T \in \mathcal{G}_2(\mathbb{R}).$$

Proposition 4.3′. If $(a,b) \in \mathbb{R}^2$, then

$$E(T(1,1)) = \{T(a,b)|\ a = b > 0\}, \quad \forall T \in \mathcal{G}_2(\mathbb{R}).$$

For a matrix $A \in M_n(\mathbb{R})$, we define an operator $\bar{A} : V^n \to V^n$ by $\bar{A}x = (\text{sgn}^*(Ax^T))^T$ for every $x \in V^n$.

Definition 4.1. For a matrix $W \in M_n(\mathbb{R})$, if the DHNN $(W,0)$ is stable (resp. unstable), then we say that the matrix W is stable (resp. unstable).

Form Theorem 3.2 we deduce that if $W \in M_n(\mathbb{R})$ is strictly diagonal dominant, then W is stable.

From Remark 1.1 we can deduce the following result.

Proposition 4.4. Let matrix $W \in M_n(\mathbb{R})$. Then W is stable if and only if there exists an integer $k_0 \geq 0$ such that $\bar{W}^k = \bar{W}^{k_0}$ for all integers $k \geq k_0$.

It is easy to check that

$$\mathbb{R}^2 = \cup_{T\in\mathcal{G}_2(\mathbb{R})} E(T(1,0)) \cup_{T\in\mathcal{G}_2(\mathbb{R})} E(T(1,1)) \cup \{0\}, \tag{4.1}$$

and so we know from Proposition 4.1, 4.2 and 4.3 that there are only nine equi-active classes of 2-dimensional vectors, which are as follows

$$E((1,0)), E((0,0)), E((1,1)), E((1,-1)), E((0,1)),$$

$$E((0,-1)), E((-1,1)), E((-1,-1)), E((-1,0)).$$

Now, we write in turn

$$(1,0), (0,0), (1,1), (1,-1), (0,1), (0,-1), (-1,1), (-1,-1), (-1,0)$$

as $\alpha_1, \alpha_2, \ldots, \alpha_9$ and

$$(0,1), (0,0), (1,1), (-1,1), (1,0), (-1,0), (1,-1), (-1,-1), (0,-1)$$

as $\beta_1, \beta_2, \ldots, \beta_9$. Set $\alpha_i = (\alpha_{i1}, \alpha_{i2})$ and $\beta_j = (\beta_{j1}, \beta_{j2})$.

Put $O_{ij} = \begin{bmatrix} \alpha_i \\ \beta_j \end{bmatrix}$, $a_{ij} = \begin{cases} 1, & O_{ij} \text{ is stable;} \\ 0, & O_{ij} \text{ is unstable,} \end{cases}$ and define $A = [a_{ij}]$ which is called the distribution matrix of convergence of DHNNs with two neurons and zero threshold.

Theorem 4.1. The distribution matrix A has the following form.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. We will complete the proof by the following steps.
Step 1. The matrix $A = (a_{ij})$ is symmetric.

Since $\beta_i = P_{12}\alpha_i$, $\alpha_i = P_{12}\beta_i$, we have $P_{12}\bar{O}_{ij} = \bar{O}_{ji}P_{12}$. Thus, $P_{12}\bar{O}_{ij}^k P_{12} = \bar{O}_{ji}^k$ for all positive integers k. By use of Proposition 4.4, we see that O_{ij} is stable if and only if O_{ji} is stable. This shows that A is symmetric.

Step 2. For each $j \in \{1, 2, \ldots, 9\}$, the matrices O_{2j}, O_{3j} and O_{4j} have the same stabilities.

