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Asymptotic Behavior of Solutions for a Delay Reaction-Diffusion Equation of Neutral Type*

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Abstract: In this paper, we consider a delay reaction-diffusion equation of neutral type of the form:

$$\frac{\partial}{\partial t}(u(t,x)+pu(t-\tau,x))+q(t,x)u(t-\sigma,x)=a^2\Delta u(t,x) \qquad (*)$$

for $(t,x)\in\mathbb{R}^+\times\Omega$ with homogeneous Neumann boundary condition:

$$\frac{\partial}{\partial \mathbf{n}}u(t,x)=0 \text{ for } (t,x)\in\mathbb{R}^+\times\partial\Omega \qquad (**)$$

and initial condition:

$$u(t,x)=\phi(t,x) \text{ for } (t,x)\in[-\lambda,0]\times\bar{\Omega}, \qquad (***)$$

where $\tau>0,\sigma\in\mathbb{R}^+,p,a\in\mathbb{R},\lambda=\max\{\tau,\sigma\},q(t,x)>0$ for $(t,x)\in\mathbb{R}^+\times\Omega,\Omega$ is a bounded open region in $\mathbb{R}^n,\partial\Omega$ is the boundary of Ω, which is piecewise smooth, $\mathbf{n}$ is the exterior normal direction to $\partial\Omega$ and Δ is the Laplacian operator. We study various cases of p in the neutral term and obtain that if $p>1$ then every nonoscillatory solution of Initial and Boundary Value Problem (*)-(***) tends uniformly in $x\in\Omega$ to zero as $t\to\infty$; if $p=-1$ then every solution of Initial and Boundary Value Problem (*)-(***) oscillates and if $p<-1$ then every nonoscillatory solution of Initial and Boundary Value Problem (*)-(***) goes uniformly in $x\in\Omega$ to infinity or minus infinity under some hypotheses.

Keywords: asymptoic behavior, reaction-diffusion equations, delays, neutral type, oscillation

Mathematics Subject Classification (2000): 35B05, 35B40

1 Introduction

Consider the delay reaction-diffusion equation of neutral type of the form:

$$\frac{\partial}{\partial t}(u(t,x)+pu(t-\tau,x))+q(t,x)u(t-\sigma,x)=a^2\Delta u(t,x) \qquad (1)$$

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for $(t,x) \in \mathbb{R}^+ \times \Omega$ with homogeneous Neumann boundary condition:

$$\frac{\partial}{\partial \mathbf{n}} u(t,x) = 0 \text{ for } (t,x) \in \mathbb{R}^+ \times \partial\Omega \tag{2}$$

and initial condition:

$$u(t,x) = \phi(t,x) \text{ for } (t,x) \in [-\lambda, 0] \times \bar{\Omega}, \tag{3}$$

where $\tau > 0, \sigma \in \mathbb{R}^+, p, a \in \mathbb{R}, \lambda = \max\{\tau, \sigma\}, q(t,x) > 0$ for $(t,x) \in \mathbb{R}^+ \times \Omega, \Omega$ is a bounded open region in $\mathbb{R}^n, \partial\Omega$ is the boundary of Ω, which is piecewise smooth, $\mathbf{n}$ is the exterior normal direction to $\partial\Omega$ and $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator.

If we do not consider the diffusive term in Equation (1), then it reduces to a delay differential equation of neutral type of the form:

$$\frac{\mathrm{d}}{\mathrm{dt}} (x(t) + px(t-\tau)) + q(t)x(t-\sigma) = 0 \text{ for } t \in \mathbb{R}^+. \tag{4}$$

The asymptotic behavior and oscillation of solutions of Equation (4) have been extentively investigated (see for instance [1-5,7,10-16]).

In this paper, we shall apply the idea developed in the above literature to study the asymptotic behavior and oscillaton of Initial and Boundary Value Problem (1)-(3) and discuss various cases for $p \geq -1$, $p = -1$ and $p < -1$. One may refer the authors [8] for the study on Volterra reaction-diffusion equations of neutral type and Wu [9] for the general theory on delay reaction-diffusion equations. The method developed in this paper may be applicable to the investigation of delay reaction-diffusion equations of neutral type.

