Nihonkai Math. J.
Vol.14(2003), 43-54

CONVERGENCE THEOREMS TO COMMON FIXED POINTS FOR INFINITE FAMILIES OF NONEXPANSIVE MAPPINGS IN STRICTLY CONVEX BANACH SPACES

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ABSTRACT. In this paper, we prove convergence theorems to common fixed points for infinite families of nonexpansive mappings in strictly convex Banach spaces. One of our results is the following: Let C be a compact convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C with $\bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n < 1$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \left(1 - \sum_{i=1}^{n} \lambda_i\right) x_n + \sum_{i=1}^{n} \lambda_i T_i x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

1. INTRODUCTION

A mapping T on a closed convex subset C of a Banach space E is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know that $F(T)$ is nonempty if E is uniformly convex and C is bounded; see Browder [1], Göhde [7], and Kirk [11]. In 1953, Mann [14] considered the following iteration scheme: $x_1 \in C$ and

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n \tag{1}$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later several authors have studied Mann's iteration process; see Edelstein and O'brien [5], Groetsch [8], Ishikawa [9], Opial [15], Outlaw [16], Reich [17] and so on. For example, Reich [17] proved the following: (1) converges weakly to a fixed point z of T if E is uniformly convex and the norm of E is Fréchet differentiable, C is closed and convex, T is nonexpansive and has a fixed point, and $\{\alpha_n\}$ satisfies $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Also Ishikawa [9] proved the following: (1) converges strongly to a fixed point z of T if C is compact and convex, T is nonexpansive, and $\{\alpha_n\}$ satisfies $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Convergence theorems for families of nonexpansive mappings are proved in Crombez [4], Ishikawa [10], Kitahara and Takahashi [12], Linhart [13], Takahashi and Tamura [21] and so on. For example, Linhart [13] proved the following:

2000 *Mathematics Subject Classification.* Primary 47H09, Secondary 47H10.
Key words and phrases. Fixed point, Nonexpansive mapping, Convergence theorem.
*The author is supported in part by Grants-in-Aid for Scientific Research, Japan Society for the Promotion of Science.

Theorem 1 (Linhart [13]). *Let C be a nonempty, compact and convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose that $T_i \circ T_j = T_j \circ T_i$ for $i, j \in \mathbb{N}$. Let $\{\alpha_n : n \in \mathbb{N}\}$ be a sequence in $(0,1)$. Put $U_i x = \alpha_i T_i x + (1-\alpha_i)x$ for $i \in \mathbb{N}$ and $x \in C$. Let f be a mapping on $\mathbb{N}$ satisfying for $k, n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $m \geq n$ and $f(m) = k$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = U_{f(n)} \circ U_{f(n-1)} \circ \cdots \circ U_{f(1)} x_1$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

In this paper, motivated by Linhart's result, we consider another iteration scheme and prove convergence theorems for infinite families of nonexpansive mappings in strictly convex Banach spaces.

2. Preliminaries

Throughout of the paper, we denote by $\mathbb{N}$ the set of positive integers. A Banach space E is called strictly convex if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called uniformly convex if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x+y\|/2 < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. It is clear that a uniformly convex Banach space is strictly convex. The norm of E is called Fréchet differentiable if for each $x \in E$ with $\|x\| = 1$, $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists and is attained uniformly in $y \in E$ with $\|y\| = 1$. A Banach space E satisfies Opial's condition if for each weakly convergent sequence $\{x_n\}$ in E with weak limit z, $\liminf_n \|x_n - z\| < \liminf_n \|x_n - y\|$ for all $y \in E$ with $y \neq z$. All Hilbert spaces and $\ell^p (1 \leq p < \infty)$ satisfy Opial's condition, while L^p with $1 < p < \infty$ and $p \neq 2$ do not.

The following lemmas are used in the proofs of main results.

Lemma 1 (Browder [2]). *Let E be a uniformly convex Banach space and let C be a bounded closed convex subset of E. Let S be a nonexpansive mapping on C. Let $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to some $z \in C$ and $\{\|Sx_n - x_n\|\}$ converges to 0. Then $Sz = z$.*

The following lemma is essentially proved by Opial [15].

Lemma 2 (Opial [15]). *Let E be a Banach space which satisfies Opial's condition. Let S be a nonexpansive mapping on a closed convex subset C of E. Let $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to some $z \in C$ and $\{\|Sx_n - x_n\|\}$ converges to 0. Then $Sz = z$.*

Bruck [3] proved the following interesting lemma.