We know from the definition of α_i that

$$\alpha_2 = (0,0),\ \alpha_3 = (1,1),\ \alpha_4 = (1-1).$$

Put

$$v_1 = (1,1),\ v_2 = (1,-1),\ v_3 = (-1,1),\ v_4 = (-1,-1). \tag{4.2}$$

Then we have

$$\text{sgn}^*\alpha_2 v_i^T = 1\ (i = 1,2,3,4);$$
$$\text{sgn}^*\alpha_3 v_i^T = 1\ (i = 1,2,3),\ \text{sgn}^*\alpha_3 v_4^T = -1;$$
$$\text{sgn}^*\alpha_4 v_i^T = 1\ (i = 1,2,4),\ \text{sgn}^*\alpha_4 v_3^T = -1.$$

Thus, we obtain for each $j \in \{1, 2, \ldots, 9\}$that

$$\bar{O}_{2j}v_m \in \{v_1, v_2\}, (m = 1,2,3,4) \tag{4.3}$$

$$\bar{O}_{3j}v_m \in \{v_1, v_2\}, (m = 1,2,3) \text{ and } \bar{O}_{3j}v_4 \in \{v_3, v_4\} \tag{4.4}$$

$$\bar{O}_{4j}v_m \in \{v_1, v_2\}, (m = 1,2,4) \text{ and } \bar{O}_{4j}v_3 \in \{v_3, v_4\} \tag{4.5}$$

If O_{2j} is stable, then there exists an integer $k_0 \geq 0$ such that $\bar{O}_{2j}^k = \bar{O}_{2j}^{k_0}$ for all integers $k \geq k_0$. We note that $\bar{O}_{2j}v_i = \bar{O}_{3j}v_i$ for $i = 1,2,3$. If $\bar{O}_{3j}v_4 = v_3$, then for all $k \geq k_0+1$, we have that $\bar{O}_{3j}^k v_i = \bar{O}_{3j}^{k_0+1}v_i$ for $i = 1,2,3$ and $\bar{O}_{3j}^k v_4 = \bar{O}_{3j}^{k-1}\bar{O}_{3j}v_4 = \bar{O}_{3j}^{k-1}v_3 = \bar{O}_{3j}^{k_0}v_3 = \bar{O}_{3j}^{k_0}\bar{O}_{3j}v_4 = \bar{O}_{3j}^{k_0+1}v_4$. Then O_{3j} is stable. If $\bar{O}_{3j}v_4 = v_4$, then we clearly have that $\bar{O}_{3j}^k = \bar{O}_{3j}^{k_0}$ and O_{3j} is also stable. Conversely, O_{3j} is stable, then we easily have O_{2j} is stable. The proof for O_{4j} is similar.

Step 3. O_{8j} and O_{9j} are unstable for each $j = 1, 2, \ldots, 9$.

We know that

$$\alpha_8 = (-1,-1),\ \alpha_9 = (-1,0)$$

and so we have

$$\text{sgn}^*\alpha_9 v_1^T = \text{sgn}^*\alpha_9 v_2^T = -1,\ \text{sgn}^*\alpha_9 v_3^T = \text{sgn}^*\alpha_9 v_4^T = 1.$$

Thus, $\bar{O}_{9j}v_1, \bar{O}_{9j}v_2 \in \{v_3, v_4\}$ and $\bar{O}_{9j}v_3, \bar{O}_{9j}v_4 \in \{v_1, v_2\}$. Therefore, we easily have that O_{9j} is unstable. Also, we have

$$\text{sgn}^*\alpha_8 v_1^T = -1,\ \text{sgn}^*\alpha_8 v_3^T = \text{sgn}^*\alpha_8 v_4^T = 1.$$

This shows v_1, v_3, v_4 are not fixed point of $\bar{O}_{8j}$ for each $j = 1, 2, \ldots, 9$. Now, we suppose O_{8j} is stable for some $j \in \{1, 2, \ldots, 9\}$.Then v_2 must be a fixed point of $\bar{O}_{8j}$. So, we have $\beta_j v_2^T = \beta_{j1} - \beta_{j2} < 0$. Thus, $j \in \{1, 4, 6\}$. But

$$\mathrm{sgn}^*(O_{81}v_1^T) = v_3^T,\ \mathrm{sgn}^*(O_{81}v_3^T) = v_1^T;$$

$$\mathrm{sgn}^*(O_{84}v_1^T) = v_3^T,\ \mathrm{sgn}^*(O_{84}v_3^T) = v_1^T;$$

$$\mathrm{sgn}^*(O_{86}v_1^T) = v_4^T,\ \mathrm{sgn}^*(O_{86}v_4^T) = v_1^T.$$

It follows from Proposition 4.4 that O_{8j} is unstable, a contradiction.