By a *regular* or *classical solution* of Initial and Boundary Value Problem (1)-(3) we mean a function $u(t,x)$, which is defined for $t \geq -\lambda$ and $x \in \bar{\Omega}$; satisfies Equation (1) for $(t,x) \in \mathbb{R}^+ \times \Omega$, satisfies homogeneous Neumann boundary condition (2) for $(t,x) \in \mathbb{R}^+ \times \partial\Omega$ and satisfies initial condition (3) for $(t,x) \in [-\lambda, 0] \times \bar{\Omega}$.

For the existence, uniqueness and regularity of solutions of Initial and Boundary Value Problem (1)-(3), one is refered to Wu [9].

A solution $u(t,x)$ of Initial and Boundary Value Problem (1)-(3) is said to be *eventually positive* or *negative* if there exists a $t^* \in \mathbb{R}^+$ such that $u(t,x) > 0$ or $u(t,x) < 0$ for $t \geq t^*$ and $x \in \Omega$.

A solution $u(t,x)$ of Initial and Boundary Value Problem (1)-(3) is said to be *oscillatory* if it is neither eventually positive nor eventually negative.

2 The Case: $p \geq -1$

THEOREM 1. *Assume that for any infinite sequence of disjoint open intervals* $I_i \subset \mathbb{R}^+$ *for* $i = 1, 2, \cdots$ *with* $\sup I_i \to \infty$ *as* $i \to \infty$ *and any sequence of open* *subregions* $\Omega_i \subset \Omega$ *for* $i = 1, 2, \cdots$ *such that for any* $i_0 \geq 1$

$$\sum_{i=i_0}^{\infty} \int_{I_i \times \Omega_i} q(t,x)\mathrm{d}x\mathrm{d}t = \infty. \tag{5}$$

Let $u(t,x)$ *be a nonoscillatory solution of Initial and Boundary Value Problem (1)-(3) with* $p \geq -1$. *Then*

$$\lim_{t\to\infty} u(t,x) = 0 \text{ uniformly in } x \in \Omega. \tag{6}$$

PROOF. Without loss of generality, we let $u(t,x)$ be an eventually positive solution of Initial and Boundary Value Problem (1)-(3). So, there exists a $t_1 \in \mathbb{R}^+$ such that $u(t,x) > 0$ for $t \geq t_1$ and $x \in \Omega$. This follows that $u(t-\tau,x) > 0$ and $u(t-\sigma,x) > 0$ for $t \geq t_2 := t_1 + \lambda$ and $x \in \Omega$. Let

$$v(t,x) = u(t,x) + pu(t-\tau,x) \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega. \tag{7}$$

Then, we have from Equation (1)

$$\frac{\partial}{\partial t}v(t,x) = -q(t,x)u(t-\sigma,x) + a^2\Delta u(t,x) \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega. \tag{8}$$

Integrating the both sides of (8) over Ω, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -\int_{\Omega} q(t,x)u(t-\sigma,x)\mathrm{d}x + a^2\int_{\Omega}\Delta u(t,x)\mathrm{d}x \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega, \tag{9}$$

where $V(t) = \int_\Omega v(t,x)\mathrm{d}x$ for $t \in \mathbb{R}^+$.

By virtue of the Green's first identity (see for instance Protter and Weinberger [6]) and homogeneous Neumann boundary condition (2), we know that

$$\int_{\Omega}\Delta u(t,x)\mathrm{d}x = \int_{\partial\Omega}\frac{\partial}{\partial \mathbf{n}}u(t,x)\mathrm{d}x = 0 \text{ for } t \in \mathbb{R}^+.$$

Therefore, we obtain from (9)

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -\int_{\Omega} q(t,x)u(t-\sigma,x)\mathrm{d}x \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega. \tag{10}$$

In the sequel, we shall consider two cases: (I). $p \in \mathbb{R}^+$ and (II). $-1 \leq p < 0$.