Lemma 3 (Bruck [3]). *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\alpha_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \alpha_n = 1$. Then a mapping S on C defined by*

$$Sx = \sum_{n=1}^{\infty} \alpha_n T_n x$$

for $x \in C$ is well-defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

The following lemma was proved by Reich [17]; see also [20].

Lemma 4 (Reich [17]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is Fréchet differentiable and let $\{U_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C with $\bigcap_{n=1}^{\infty} F(U_n) \neq \varnothing$. Let $x \in C$ and $W_n = U_n U_{n-1} \cdots U_1$ for all $n \in \mathbb{N}$. Then the set $\left(\bigcap_{n=1}^{\infty} \overline{co}\{W_m x : m \geq n\}\right) \bigcap \left(\bigcap_{n=1}^{\infty} F(U_n)\right)$ consists of at most one point, where $\overline{co}\{W_m x : m \geq n\}$ is the closure of the convex hull of $\{W_m x : m \geq n\}$.*

3. Lemmas

To prove our main results, we also need the following lemmas.

Lemma 5. *Let $\{z_n\}$ and $\{w_n\}$ be sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\limsup_n \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n) z_n$ for all $n \in \mathbb{N}$,*

$$\limsup_{n\to\infty} \|w_n - w_{n+k}\| - \|z_n - z_{n+k}\| \leq 0$$

for all $k \in \mathbb{N}$, and the limit of $\{\|w_n - z_n\|\}$ exists. Then

$$\lim_{n\to\infty} \left| \|w_{n+k} - z_n\| - (1 + \alpha_n + \cdots + \alpha_{n+k-1}) \cdot \lim_{j\to\infty} \|w_j - z_j\| \right| = 0$$

hold for all $k \in \mathbb{N}$.

Proof. Put $d = \lim_n \|w_n - z_n\|$ and $a = (1 - \limsup_n \alpha_n)/2$, and fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $d - \varepsilon \leq \|w_n - z_n\| \leq d + \varepsilon$, $0 \leq \alpha_n \leq 1 - a$ and $\|w_{n+j} - w_n\| - \|z_{n+j} - z_n\| \leq \varepsilon$ for all $n \geq n_0$ and $j \in \{1, 2, \cdots, k\}$. Fix $n \in \mathbb{N}$ with $n \geq n_0$. We first show

$$\|w_{n+k} - z_{n+j}\| \geq (1 + \alpha_{n+j} + \cdots + \alpha_{n+k-1}) \cdot d - \frac{(k-j)(k+2)}{a^{k-j}} \varepsilon \tag{2}$$

for $j \in \{0, 1, 2, \cdots, k-1\}$. From

$$\begin{aligned}
& d - \varepsilon \\
& \leq \|w_{n+k} - z_{n+k}\| \\
& \leq \alpha_{n+k-1} \|w_{n+k} - w_{n+k-1}\| + (1 - \alpha_{n+k-1}) \|w_{n+k} - z_{n+k-1}\| \\
& \leq \alpha_{n+k-1} \|z_{n+k} - z_{n+k-1}\| + \varepsilon + (1 - \alpha_{n+k-1}) \|w_{n+k} - z_{n+k-1}\| \\
& = \alpha_{n+k-1}^2 \|w_{n+k-1} - z_{n+k-1}\| + \varepsilon + (1 - \alpha_{n+k-1}) \|w_{n+k} - z_{n+k-1}\| \\
& \leq \alpha_{n+k-1}^2 d + 2\varepsilon + (1 - \alpha_{n+k-1}) \|w_{n+k} - z_{n+k-1}\|,
\end{aligned}$$

we obtain

$$\begin{aligned}
\|w_{n+k} - z_{n+k-1}\| &\geq \frac{(1 - \alpha_{n+k-1}^2) d - 3\varepsilon}{1 - \alpha_{n+k-1}} \\
&\geq (1 + \alpha_{n+k-1}) d - \frac{k+2}{a} \varepsilon.
\end{aligned}$$