Step 4. O_{2j} is stable for each $j = 1, 2, \ldots, 7$.

First we know from the definition of β_j and α_i that $\beta_{j2} \geq 0$ when $j \leq 6$ and that $\mathrm{sgn}^*\alpha_2 v_i^T = 1 (i = 1, 2, 3, 4)$. Suppose that O_{2j} is unstable for some $j \in \{1, 2, \ldots, 6\}$. Then we have that

$$\mathrm{sgn}^*(O_{2j}v_1^T) = \mathrm{sgn}^*(O_{2j}v_2^T),\ \mathrm{sgn}^*(O_{2j}v_2^T) = \mathrm{sgn}^*(O_{2j}v_1^T).$$

Hence

$$\beta_{j1} + \beta_{j2} < 0,\ \beta_{j1} - \beta_{j2} \geqslant 0.$$

Then $\beta_{j1} = \beta_{j2} = 0$, a contradiction. So O_{2j} is stable for each $j = 1, 2, \ldots, 6$. For O_{27}, by a direct calculus, we have the following.

$$\bar{O}_{27}v_1 = v_1, \bar{O}_{27}v_2 = v_1, \bar{O}_{27}v_3 = v_2 \text{ and } \bar{O}_{27}v_4 = v_1.$$

Hence, we have $\bar{O}_{27}^k = \bar{O}_{27}^3$ for all integers $k \geq 3$. This shows that O_{27} is stable.

Step 5. O_{1j} $(j = 1, 2, \ldots, 6)$ are stable and O_{17} is unstable.

From Step 1, 2 and 3, it suffices to prove that O_{11}, O_{15}, O_{16} are stable. By Propositions 2.3 and 4.2, we have

$$O_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ O_{15} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \approx \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix},\ O_{16} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \approx \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}.$$

Clearly,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix},\ \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$$

are positive definite. It follows from Corollary 2.1 that O_{1j} $(j = 1, 2, \ldots, 6)$ are stable. Since $\bar{O}_{17}v_3 = v_4, \bar{O}_{17}v_4 = v_3$, it follows from Proposition 4.4 that O_{17} is unstable.

Step 6. O_{ij} is unstable for $i \in \{5, 6, 7\}, j \in \{5, 6, 7\}$.

From Step 1, we need only to prove that $O_{55}, O_{66}, O_{77}, O_{56}, O_{57}$ and O_{67} are unstable. It is easy to see that

$$\mathrm{sgn}^*(O_{55}v_3^T) = v_2^T,\ \mathrm{sgn}^*(O_{55}v_2^T) = v_3^T;\ \mathrm{sgn}^*(O_{66}v_1^T) = v_4^T, \mathrm{sgn}^*(O_{66}v_4^T) = v_1^T;$$

$$\mathrm{sgn}^*(O_{57}v_3^T) = v_2^T,\ \mathrm{sgn}^*(O_{57}v_2^T) = v_3^T;\ \mathrm{sgn}^*(O_{77}v_3^T) = v_2^T,\ \mathrm{sgn}^*(O_{77}v_2^T) = v_3^T;$$
$$\mathrm{sgn}^*(O_{56}v_1^T) = v_2^T,\ \mathrm{sgn}^*(O_{56}v_2^T) = v_4^T,\ \mathrm{sgn}^*(O_{56}v_4^T) = v_3^T,\ \mathrm{sgn}^*(O_{56}v_3^T) = v_1^T;$$
$$\mathrm{sgn}^*(O_{67}v_1^T) = v_3^T,\ \mathrm{sgn}^*(O_{67}v_3^T) = v_4^T,\ \mathrm{sgn}^*(O_{67}v_4^T) = v_1^T.$$

It follows from Proposition 4.4 that $O_{55}, O_{66}, O_{77}, O_{56}, O_{57}$ and O_{67} are unstable. This completes the proof.

Now, we can get the following theorem by using Propositions 4.1, 4.2 and 4.3 as well as Theorem 4.1.

Theorem 4.2. If $W = (w_{ij}) \in M_{2,2}(\mathbf{R})$, then the DHNN $(W, 0)$ is stable if and only if one of the following conditions holds.