Case I. $p \in \mathbb{R}^+$

¿From (7) and (10), we know that $v(t,x) > 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}V(t) < 0$ for $t \geq t_2$ and $x \in \Omega$. In what follows, we want to show that (6) holds. If it is not the case, then there exists a sequence of points $\left(t^{(i)}, x^{(i)}\right) \in \mathbb{R}^+ \times \Omega$ for $i = 1, 2, \cdots$ such that $\lim_{i\to\infty} u\left(t^{(i)} - \sigma, x^{(i)}\right) = \delta = \text{const.} > 0$.

We may let $\{t^{(i)}\}$ be an increasing sequence. Hence, we can select an $i_1 \geq 1$ such that $t^{(i)} \geq t_2$ for $i \geq i_1$ and $u\left(t^{(i)} - \sigma, x^{(i)}\right) > \frac{\delta}{2}$ for $i \geq i_1$. From the continuity of the solution $u(t,x)$, there must exist an infinite sequence of disjoint intervals $I_i \subset \mathbb{R}^+$ with the property: $\inf I_{i_1} \geq t_2, \sup I_i \to \infty$ as $i \to \infty$ and a sequence of open regions $\Omega_i \subset \Omega$ such that $\left(t^{(i)}, x^{(i)}\right) \in I_i \times \Omega_i$ for $i \geq i_1$ and

$$u(t-\sigma, x) > \frac{\delta}{2} \text{ for } (t,x) \in I_i \times \Omega_i, i \geq i_1. \tag{11}$$

For any $t \geq t_2$, there exists an $i_2 \geq i_1$ such that $\sup I_{i_2} \leq t \leq \sup I_{i_2+1}$. This follows from (10) and (11) by noting that $V(t)$ is nonincreasing

$$V(\sup I_{i_1}) - V(\inf I_{i_1}) \leq -\frac{\delta}{2} \int\limits_{I_{i_1}\times\Omega_{i_1}} q(t,x)\mathrm{d}x\mathrm{d}t,$$

$$V(\sup I_{i_1+1}) - V(\inf I_{i_1+1}) \leq -\frac{\delta}{2} \int\limits_{I_{i_1+1}\times\Omega_{i_1+1}} q(t,x)\mathrm{d}x\mathrm{d}t,$$

$$\cdots\cdots$$

$$V(t) - V(\inf I_{i_2}) \leq V(\sup I_{i_2}) - V(\inf I_{i_2}) \leq -\frac{\delta}{2} \int\limits_{I_{i_2}\times\Omega_{i_2}} q(t,x)\mathrm{d}x\mathrm{d}t.$$

Summing up the above $i_2 - i_1 + 1$ inequalities, we have

$$V(t) - V(\inf I_{i_1}) \leq -\frac{\delta}{2} \sum_{i=i_1}^{i_2} \int\limits_{I_i\times\Omega_i} q(t,x)\mathrm{d}x\mathrm{d}t.$$

This follows that $V(t) \to -\infty$ as $t \to \infty$ and contradicts that $v(t,x) > 0$ for $t \geq t_2$ and $x \in \Omega$ and the definition of $V(t)$ for $t \in \mathbb{R}^+$.

Case II. $-1 \leq p < 0$

We first prove that $V(t) \geq 0$ for $t \geq t_2$. If it is not the case, by noting that $V(t)$ in nonincreasing for $t \geq t_2$, we can suppose that there exists a $t_3 \geq t_2$ such that $V(t) < 0$ for $t \geq t_3$. Therefore, there exists a $\mu = \text{const.} > 0$ such that $V(t) \leq -\mu$ for $t \geq t_3$. So, we have from (7)

$$U(t) \leq -\mu - pU(t-\tau) \text{ for } t \geq t_3, \tag{12}$$

where $U(t) = \int_\Omega u(t,x)\mathrm{d}x$ for $t \in \mathbb{R}^+$.

If $p = -1$, we have from (12) $U(t_3 + i\tau) \leq -(i+1)\mu + U(t_3 - \tau) \to -\infty$. This is a contradiction.

If $-1 < p < 0$, we have from (12)

$$U(t_3 + i\tau) \le -\mu \cdot \frac{1-(-p)^{i+1}}{1+p} + (-p)^{i+1} U(t_3 - \tau) \to -\frac{\mu}{1+p} < 0.$$

This is also a contradiction.

In what follows, we can prove that (6) holds by using the same arguments as that in Case I. We omit it. This completes the proof.