So, (2) holds in the case of $j = k-1$. If (2) holds for some $j \in \{1, 2, \cdots, k-1\}$, then from

$$
\begin{aligned}
&\left(1+\sum_{i=j}^{k-1} \alpha_{n+i}\right) \cdot d - \frac{(k-j)(k+2)}{a^{k-j}}\varepsilon \\
&\le \|w_{n+k} - z_{n+j}\| \\
&\le \alpha_{n+j-1}\|w_{n+k} - w_{n+j-1}\| + (1-\alpha_{n+j-1})\|w_{n+k} - z_{n+j-1}\| \\
&\le \alpha_{n+j-1}\|z_{n+k} - z_{n+j-1}\| + \varepsilon + (1-\alpha_{n+j-1})\|w_{n+k} - z_{n+j-1}\| \\
&\le \alpha_{n+j-1} \sum_{i=j-1}^{k-1} \|z_{n+i+1} - z_{n+i}\| + \varepsilon \\
&\quad + (1-\alpha_{n+j-1})\|w_{n+k} - z_{n+j-1}\| \\
&= \alpha_{n+j-1} \sum_{i=j-1}^{k-1} \alpha_{n+i}\|w_{n+i} - z_{n+i}\| + \varepsilon \\
&\quad + (1-\alpha_{n+j-1})\|w_{n+k} - z_{n+j-1}\| \\
&\le \alpha_{n+j-1} \sum_{i=j-1}^{k-1} \alpha_{n+i}(d+\varepsilon) + \varepsilon + (1-\alpha_{n+j-1})\|w_{n+k} - z_{n+j-1}\| \\
&\le \alpha_{n+j-1} \sum_{i=j-1}^{k-1} \alpha_{n+i} d + (k+1)\varepsilon + (1-\alpha_{n+j-1})\|w_{n+k} - z_{n+j-1}\|,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\|w_{n+k} - z_{n+j-1}\| \\
&\ge \frac{1+\sum_{i=j}^{k-1} \alpha_{n+i} - \alpha_{n+j-1}\sum_{i=j-1}^{k-1} \alpha_{n+i}}{1-\alpha_{n+j-1}} d \\
&\quad - \frac{\frac{(k-j)(k+2)}{a^{k-j}} + (k+1)}{1-\alpha_{n+j-1}}\varepsilon \\
&\ge \left(1+\sum_{i=j-1}^{k-1} \alpha_{n+i}\right) d - \frac{(k-j+1)(k+2)}{a^{k-j+1}}\varepsilon.
\end{aligned}
$$

So, (2) holds for all $j \in \{0, 1, 2, \cdots, k-1\}$. Specially, we have

$$
\|w_{n+k} - z_n\| \ge (1 + \alpha_n + \cdots + \alpha_{n+k-1}) \cdot d - \frac{k(k+2)}{a^k}\varepsilon. \tag{3}
$$

On the other hand, we have

$$
\|w_{n+k} - z_n\| \le \|w_{n+k} - z_{n+k}\| + \sum_{i=0}^{k-1} \|z_{n+i+1} - z_{n+i}\| \tag{4}
$$

$$= \|w_{n+k} - z_{n+k}\| + \sum_{i=0}^{k-1} \alpha_{n+i}\|w_{n+i} - z_{n+i}\|$$

$$\le d + \varepsilon + \sum_{i=0}^{k-1} \alpha_{n+i}(d+\varepsilon)$$

$$\le d + \sum_{i=0}^{k-1} \alpha_{n+i} d + (k+1)\varepsilon.$$

From (3) and (4), we obtain

$$\big| \|w_{n+k} - z_n\| - (1 + \alpha_n + \cdots + \alpha_{n+k-1})d \big| \le \frac{k(k+2)}{a^k}\varepsilon.$$

This completes the proof. □

By using Lemma 5, we obtain the following.

Lemma 6. *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n) z_n$ for all $n \in \mathbb{N}$,*

$$\limsup_{n\to\infty} \|w_n - w_{n+k}\| - \|z_n - z_{n+k}\| \le 0$$

for all $k \in \mathbb{N}$, and the limit of $\{\|w_n - z_n\|\}$ exists. Then $\lim_n \|w_n - z_n\| = 0$.

Proof. We put $a = \liminf_n \alpha_n > 0$, $M = 2 \cdot \sup\{\|z_n\| + \|w_n\| : n \in \mathbb{N}\}$ and $d = \lim_n \|w_n - z_n\|$. We assume $d > 0$ and fix $k \in \mathbb{N}$ with $(1+ka)d > M$. By Lemma 5, we have

$$\lim_{n\to\infty} \big| \|w_{n+k} - z_n\| - (1 + \alpha_n + \cdots + \alpha_{n+k-1}) \cdot d \big| = 0$$

and hence

$$\limsup_{n\to\infty} \|w_{n+k} - z_n\| \ge (1+ka)d > M.$$

This is a contradiction. Therefore $d = 0$. □

4. Main results

In this section, we state our main results. We first prove a strong convergence theorem.