(1) $w_{11} \geq |w_{12}|$ and $w_{22} \geq 0$;
(2) $w_{22} \geq |w_{21}|$ and $w_{11} \geq 0$;
(3) $w_{11} = |w_{12}|$ and $-w_{22} = w_{21} > 0$;
(4) $w_{22} = |w_{21}|$ and $-w_{11} = w_{12} > 0$.

Proof. Suppose that one of the four conditions holds.

Case 1. Let (1) hold. From Propositions 4.1, 4.2 and 4.3 we see that

$$w_1 := (w_{11}, w_{12}) \in E((1,0)) \cup E((0,0)) \cup E((1,1)) \cup E((1,-1));$$

$$w_2 := (w_{21}, w_{22}) \notin E((1,-1)) \cup (E((-1,-1)) \cup E((0,-1))).$$

Thus, $W \approx O_{ij}$ for some $i \in \{1,2,3,4\}$ and $j \in \{1,2,3,4,5,6\}$. Therefore, we get from the matrix A of Theorem 4.1 that $(W,0)$ is stable.

Case 2. Let (2) hold. Then by a method similar to Case 1, we obtain that $W \approx O_{ij}$ for some $i \in \{1,2,3,4,5,6\}$ and $j \in \{1,2,3,4\}$. Therefore, we get from the matrix A of Theorem 4.1 that $(W,0)$ is stable.

Case 3. Let (3) hold. Then from Proposition 4.1, 4.2 and 4.3, we have

$$w_1 \in E((0,0)) \cup E((1,1)) \cup E((1,-1)),\ \ w_2 \in E((1,-1)).$$

Thus, $W \approx O_{i7}$ for some $i \in \{2,3,4\}$. Therefore, we get from the matrix A that $(W,0)$ is stable.

Case 4. Let (4) hold. Then $W \approx O_{7j}$ for some $j \in \{2,3,4\}$. Therefore, we get from the matrix A of Theorem 4.1 that $(W,0)$ is stable.

Conversely, suppose that $(W,0)$ is stable and that $W \approx O_{ij}$ for some $i,j \in \{1,2,\ldots,9\}$. From Theorem 4.1, we can see that one of the following conditions is satisfied.

(1) $1 \leq i \leq 4$ and $1 \leq j \leq 6$;
(2) $1 \leq i \leq 6$ and $1 \leq j \leq 4$;
(3) $2 \leq i \leq 4$ and $j = 7$;

(4) $i = 7$ and $2 \leq j \leq 4$.

This shows that one of the four conditions in the theorem is satisfied. This completes the proof.

Corollary 4.1. If $W = (w_{ij}) \in M_{2,2}(\mathbb{R})$ is positive semidefinite, then W is stable.

Proof. If W is positive semidefinite, then we have $w_{11} \geq 0, w_{22} \geq 0$ and $w_{11}w_{22} \geq w_{12}w_{21} = w_{12}^2$. Hence, $w_{11} \geq |w_{12}|$ or $w_{22} \geqslant |w_{21}|$. It follows from Theorem 4.2 that $(W, 0)$ is stable. This completes the proof.

Remark 4.1. It is easy to check that if $W \in M_{2,2}(\mathbb{R})$ is positive definite, then $W \approx O_{1j}$ for some $j \in \{1, 3, 4, 5, 6\}$, or $W \approx O_{i1}$ for some $i \in \{1, 3, 4, 5, 6\}$.

5. Conclusion

We have introduced a new method to study the stability of DHNNs. By the method, a sufficient and necessary condition for a DHNN with two neurons in synchronous mode to be stable is given. As we know, it may be a unique characterization of stability of DHNNs. Then the stability of DHNNs with two neurons in synchronous mode has been completely solved when $\Theta = 0$. Although, our main result(Theorem 4.2) is only about the DHNNs with two neurons, many information about the DHNNs with more than 2 neurons have been also revealed. For example, we can see from Remark 4.1 that the positive definite matrices account for a small part of the stable matrix in the distribution matrix A of DHNNs with two neurons and zero threshold.

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