REMARK 1. We now show that Initial and Boundary Value Problem (1)-(3) with $p > -1$ has an eventually positive solution which tends uniformly in $x \in \Omega$ to zero as $t \to \infty$ by an example.

Consider the equation:

$$\frac{\partial}{\partial t}(u(t,x) + pu(t-1,x)) + q(t,x)u(t-2,x) = \Delta u(t,x) \tag{13}$$

for $t \ge 3$ and $x \in (1,2)$ with homogeneous Neumann boundary condition:

$$\frac{\partial}{\partial x}u(t,1) = \frac{\partial}{\partial x}u(t,2) \text{ for } t \ge 3 \tag{14}$$

and initial condition:

$$u(t,x) = \frac{x}{t} \text{ for } (t,x) \in [1,3] \times [1,2], \tag{15}$$

where $p > -1$ and $q(t,x) = \frac{1+p}{t-1} - \frac{(2+p)t^2 - t + 1}{t^2(t-1)}$.

It can be verified that $u(t,x) = \frac{x}{t}$ is a positive solution of Initial and Boundary Value Problem (13)-(15). Obviously, if $p > -1$, then $q(t,x)$ is not infinitely integrable. On the other hand, if $p = -1$, then $q(t,x)$ is infinitely integrable.

3 The Case: $p = -1$

THEOREM 2. *Assume that for any increasing sequence of disjoint open intervals $I_i \subset \mathbb{R}^+$ for $i = 1, 2, \cdots$ with $\sup I_i \to \infty$ as $i \to \infty$ and any sequence of subregions $\Omega_i \subset \Omega$ for $i = 1, 2, \cdots$ such that for any $i_0 \ge 1$*

$$\int_\sigma^\infty tq^*(t) \sum_{i=i_0}^{\infty} \int_{([t-\sigma, \sup I_i] \bigcap I_i) \times \Omega_i} q(s,x)\mathrm{d}x\mathrm{d}s = \infty, \tag{16}$$

where $q^*(t) := \inf_{x \in \Omega} q(t,x)$ for $t \in \mathbb{R}^+$. Then every solution of Initial and Boundary Value Problem (1)-(3) with $p = -1$ oscillates.

PROOF. Suppose, for the sake of contradiction, that Initial and Boundary Value Problem (1)-(3) has an eventually positive solution $u(t,x)$. Take a $t_1 \in \mathbb{R}^+$ such that $u(t,x) > 0$ for $t \geq t_1$ and $x \in \Omega$. So, $u(t-\tau,x), u(t-\sigma,x) > 0$ for $t \geq t_2 := t_1 + \lambda$ and $x \in \Omega$. Let

$$v(t,x) = u(t,x) - u(t-\tau,x) \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega. \tag{17}$$

Then by Equation (1), we have

$$\frac{\partial}{\partial t} v(t,x) = -q(t,x)u(t-\sigma,x) + a^2 \Delta u(t,x) \text{ for } (t,x) \in \mathbb{R}^+ \times \Omega. \tag{18}$$

Integrating the both sides of (18) over Ω, we have from Green's first identity that (10) holds. From (10), we know that $V(t)$ is nonincreasing for $t \geq t_2$. By using the similar arguments as in the proof of Theorem 1, we know that $V(t) \geq 0$ for $t \geq t_2$. This follows from (17) that

$$U(t) \geq U(t-\tau) \text{ for } t \geq t_2. \tag{19}$$

Therefore, there exist an $M' = \text{const.} > 0$ and a $t_3 \geq t_2$ such that $U(t) \geq M'$ for $t \geq t_3$ and $U(t-\tau), U(t-\sigma) \geq M'$ for $t \geq t_4 := t_3 + \lambda$.

By the definition of $U(t)$, there exists a subsequence $\left(t^{(i)}, x^{(i)}\right) \in I_i \times \Omega_i \subset \mathbb{R}^+ \times \Omega$ such that $u(t-\sigma,x) \geq M := \frac{M'}{\text{mes}\Omega}$ for $(t,x) \in I_i \times \Omega_i \subset \mathbb{R}^+ \times \Omega$, where $\inf I_i \geq t_4$ and $\Omega_i \subset \Omega$ for $i = 1, 2, \cdots$. It is obvious that we can let I_i be an infinite sequence of disjoint open intervals with $\sup I_i \to \infty$ as $i \to \infty$. In fact, if it is not the case, then we consider $t \geq \sup I_i$. This will lead a contradiction.