Theorem 2. *Let C be a compact convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C with $\bigcap_{n=1}^{\infty} F(T_n) \ne \varnothing$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n < 1$, and let $\{I_n\}$ be a sequence of subsets of $\mathbb{N}$ satisfying $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} I_n = \mathbb{N}$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \left(1 - \sum_{i\in I_n} \lambda_i\right) x_n + \sum_{i \in I_n} \lambda_i T_i x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

Before proving Theorem 2, we prove the following.

Lemma 7. *Let C be a closed convex subset of a Banach space E. Let $\{T_n : n \in \mathbb{N}\}$, $\{\lambda_n\}$, $\{I_n\}$ and $\{x_n\}$ as in Theorem 2. Then the following hold:*

(i) *For $w \in \bigcap_{j=1}^{\infty} F(T_j)$ and $n \in \mathbb{N}$, $\|x_{n+1} - w\| \leq \|x_n - w\|$;*

(ii) *$\{x_n : n \in \mathbb{N}\}$ and $\{T_k x_n : k, n \in \mathbb{N}\}$ are bounded;*

(iii)

$$\lim_{n\to\infty} \left\| x_n - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| = \lim_{n\to\infty} \left\| x_n - \frac{\sum_{i=1}^{\infty} \lambda_i T_i x_n}{\sum_{i=1}^{\infty} \lambda_i} \right\| = 0.$$

Proof. We note that $\{x_n\}$ is well defined by Lemma 3. Without loss of generality, we may assume that $I_1 \neq \varnothing$. We have

$$\begin{aligned}
\|w - x_{n+1}\| &\leq \left(1 - \sum_{i\in I_n} \lambda_i\right) \|w - x_n\| + \sum_{i\in I_n} \lambda_i \|w - T_i x_n\| \\
&\leq \left(1 - \sum_{i\in I_n} \lambda_i\right) \|w - x_n\| + \sum_{i\in I_n} \lambda_i \|w - x_n\| \\
&= \|w - x_n\|
\end{aligned}$$

for $w \in \bigcap_{j=1}^{\infty} F(T_j)$ and $n \in \mathbb{N}$. So (i) is shown. We fix $k, n \in \mathbb{N}$ and $w \in \bigcap_{j=1}^{\infty} F(T_j)$, and put $M = \|x_1 - w\| + \|w\|$. Then we have

$$\|T_k x_n\| \leq \|T_k x_n - w\| + \|w\| \leq \|x_n - w\| + \|w\| \leq \|x_1 - w\| + \|w\| = M.$$

Hence (ii) is shown. We note that

$$x_{n+1} = \left(1 - \sum_{i\in I_n} \lambda_i\right) x_n + \left(\sum_{i\in I_n} \lambda_i\right) \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i}$$

for $n \in \mathbb{N}$. Since

$$\begin{aligned}
&\left\| x_{n+1} - \frac{\sum_{i\in I_{n+1}} \lambda_i T_i x_{n+1}}{\sum_{i\in I_{n+1}} \lambda_i} \right\| \\
&\leq \left\| x_{n+1} - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| + \left\| \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} - \frac{\sum_{i\in I_{n+1}} \lambda_i T_i x_n}{\sum_{i\in I_{n+1}} \lambda_i} \right\| \\
&\quad + \left\| \frac{\sum_{i\in I_{n+1}} \lambda_i T_i x_n}{\sum_{i\in I_{n+1}} \lambda_i} - \frac{\sum_{i\in I_{n+1}} \lambda_i T_i x_{n+1}}{\sum_{i\in I_{n+1}} \lambda_i} \right\| \\
&\leq \left\| x_{n+1} - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| + \frac{\sum_{i\in I_{n+1}\setminus I_n} \lambda_i}{\left(\sum_{i\in I_n} \lambda_i\right) \cdot \left(\sum_{i\in I_{n+1}} \lambda_i\right)} \left\| \sum_{i\in I_n} \lambda_i T_i x_n \right\| \\
&\quad + \frac{1}{\sum_{i\in I_{n+1}} \lambda_i} \left\| \sum_{i\in I_{n+1}\setminus I_n} \lambda_i T_i x_n \right\| + \frac{1}{\sum_{i\in I_{n+1}} \lambda_i} \sum_{i\in I_{n+1}} \lambda_i \|T_i x_n - T_i x_{n+1}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| x_{n+1} - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| + \frac{2M \sum_{i\in I_{n+1}\setminus I_n} \lambda_i}{\sum_{i\in I_{n+1}} \lambda_i} + \|x_n - x_{n+1}\| \\
&= \left\| x_n - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| + \frac{2M \sum_{i\in I_{n+1}\setminus I_n} \lambda_i}{\sum_{i\in I_{n+1}} \lambda_i} \\
&\leq \left\| x_n - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| + 2M \sum_{i\in I_{n+1}\setminus I_n} \lambda_i
\end{aligned}
$$