By virtue of (10), we can get

$$\frac{\text{d}}{\text{d}t} V(t) \leq -\int\limits_{\Omega_i} q(t,x)u(t-\sigma,x)\text{d}x \leq -M \int\limits_{\Omega_i} q(t,x)\text{d}x \text{ for } t \in I_i, i = 1, 2, \cdots.$$

For any $t \geq t_4$, we select the least i_1 such that $t \leq \inf I_{i_1}$. This yields by noting that $V(t)$ is nonincreasing

$$\begin{aligned} V(\sup I_{i_1}) - V(t) &\leq V(\sup I_{i_1}) - V(\inf I_{i_1}) \\ &\leq -M \int\limits_{([t,\sup I_{i_1}] \bigcap I_{i_1}) \times \Omega_{i_1}} q(s,x)\text{d}x\text{d}s \text{ for } t \geq t_4, \end{aligned}$$

$$V(\sup I_{i_1+1}) - V(\inf I_{i_1+1}) \leq -M \int\limits_{([t,\sup I_{i_1+1}] \bigcap I_{i_1+1}) \times \Omega_{i_1+1}} q(s,x)\text{d}x\text{d}s \text{ for } t \geq t_4,$$

$$\cdots\cdots,$$

$$V(\sup I_i) - V(\inf I_i) \leq -M \int\limits_{([t,\sup I_i] \bigcap I_i) \times \Omega_i} q(s,x)\text{d}x\text{d}s \text{ for } t \geq t_4$$

$$\cdots\cdots.$$

Summing up the above inequalities, we obtain that

$$V(t) \geq M \sum_{i=i_1}^{\infty} \int_{([t,\sup I_i] \bigcap I_i)\times\Omega_i} q(s,x)\mathrm{d}x\mathrm{d}s \ \text{ for } \ t \geq t_4.$$

Now, we let T be such that $T = [\frac{t-t_4}{\tau}]$ for $\inf I_{i_1} \geq t \geq t_4$, where $[\cdot]$ is the greatest integer function. Then, we have from (17)

$$\begin{aligned} U(t) &\geq U(t-\tau) + M \sum_{i=i_1}^{\infty} \int_{([t,\sup I_i] \bigcap I_i)\times\Omega_i} q(s,x)\mathrm{d}x\mathrm{d}s \\ &\geq U(t-T\tau) + M \left(\sum_{i=i_1}^{\infty} \int_{([t,\sup I_i] \bigcap I_i)\times\Omega_i} q(s,x)\mathrm{d}x\mathrm{d}s + \cdots \right. \\ &\quad \left. + \sum_{i=i_1}^{\infty} \int_{([t-(T-1)\tau,\sup I_i] \bigcap I_i)\times\Omega_i} q(s,x)\mathrm{d}x\mathrm{d}s \right) \\ &\geq MT \sum_{i=i_1}^{\infty} \int_{([t,\sup I_i] \bigcap I_i)\times\Omega_i} q(s,x)\mathrm{d}x\mathrm{d}s \ \text{ for } \ \inf I_{i_1} \geq t \geq t_4. \end{aligned}$$

Again, by making use of (10) and (19), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} V(t) &\leq -q^*(t)U(t-\sigma) \\ &\leq -q^*(t)MT \sum_{i=i_1}^{\infty} \int_{([t-\sigma,\sup I_i] \bigcap I_i)\times\Omega_i} q(s,x)\mathrm{d}x\mathrm{d}s \\ &:= -H(t) \ \text{ for } \ \inf I_{i_1} \geq t \geq t_4. \end{aligned} \tag{20}$$

Noting that $\frac{T}{t} \to \frac{1}{\tau}$ as $t \to \infty$, we have

$$\frac{H(t)}{tq^*(t) \sum_{i=i_1}^{\infty} \int_{([t-\sigma,\sup I_i] \bigcap I_i)\times\Omega_i} q(s,x)\mathrm{d}x\mathrm{d}s} = \frac{MT}{t} \to \frac{M}{\tau} \ \text{ as } \ t \to \infty. \tag{21}$$

Hence, we have from (16) and (21), $\int_{\sigma}^{\infty} H(t)\mathrm{d}t = \infty$. This yields from (20) that $V(t) \to -\infty$ as $t \to \infty$. This is a contradiction. The proof is complete.