for $n \in \mathbb{N}$, we have

$$
\left\| x_m - \frac{\sum_{i\in I_m} \lambda_i T_i x_m}{\sum_{i\in I_m} \lambda_i} \right\| \leq \left\| x_n - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| + 2M \sum_{i\in I_m\setminus I_n} \lambda_i
$$

for $m, n \in \mathbb{N}$ with $m > n$. Hence

$$
\limsup_{m\to\infty} \left\| x_m - \frac{\sum_{i\in I_m} \lambda_i T_i x_m}{\sum_{i\in I_m} \lambda_i} \right\| \leq \left\| x_n - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| + 2M \sum_{i\in \mathbb{N}\setminus I_n} \lambda_i
$$

for $n \in \mathbb{N}$. Therefore

$$
\limsup_{m\to\infty} \left\| x_m - \frac{\sum_{i\in I_m} \lambda_i T_i x_m}{\sum_{i\in I_m} \lambda_i} \right\| \leq \liminf_{n\to\infty} \left\| x_n - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\|,
$$

i.e., the limit of

$$
\left\{ \left\| x_n - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| \right\}
$$

exists. For each $k, n \in \mathbb{N}$, we also have

$$
\begin{aligned}
&\left\| \frac{\sum_{i\in I_{n+k}} \lambda_i T_i x_{n+k}}{\sum_{i\in I_{n+k}} \lambda_i} - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| - \|x_{n+k} - x_n\| \\
&\leq \left\| \frac{\sum_{i\in I_{n+k}} \lambda_i T_i x_{n+k}}{\sum_{i\in I_{n+k}} \lambda_i} - \frac{\sum_{i\in I_n} \lambda_i T_i x_{n+k}}{\sum_{i\in I_n} \lambda_i} \right\| \\
&\quad + \left\| \frac{\sum_{i\in I_n} \lambda_i T_i x_{n+k}}{\sum_{i\in I_n} \lambda_i} - \frac{\sum_{i\in I_n} \lambda_i T_i x_n}{\sum_{i\in I_n} \lambda_i} \right\| - \|x_{n+k} - x_n\| \\
&\leq \left\| \frac{\sum_{i\in I_{n+k}} \lambda_i T_i x_{n+k}}{\sum_{i\in I_{n+k}} \lambda_i} - \frac{\sum_{i\in I_n} \lambda_i T_i x_{n+k}}{\sum_{i\in I_n} \lambda_i} \right\| \\
&\leq \frac{\sum_{i\in I_{n+k}\setminus I_n} \lambda_i}{\left(\sum_{i\in I_{n+k}} \lambda_i\right) \left(\sum_{i\in I_n} \lambda_i\right)} \sum_{i\in I_n} \lambda_i \|T_i x_{n+k}\| + \frac{1}{\sum_{i\in I_{n+k}} \lambda_i} \sum_{i\in I_{n+k}\setminus I_n} \lambda_i \|T_i x_{n+k}\| \\
&\leq \frac{2 \sum_{i\in I_{n+k}\setminus I_n} \lambda_i}{\sum_{i\in I_{n+k}} \lambda_i} M.
\end{aligned}
$$

Hence we obtain

$$\limsup_{n\to\infty}\left\|\frac{\sum_{i\in I_{n+k}}\lambda_i T_i x_{n+k}}{\sum_{i\in I_{n+k}}\lambda_i}-\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}\right\|-\|x_{n+k}-x_n\|\le 0$$

for $k\in\mathbb{N}$. By Lemma 6, we have

$$\lim_{n\to\infty}\left\|x_n-\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}\right\|=0.$$