REMARK 2. In a matter of fact, there is no nonoscillatory solution for Initial and Boundary Value Problem (1)-(3) with $p = -1$. It can be also seen from the example in Remark 1.

4 The Case: $p < -1$

THEOREM 3. *Assume that the assumptions in Theorem 1 hold. Let $u(t,x)$ be a nonoscillatory solution of Initial and Boundary Value Problem (1)-(3) with $p < -1$. Then*

$$\lim_{t\to\infty} u(t,x) = \infty \ \text{or} \ -\infty \ \text{uniformly in} \ x \in \Omega.$$

PROOF. Without loss of generality, we let $u(t,x)$ be an eventually positive solution of Initial and Boundary Value Problem (1)-(3). We first show that (6) does not hold. If it is not the case, then $\lim_{t\to\infty} v(t,x) = 0$ uniformly in $x \in \Omega$ by (7). This follows that $\lim_{t\to\infty} V(t) = 0$. But, we know that $V(t)$ is nonincreasing from (10). Hence, $V(t) \geq 0$ for $t \geq t_2$. From the definition of $V(t)$, there exists a sequence of points $\left(t^{(i)}, x^{(i)}\right) : t^{(i)} \geq t_2$ and $x^{(i)} \in \Omega$ for $i = 1, 2, \cdots$ such that $v\left(t^{(i)}, x^{(i)}\right) \geq 0$. Thus, we have from (7)

$$u\left(t^{(i)}, x^{(i)}\right) \geq -pu\left(t^{(i)} - \tau, x^{(i)}\right) \ \text{for} \ t^{(i)} \geq t_2 \ \text{and} \ x^{(i)} \in \Omega, i = 1, 2, \cdots.$$

This contradicts (6) by noting that $p < -1$ or $-p > 1$.

In the sequel, we shall prove that

$$\lim_{t\to\infty} u(t,x) = \infty \ \text{uniformly in} \ x \in \Omega. \tag{22}$$

If it is not the case, then there exists a sequence of points $\left(t^{(i)}, x^{(i)}\right) \in \mathbb{R}^+ \times \Omega$ for $i = 1, 2, \cdots$ such that $0 < \lim_{i\to\infty} u\left(t^{(i)} - \sigma, x^{(i)}\right) = L < \infty$.

We may let $\{t^{(i)}\}$ be an increasing sequence. So, we can choose an $i_1 \geq 1$ such that $t^{(i)} \geq t_3$ for $i \geq i_1$ and $0 < u\left(t^{(i)} - \sigma, x^{(i)}\right) < \frac{L}{2}$ for $i \geq i_1$.

By virtue of the continuity of solution $u(t,x)$, we can take an infinite sequence of disjoint open intervals $I_i \subset \mathbb{R}^+$ with $\inf I_{i_1} \geq t_2, \sup I_i \to \infty$ as $i \to \infty$ and a sequence of subregions $\Omega_i \subset \Omega$ for $i = 1, 2, \cdots$ such that $\left(t^{(i)}, x^{(i)}\right) \in I_i \times \Omega_i$ and

$$0 < u(t - \sigma, x) < \frac{L}{2} \ \text{for} \ (t,x) \in I_i \times \Omega_i, i \geq i_1.$$

As in the proof of Theorem 1, we have $\lim_{t\to\infty} V(t) = -\infty$. Consequently,

$$\lim_{t\to\infty} v(t,x) = -\infty \ \text{uniformly in} \ x \in \Omega. \tag{23}$$

In a matter of fact, noting that $v(t,x) \geq 0$ can not hold for all $(t,x) \in \mathbb{R}^+ \times \Omega$, if (23) does not hold, then there exists a sequence of points $\left(t^{(i)}, x^{(i)}\right) \in \mathbb{R}^+ \times \Omega$ such that $0 > \lim_{i\to\infty} v\left(t^{(i)}, x^{(i)}\right) = -M^* > -\infty$, where $M^* = \text{const.} > 0$.

We may let $\{t^{(i)}\}$ be an increasing sequence. So, there exists an $i_2 \geq i_1$ such that $v\left(t^{(i)}, x^{(i)}\right) > -M^* - 1$ for $i \geq i_2$. By continuity of $v(t,x)$, there exist an infinite sequence of disjoint open intervals $I_i \subset \mathbb{R}^+$ for $i = 1, 2, \cdots$ with $\inf I_{i_2} \geq t_2, \sup I_i \to \infty$ as $i \to \infty$ and a sequence of subregion $\Omega_i \subset \Omega$ for $i = 1, 2, \cdots$ such that $0 > v(t,x) > -M^* - 1$ for $(t,x) \in I_i \times \Omega_i, i \geq i_2$.

Let $I_i^0 \times \Omega_i^0 \subset \mathbb{R}^+ \times \Omega$ for $i = 1, 2, \cdots$ be such that

$$v(t,x) \begin{cases} < 0 & \text{for } (t,x) \in I_i^0 \times \Omega_i^0, i = 1, 2, \cdots, \\ \geq 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} V(t) &= \int_\Omega v(t,x)\mathrm{d}x = \int_{\Omega \setminus \Omega_i^0} v(t,x)\mathrm{d}x + \int_{\Omega_i^0} v(t,x)\mathrm{d}x \\ &\geq \int_{\Omega_i^0} v(t,x)\mathrm{d}x \quad \text{for } t \in I_i^0, i = 1, 2, \cdots. \end{aligned}$$

On the other hand, it is easy to see that $\bigcup I_i \subset \bigcup I_i^0$ and $\bigcup \Omega_i \subset \bigcup \Omega_i^0$ for $i \geq i_2$. Hence, we obtain that $V(t) \geq \int_{\Omega_i} v(t,x)\mathrm{d}x \geq -(M^* + 1)\mathrm{mes}\Omega_i$ for $t \in I_i, i \geq i_2$, where $\mathrm{mes}\Omega_i$ means the measure of Ω_i for $i = 1, 2, \cdots$. This is a contradiction if we let $t \to \infty$.

Now, dividing by $u(t-\tau, x)$ in the both sides of (7), we have

$$\frac{v(t,x)}{u(t-\tau,x)} = \frac{u(t,x)}{u(t-\tau,x)} + p \geq p \quad \text{for } t \geq t_2 \text{ and } x \in \Omega. \tag{24}$$

Taking $\left(t^{(i)} - \tau + \sigma, x^{(i)}\right) \in I_i \times \Omega_i$ for $i = 1, 2, \cdots$, we know that $\lim_{i \to \infty} \frac{v\left(t^{(i)}, x^{(i)}\right)}{u\left(t^{(i)} - \tau, x^{(i)}\right)} = -\infty$. This contradicts (22). Thus, the proof is complete.

REMARK 3. We can show by an example that Initial and Boundary Value Problem (1)-(3) with $p < -1$ has an eventually positive solution $u(t,x)$ which goes uniformly in $x \in \Omega$ to infinity as $t \to \infty$.

Consider the equation:

$$\frac{\partial}{\partial t}\left(u(t,x) + pu(t-1,x)\right) + q(t,x)u(t-2,x) = \Delta u(t,x) \tag{25}$$

for $t \geq 3$ and $x \in (1,2)$ with homogeneous Neumann boundary condition:

$$\frac{\partial}{\partial x}u(t,1) = \frac{\partial}{\partial x}u(t,2) \quad \text{for } t \geq 3 \tag{26}$$

and initial condition:

$$u(t,x) = xt \quad \text{for } (t,x) \in [1,3] \times [1,2], \tag{27}$$

where $p < -1$ and $q(t,x) = -\frac{1+p}{t-2}$.

It can be verified that $u(t,x) = xt$ is a positive solution of Initial and Boundary Value Problem (25)-(27). Obviously, if $p < -1$, then $q(t,x)$ is not infinitely integrable.

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