For each $n\in\mathbb{N}$, we also have

$$\begin{aligned}
&\left\|x_n-\frac{\sum_{i=1}^{\infty}\lambda_i T_i x_n}{\sum_{i=1}^{\infty}\lambda_i}\right\|\\
\le&\left\|x_n-\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}\right\|+\left\|\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}-\frac{\sum_{i=1}^{\infty}\lambda_i T_i x_n}{\sum_{i=1}^{\infty}\lambda_i}\right\|\\
\le&\left\|x_n-\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}\right\|+\left\|\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}-\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i=1}^{\infty}\lambda_i}\right\|\\
&+\frac{1}{\sum_{i=1}^{\infty}\lambda_i}\left\|\sum_{i\in\mathbb{N}\setminus I_n}\lambda_i T_i x_n\right\|\\
\le&\left\|x_n-\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}\right\|+\frac{\sum_{i\in\mathbb{N}\setminus I_n}\lambda_i}{(\sum_{i\in I_n}\lambda_i)\cdot(\sum_{i=1}^{\infty}\lambda_i)}\sum_{i\in I_n}\lambda_i\|T_i x_n\|\\
&+\frac{1}{\sum_{i=1}^{\infty}\lambda_i}\sum_{i\in\mathbb{N}\setminus I_n}\lambda_i\|T_i x_n\|\\
\le&\left\|x_n-\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}\right\|+\frac{\sum_{i\in\mathbb{N}\setminus I_n}\lambda_i}{(\sum_{i\in I_n}\lambda_i)\cdot(\sum_{i=1}^{\infty}\lambda_i)}\sum_{i\in I_n}\lambda_i M\\
&+\frac{\sum_{i\in\mathbb{N}\setminus I_n}\lambda_i}{\sum_{i=1}^{\infty}\lambda_i}M\\
=&\left\|x_n-\frac{\sum_{i\in I_n}\lambda_i T_i x_n}{\sum_{i\in I_n}\lambda_i}\right\|+\frac{2\sum_{i\in\mathbb{N}\setminus I_n}\lambda_i}{\sum_{i=1}^{\infty}\lambda_i}M
\end{aligned}$$

Therefore we obtain

$$\lim_{n\to\infty}\left\|x_n-\frac{\sum_{i=1}^{\infty}\lambda_i T_i x_n}{\sum_{i=1}^{\infty}\lambda_i}\right\|=0.$$

Therefore we have shown (iii). This completes the proof. □

Proof of Theorem 2. Define a nonexpansive mapping S on C by

$$Sx=\frac{\sum_{n=1}^{\infty}\lambda_n T_n x}{\sum_{n=1}^{\infty}\lambda_n}$$

for $x\in C$. By Lemma 7, we have

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0.$$

Since C is compact, there exists a strongly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with strong limit $z \in C$. Since

$$\begin{aligned}
\|Sz - z\| &= \limsup_{k\to\infty} \left(\|Sz - Sx_{n_k}\| + \|Sx_{n_k} - x_{n_k}\| + \|x_{n_k} - z\|\right) \\
&\le \limsup_{k\to\infty} \left(2\|x_{n_k} - z\| + \|Sx_{n_k} - x_{n_k}\|\right) \\
&= 0,
\end{aligned}$$

z is a fixed point of S and hence is a common fixed point of $\{T_n : n \in \mathbb{N}\}$ by Lemma 3. So using Lemma 7 again, we have $\|x_{n+1} - z\| \le \|x_n - z\|$ for $n \in \mathbb{N}$. Therefore $\{x_n\}$ converges strongly to z. □

As a direct consequence of Theorem 2, we obtain the following.

Corollary 1. *Let C be a compact convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C with $\bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n < 1$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = \left(1 - \sum_{i=1}^{n} \lambda_i\right) x_n + \sum_{i=1}^{n} \lambda_i T_i x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

Proof. We put $I_n = \{1, 2, \cdots, n\}$ for $n \in \mathbb{N}$. Then by Theorem 2, we obtain the desired result. □

We next prove a weak convergence theorem.

Theorem 3. *Let E be a Banach space. Suppose either of the following holds:*

(i) *E is strictly convex and satisfies Opial's condition; or*
(ii) *E is uniformly convex and its norm is Fréchet differentiable.*

Let C be a weakly compact convex subset of E and let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C with $\bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n < 1$, and let $\{I_n\}$ be a sequence of subsets of $\mathbb{N}$ satisfying $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} I_n = \mathbb{N}$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \left(1 - \sum_{i \in I_n} \lambda_i\right) x_n + \sum_{i \in I_n} \lambda_i T_i x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

Proof. Define a nonexpansive mapping S on C by

$$Sx = \frac{\sum_{n=1}^{\infty} \lambda_n T_n x}{\sum_{n=1}^{\infty} \lambda_n}$$

for $x \in C$. By Lemma 7, we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Assume that $\{x_n\}$ is not a weak convergent sequence. Then since C is weakly compact, there exist two distinct weak sequential limits z_1 and z_2 of the subsequence $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ respectively. By Lemma 1 and Lemma 2, z_1 is a fixed point of S and hence z_1 is a common fixed point of $\{T_n : n \in \mathbb{N}\}$ by Lemma 3. So is z_2. In the case of (i), using Lemma 7 again, we obtain

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{j\to\infty} \|x_{n_j} - z_1\| < \lim_{j\to\infty} \|x_{n_j} - z_2\| = \lim_{n\to\infty} \|x_n - z_2\| \\ &= \lim_{k\to\infty} \|x_{n_k} - z_2\| < \lim_{k\to\infty} \|x_{n_k} - z_1\| = \lim_{n\to\infty} \|x_n - z_1\|.\end{aligned}$$

This is a contradiction. In the case of (ii), for each $n \in \mathbb{N}$, we define a nonexpansive mapping U_n on C by

$$U_n x = \left(1 - \sum_{i\in I_n} \lambda_i\right) x + \sum_{i\in I_n} \lambda_i T_i x$$

for $x \in C$. Then $\{x_n\}$ can be written as $x_{n+1} = U_n U_{n-1} \cdots U_1 x_1$. By Lemma 3, we have $F(U_n) = \bigcap_{i\in I_n} F(T_i)$ for $n \in \mathbb{N}$ and hence

$$F(S) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(U_n).$$

Since z_1 and z_2 are weak subsequential limit and belong to $F(S)$, we have

$$z_1, z_2 \in \left(\bigcap_{n=1}^{\infty} \overline{co}\{W_m x : m \geq n\}\right) \bigcap \left(\bigcap_{n=1}^{\infty} F(U_n)\right),$$

where $W_n = U_n U_{n-1} \cdots U_1$ for $n \in \mathbb{N}$. By Lemma 4, we get a contradiction. This completes the proof. □

As a direct consequence of Theorem 3, we obtain the following.

Corollary 2. *Let E be a Banach space. Suppose either of the following holds:*

(i) *E is strictly convex and satisfies Opial's condition; or*
(ii) *E is uniformly convex and its norm is Fréchet differentiable.*

Let C be a weakly compact convex subset of E and let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n < 1$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \left(1 - \sum_{i=1}^{n} \lambda_i\right) x_n + \sum_{i=1}^{n} \lambda_i T_i x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

5. Appendix

Concerning Lemma 3, we prove the following characterization theorem.

Theorem 4. *Let E be a Banach space. Then the following are equivalent:*

(i) *E is not strictly convex;*

(ii) *there exist affine and nonexpansive mappings T_1 and T_2 defined on a compact convex subset C of E satisfying*

$$F\left(\frac{T_1+T_2}{2}\right) \neq F(T_1) \cap F(T_2).$$

Proof. By Lemma 3, (ii) implies (i). So, we shall show (i) implies (ii). Assume that E is not strictly convex. Then there exist u and v in E such that $\|u\| = \|v\| = \|u+v\|/2 = 1$ and $u \neq v$. We define a compact convex subset C of E by

$$C = \{\alpha u + \beta v : \alpha \geq 0, \beta \geq 0, \alpha + \beta \leq 1\}$$

and affine mappings T_1 and T_2 on C by

$$T_1(\alpha u + \beta v) = (\alpha + \beta)u \quad \text{and} \quad T_2(\alpha u + \beta v) = (\alpha + \beta)v.$$

Then T_1 and T_2 are nonexpansive mappings; see [18]. It is clear that

$$F\left(\frac{T_1+T_2}{2}\right) = \left\{\frac{\alpha}{2}(u+v) : \alpha \in [0,1]\right\} \quad \text{and}$$
$$F(T_1) \cap F(T_2) = \{0\}.$$

This completes the proof. □

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Received November 29, 2002 Revised January 28, 2003